Topology Vol. 12, pp. 381–388. Pergamon Press 1973, Printed in Great Britain

DEFORMATIONS OF SECONDARY CHARACTERISTIC CLASSES

JAMES L. HEITSCH

(*Received* 5 *October* 1972; *revised* 24 *February* 1973)

§1. INTRODUCTION

IN THIS paper we exhibit a formula for the derivative of a deformation of a secondary characteristic class. These classes were originally defined by Godbillon and Vey [7] and Chern and Simons [6] and have been the subject of great interest in the theory of foliations. See, for example, Bott and Haefliger [4] and Simons [10].

The formula implies that certain of the generalized Godbillon and Vey characteristic classes for foliations are invariants of the connected components of the space of foliations on a manifold. In [12], D. Lehmann gives a proof of this theorem without using the derivative formula. We remark that the cohomology classes of a codimension q foliation coming from the Simons characters [10] are also constant on connected components unless the character occurs in dimension $2q + 1$. For real foliations, the examples of Thurston [11] show that in dimension $2q + 1$ it is possible to have classes which vary continuously. For examples in the complex case see Bott [2].

The formula is a generalization of a formula of Chern and Simons [6]. However, their formula was for forms defined on the principal bundle over a manifold, while ours is for forms defined on the manifold itself and so can be applied to the theory of secondary classes. The reader should also note that while the results here are stated only for $B\Gamma_q$ and WO_q, similar results hold for the spaces associated to complex foliations and foliations with trivial normal bundles.

References for our point of view towards characteristic classes in general and secondary characteristic classes in particular are [3], [5], [8] and [9]. Finally, I would like to thank H. B. Lawson and R. Bott for helpful conversations, and J. Vey for permitting me to include his results on $H^*(WO_q)$.

§2. THE FORMULA AND ITS APPLICATIONS

Let G be a compact Lie group, g its Lie algebra, and $I(G)$ the ring of ad G invariant symmetric multilinear real valued functions on g. Let M be a manifold and E a principal G bundle over M. Let ω_0 be a fixed connection form on E and ω_1^s a differentiable family of connection forms on E depending on $s \in \mathbb{R}$.

THEOREM 1. *Suppose $f \in I(G)$ is homogeneous of degree k and let*

$$\omega_t^s = t\omega_1^s + (1-t)\omega_0 \qquad t \in [0, 1]$$

$$\Omega_t^s = \mathrm{d}\omega_t^s - \frac{1}{2}[\omega_t^s, \omega_t^s] = \textit{curvature of } \omega_t^s$$

$$\psi_s = \partial/\partial s(\omega_1^s)$$

$$\Delta_f(\omega_1^s, \omega_0) = k\int_0^1 f(\omega_1^s - \omega_0, \Omega_t^s, \ldots, \Omega_t^s)\,\mathrm{d}t,$$

and

$$V = \int_0^1 tf(\psi_s, \omega_1^s - \omega_0, \Omega_t^s, \ldots, \Omega_t^s)\,\mathrm{d}t.$$

Then

$$(*) \qquad \partial/\partial s\,(\Delta_f(\omega_1^s, \omega_0)) = k(k-1)\,\mathrm{d}V + kf(\psi_s, \Omega_1^s, \ldots, \Omega_1^s).$$

The proof appears in Section 3. Note that $\Delta_f(\omega_1^s, \omega_0)$ is a real valued $2k-1$ form on M and that

$$(**) \qquad \mathrm{d}(\Delta_f(\omega_1^s, \omega_0)) = f(\Omega_1^s, \ldots, \Omega_1^s) - f(\Omega_0^s, \ldots, \Omega_0^s).$$

The classical importance of $\Delta_f(\omega_1^s, \omega_0)$ is the formula (**) for it shows that the Weil homomorphism $W\colon I(G) \to H^*(M, \mathbb{R})$ is independent of the choice of connection. With the discovery of the Bott vanishing theorem [1] this form has been of great importance in the study of foliations, and the fact that it is not independent of the choice of connection is precisely why it is so interesting.

COROLLARY 1. *Let f, ω_1^s, Ω_1^s and ψ_s be as in Theorem 1. Then*

$$\partial/\partial s(f(\Omega_1^s, \ldots, \Omega_1^s)) = \mathrm{d}(kf(\psi_s, \Omega_1^s, \ldots, \Omega_1^s)).$$

In [4], Bott and Haefliger construct a differential graded complex denoted by WO_q. We recall their construction. Let $\mathbb{R}[c_1, \ldots, c_q]$ be the polynomial ring over $\mathbb{R}$ in the variables $c_1, \ldots, c_q$ with degree $c_i = 2i$. Let $\Lambda(h_1, h_3, \ldots, h_{2k+1})$ be the exterior algebra on the indicated h's where degree $h_i = 2i - 1$ and $2k+1$ is the largest odd integer $\leq q$. Let $\mathbb{R}[c_1, \ldots, c_q]$ be the ring $\mathbb{R}[c_1, \ldots, c_q]$/(elements of degree $> 2q$). Then WO_q is the differential complex $\Lambda(h_1, h_3, \ldots, h_{2k+1}) \otimes \mathbb{R}[c_1, \ldots, c_q]$ where $\mathrm{d}(1 \otimes c_i) = 0$ and $\mathrm{d}(h_i \otimes 1) = 1 \otimes c_i$.

For any foliation F of codimension q on a manifold M they obtain a map $\alpha_F : WO_q \to A(M)$ where $A(M)$ is the ring of differential forms on M. The map α_F may be described as follows. Let ω_1 be a torsion free connection [3] and ω_0 a riemannian connection on the normal bundle of F. Then

$$\alpha_F(h_i) = \Delta_{c_i}(\omega_1, \omega_0)$$

$$\alpha_F(c_i) = c_i(\Omega_1, \ldots, \Omega_1)$$

where the c_i's on the right are the Chern polynomials in $I(GL_q)$ and Ω_1 is the curvature of ω_1. It is the essence of this construction that the induced map $\alpha_F^*\colon H(WO_q) \to H^*(M, \mathbb{R})$ depends only on the foliation and so there is a well-defined map

$$\alpha^* : H^*(WO_q) \to H^*(B\Gamma_q, \mathbb{R})$$

making the diagram

$$\begin{array}{ccc} & H^*(WO_q) & \\ {\scriptstyle a_F^*}\swarrow & & \searrow{\scriptstyle a^*} \\ H^*(M, \mathbb{R}) & \xleftarrow{\;f^*\;} & H^*(B\Gamma_q, \mathbb{R}) \end{array}$$

commute, where $f: M \to B\Gamma_q$ is the classifying map of F.

Jacques Vey has given the following characterization of $H^*(WO_q)$.

Let $\mathscr{J}$ be the collection of finite non-decreasing sequences of positive integers with sum $\leq q$. For $J \in \mathscr{J}$ denote this sum by $|J|$. Let $\mathscr{I} = \{1, 3, \ldots, 2k+1\}$. To each element $J \in \mathscr{J}$ and each subset $I \subseteq \mathscr{I}$ we associate the tensor $h_I c_J \in WO_q$ defined as follows: let $i_1 < i_2 < \cdots < i_l$ be the elements of I and let $J = (j_1, \ldots, j_m)$ then

$$h_I c_J = h_{i_1} \wedge \cdots \wedge h_{i_l} \otimes c_{j_1} \cdots c_{j_m}.$$

If either I or J is empty the corresponding factor is 1. It is clear that the $h_I c_J$ form a basis of WO_q.

For each $I \subseteq \mathscr{I}$ denote by i^0 the smallest element of I, if $I = \phi$ then $i^0 = \infty$. For each $J \in \mathscr{J}$ denote by j^0 the smallest element of $J \cap \mathscr{I}$ and if $J \cap \mathscr{I} = \phi$, $j^0 = \infty$.

THEOREM 2 (J. Vey). *A basis for $H^*(WO_q)$ is given by the classes of the tensors $h_I c_J$ which satisfy*

(1) $i^0 + |J| > q$;

(2) $i^0 \leq j^0$.

Note that condition (1) says that $h_I c_J$ is a cocycle and $J \neq \phi$ if $h_I c_J \neq 1$. Condition (2) assures that the $h_I c_J$ are independent (e.g. in WO_3, $h_1 c_3$ and $h_3 c_1$ represent the same class but the theorem disallows $h_3 c_1$ since $i^0 = 3$ and $j^0 = 1$). We call these cocycles the basic cocycles.

Proof (J. Vey). To prove the result we filter WO_q by the degrees of the c_i's. Then $E_2 = WO_q$ and the differentials are defined as follows. If $s \in \mathscr{I}$ then $d_{2s} h_s = c_s$ and d_{2s} is zero on all the other h_i and all the c_j; if $r \neq 2s$ where $s \in \mathscr{I}$ then $\mathrm{d}r = 0$. The theorem is immediate from the following.

LEMMA. *A basis for E_{2s} is given by the classes of the tensors $h_I c_J$ which satisfy one of the conditions*

$$A_s : i^0 < s,\ i^0 + |J| > q \text{ and } i^0 \leq j^0$$

$$B_s : i^0 \geq s \text{ and } j^0 \geq s.$$

Proof. The proof is by induction. For $s = 1$ all the $h_I c_J$ are of type B_1. Suppose the lemma is true for s, to prove it for $s + 1$ we split B_s into four types.

$$\begin{array}{llll} B_s^1: & i^0 = s & i^0 + |J| \leq q & j^0 \geq s \\ B_s^2: & i^0 = s & i^0 + |J| > q & j^0 \geq s \\ B_s^3: & i^0 > s & & j^0 = s \\ B_s^4: & i^0 > s & & j^0 > s. \end{array}$$

Observe that d_{2s} is zero on tensors of type A_s, B_s^2, B_s^3 and B_s^4, that if $s \notin \mathscr{I}$ the sets B_s^1, B_s^2 and B_s^3 are empty and that $A_{s+1} = A_s \cup B_s^2$ and $B_{s+1} = B_s^4$. Thus we need only show that $d_{2s}(B_s^1) = B_s^3$ for then we will have that a basis of $E_{2s+1} \simeq E_{2s+2}$ is given by tensors of type A_s, B_s^2 and B_s^4.

Suppose $h_I c_J \in B_s^1$ and $s \in \mathscr{I}$. Then $h_I c_J = h_s \wedge h_{i_2} \wedge \cdots \wedge h_{i_l} \otimes c_{j_1} \cdots c_{j_m}$ and $d_{2s}(h_I c_J) = h_{i_2} \wedge \cdots \wedge h_{i_l} \otimes c_{j_1} \cdots c_{j_\alpha} c_s c_{j_{\alpha+1}} \cdots c_{j_m}$ where j_α is the largest element of J less than s. The set $J' = (j_1, \ldots, j_\alpha, j_s, j_{\alpha+1}, \ldots, j_m)$ is in $\mathscr{I}$ since $|J'| = s + |J| = i^0 + |J| \leq q$ and $(j')^0 = s$. Thus $d_{2s}(h_I c_J) \in B_s^3$ and it is easy to see that d_{2s} gives a bijection between the two types. *Q.E.D.*

Notice that the elements of $H^*(WO_q)$ of type A_{q+1} are the exotic characteristic classes while those of type B_{q+1} are the Chern classes.

The next result has also been remarked by J. Simons. Compare [10, Proposition 3.14].

THEOREM 3. *Let F_s, $s \in [0, 1]$, be a differentiable family of foliations of codimension q. Let $h_I c_J$ be a basic cocycle and $\langle h_I c_J \rangle$ its cohomology class. Then*

$$\alpha_{F_0}{}^*\langle h_I c_J \rangle = \alpha_{F_1}{}^*\langle h_I c_J \rangle$$

unless

$$i^0 + |J| = i_1 + j_1 + \cdots + j_m = q + 1.$$

Proof. By a differentiable family of foliations we mean a family of foliations whose associated family of normal bundles is differentiable. Since all the normal bundles are isomorphic we may assume that we have one bundle, one reimannian connection ω_0 and a family of connections $\omega_1{}^s$ with the property that $(\Omega_1{}^s)^{q+1} \equiv 0$.

If $I = (i_1, \ldots, i_l)$ we denote $(i_1, \ldots, \hat{i}_p, \ldots, i_l)$ by I_p where $\wedge$ means that entry is deleted and use the same meaning for J_q. Then from Theorem 1 and Corollary 1 we have that modulo exact forms

$$(*)\partial/\partial_s(\alpha_{Fs}(h_I c_J))$$
$$= \sum_{p=1}^{l} i_p \alpha_{F_s}(h_{I_p} c_J) \wedge c_{i_p}(\psi_s, \Omega_1{}^s, \ldots, \Omega_1{}^s) + \sum_{q=1}^{m} i_q \alpha_{F_s}(h_I c_{J_q}) \wedge d(c_{j_q}(\psi_s, \Omega_1{}^s, \ldots, \Omega_1{}^s)).$$

Now

$$\alpha_{F_s}(h_I c_{J_q}) \wedge d(c_{j_q}(\psi_s, \Omega_1{}^s, \ldots, \Omega_1{}^s))$$
$$= \pm \alpha_{F_s}(h_{I_1} c_{J_q}) \wedge c_{i_1}(\Omega_1{}^s, \ldots, \Omega_1{}^s) \wedge c_{j_q}(\psi_s, \Omega_1{}^s, \ldots, \Omega_1{}^s).$$

Applying the vanishing theorem of Bott to the right side of (*) shows that each term in the first sum $= 0$ unless $i_p + |J| \leq q + 1$ and each term in the second sum $= 0$ unless $i_1 + |J| \leq q + 1$. Because of the restrictions on the indices which may appear in the basic cocycles the right side of (*) is non zero only if $i_1 + |J| = q + 1$.

Comment 1. Actually a slightly more general theorem holds. We say a connection ω is q-flat if its curvature Ω satisfies $\Omega^{q+1} \equiv 0$. Notice that a q-flat connection gives rise to a map $\alpha_\omega{}^* : H^*(WO_q) \to H^*(M, \mathbb{R})$. Let P be a bundle and $F_q(P)$ the space of q-flat connections on P.

THEOREM 4. *If ω_0 and ω_1 lie in the same path component of $F_q(P)$ then $\alpha_{\omega_0}{}^*(\langle h_I c_J\rangle) = \alpha_{\omega_1}{}^*(\langle h_I c_J\rangle)$ unless $i^0 + |J| = q + 1$.*

Comment 2. We can give the following interpretation to the theorem. There is a natural map ρ: $WO_{q+1} \to WO_q$ given by $\rho(c_{q+1}) = 0$, $\rho(h_{q+1}) = 0$ if h_{q+1} exists and $\rho(c_i) = c_i$ $\rho(h_j) = h_j$ for the other c_i and h_j. This gives a map $\rho^*: H^*(WO_{q+1}) \to H^*(WO_q)$ and the classes in $H^*(WO_q)$ which can move are precisely those in the cokernel of ρ^*.

The cokernel of ρ^* may be thought of as follows

$$\text{coker } \rho^* = \Lambda(h_1, h_3, \ldots, h_{2k+1}) \otimes H^{2q+1}(WO_q)$$

where $h_{i_1} \otimes h_{i_2} c_J = 0$ if $i_1 \leq i_2$. We must throw out these tensors because they do not represent cohomology classes. More succinctly

$$\text{coker } \rho^* = [\Lambda(h_1, \ldots, h_{2k+1}) \otimes H^{2q+1}(WO_q)] \cap H^*(WO_q).$$

Comment 3. It is now clear how to generalize this result to $H^*(B\Gamma_q)$. Let i: $B\Gamma_q \to B\Gamma_{q+1}$ be the map induced by i: $\mathbb{R}^q \to \mathbb{R}^{q+1}$. Suppose F_s is a differentiable family of foliations on M of codimension q. Then for each s there is a map $f_s: M \to B\Gamma_q$ and a map $f: M \times I \to B\Gamma_{q+1}$. f classifies the foliation F on $M \times I$ which restricts to F_s on $M \times \{s\}$. Consider the following diagram

$$\begin{array}{ccc} B\Gamma_{q+1} & \xleftarrow{\;i\;} & B\Gamma_q \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f_s} \\ M \times I & \underset{i_s}{\overset{\pi}{\rightleftarrows}} & M. \end{array}$$

The diagram commutes since the foliation induced by f is integrably homotopic to the one induced by $i \circ f_s \circ \pi$. Thus if $\alpha \in H^*(B\Gamma_{q+1})$ then $f_1{}^* \circ {}_i{}^*(\alpha) = f_0{}^* \circ i^*(\alpha)$ and so the only elements of $H^*(B\Gamma_q)$ which may move are those in the cokernel of i^*. The reader may be tempted to remark that this trivializes Theorem 3. The power of Theorem 3 is that it makes clear precisely which classes are rigid while the above proof does not.

Theorem 3 contrasts sharply with what may happen in the case $i^0 + |J| = q + 1$.

THEOREM 5 (Thurston [11]). *There is a differentiable family F_s of codimension 1 foliations on S^3 for which $\alpha_{F_s}{}^*(\langle h_1 c_1\rangle)[S^3]$ takes on a continuum of values.*

This is the only known family of real foliations of this type and so it is an open question whether there are families of codimension q foliations $q > 1$ for which the secondary classes take on a continuum of values. One would also like to have examples of foliations to show that the rigid classes are not bundle invariants.

In [10], J. Simons constructs characters associated to connections on G bundles. For example, when $G = GL_q$ he gets characters denoted by S_p, $p \in \mathbb{Z}[c_1, \ldots, c_q]$ where the c_i are the integral Chern polynomials in $I(GL_q)$. For a given connection ω on a bundle over a manifold M, $S_p(\omega)$ is a homomorphism from singular cycles on M to $\mathbb{R}/\mathbb{Z}$, i.e. if p is a monomial of degree l

$$S_p(\omega): \mathbb{Z}_{2l-1}(M) \to \mathbb{R}/\mathbb{Z}.$$

When $p(\Omega, \ldots, \Omega) \equiv 0$ these homomorphisms define $\mathbb{R}/\mathbb{Z}$ cohomology classes denoted by $S_p[\omega]$. This is the case for torsion free connections associated to a foliation and the Simons class is independent of the choice of connection. Some of these classes are the mod $\mathbb{Z}$ reduction of the generalized Godbillon–Vey classes. The advantage of this construction is that it yields some new classes. (In the case of codimension 3 foliations the class $S_{c_2}{}^2$ for example).

A result similar to Theorem 3 holds for these classes.

THEOREM 3′. *Let F_s, $s \in [0, 1]$ be a differentiable family of foliations of codimension q, with torsion free connections ω_s. Let $p \in I(GL_q)$ be integral and homogeneous with $p(\Omega_s, \ldots, \Omega_s) \equiv 0$. Then $S_p[\omega_0] = S_p[\omega_1]$, unless degree $p = q + 1$.*

This is a corollary of the following.

THEOREM 6 (Simons [10, Proposition 3.14]). *Let ω_s, $s \in [0, 1]$ be a smooth family of connections, Ω_s their curvature and $\psi_s = \partial/\partial_s(\omega_s)$. Let ρ: $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the obvious homomorphism and let $p \in I(GL_q)$ be homogeneous of degree l. Then*

$$S_p(\omega_1) - S_p(\omega_0) = \rho \circ \left(l \int_0^1 p(\psi_s, \Omega_s, \ldots, \Omega_s)\, ds\right) \Big| \, \mathbb{Z}_{2l-1}(M).$$

§3. PROOF OF THEOREM 1

Following Chern [5], we will use the convention that if f contains less than k arguments the last one is repeated a number of times to make f a function of k arguments. Thus

$$f(\omega_1{}^s - \omega_0, \Omega_t{}^s) = f(\omega_1{}^s - \omega_0, \underbrace{\Omega_t{}^s, \ldots, \Omega_t{}^s}_{k-1}).$$

With this in mind we have

$$\begin{aligned}&\partial/\partial_s \Delta_f(\omega_1{}^s, \omega_0)\\ &= \partial/\partial_s\left(k \int_0^1 f(\omega_1{}^s - \omega_0, \Omega_t{}^s)\, dt\right) = k \int_0^1 f(\psi_s, \Omega_t{}^s)\, dt + k(k-1) \int_0^1 f(\omega_1{}^s - \omega_0,\\ &\qquad \partial/\partial_s(\Omega_t{}^s), \Omega_t{}^s)\, dt.\end{aligned}$$

Now the equation

$$d\Omega_t{}^s = -[\Omega_t{}^s, \omega_t{}^s]$$

gives

$$\begin{aligned}d(f(\psi_s, \omega_1{}^s - \omega_0, \Omega_t{}^s)) &= f(d\psi_s, \omega_1{}^s - \omega_0, \Omega_t{}^s) - f(\psi_s, d(\omega_1{}^s - \omega_0), \Omega_t{}^s)\\ &\quad - (k-2) f(\psi_s, \omega_1{}^s - \omega_0, [\Omega_t{}^s, \omega_t{}^s], \Omega_t{}^s).\end{aligned}$$

The invariance of f implies (see [5]) that the last term equals

$$-f([\psi_s, \omega_t{}^s], \omega_1{}^s - \omega_0, \Omega_t{}^s) + f(\psi_s, [\omega_1{}^s - \omega_0, \omega_t{}^s], \Omega_t{}^s)$$

so

$$\begin{aligned}d(f(\psi_s, \omega_1{}^s - \omega_0, \Omega_t{}^s)) &= f(d\psi_s - [\psi_s, \omega_t{}^s], \omega_1{}^s - \omega_0, \Omega_t{}^s)\\ &\quad - f(\psi_s, d(\omega_1{}^s - \omega_0) - [\omega_1{}^s - \omega_0, \omega_t{}^s], \Omega_t{}^s).\end{aligned}$$

Using $\partial/\partial_s(\Omega_t^s) = t(\mathrm{d}\psi_s - [\psi_s, \Omega_t^s])$ we see that

$$\partial/\partial_s \Delta_f(\omega_1^s, \omega_0) - k(k-1)\,\mathrm{d}V = k\int_0^1 f(\psi_s, \Omega_t^s)\,\mathrm{d}t$$

$$+ k(k-1)\int_0^1 tf(\psi_s, \mathrm{d}(\omega_1^s - \omega_0) - [\omega_1^s - \omega_0, \omega_t^s], \Omega_t^s)\,\mathrm{d}t$$

$$= k\int_0^1 f(\psi_s, \Omega_t^s + t(k-1)\{\mathrm{d}(\omega_1^s - \omega_0) - [\omega_1^s - \omega_0, \omega_t^s]\}, \Omega_t^s)\,\mathrm{d}t.$$

Since

$$\Omega_t^s = t\Omega_1^s + (1-t)\Omega_0^s + \tfrac{1}{2}t(1-t)[\omega_1^s - \omega_0, \omega_1^s - \omega_0]$$

we have

$$\Omega_t^s + t(k-1)\{\mathrm{d}(\omega_1^s - \omega_0) - [\omega_1^s - \omega_0, \omega_t^s]\}$$

$$= tk\left\{\Omega_1^s + \frac{1-kt}{kt}\Omega_0^s + \frac{1}{2}\left(1 - \frac{2k-1}{k}t\right)[\omega_1^s - \omega_0, \omega_1^s - \omega_0]\right\}$$

and so

$$\partial/\partial_s \Delta_f(\omega_1^s, \omega_0) - k(k-1)\,\mathrm{d}V$$

$$= k^2\int_0^1 tf\left(\psi_s, \Omega_1^s + \frac{1-kt}{kt}\Omega_0^s + \frac{1}{2}\left(1 - \frac{2k-1}{k}t\right)[\omega_1^s - \omega_0, \omega^s{}_1 - \omega_0],\right.$$

$$\left. t\Omega_1^s + (1-t)\Omega_0^s + \frac{1}{2}t(1-t)[\omega_1^s - \omega_0, \omega_1^s - \omega_0]\right)\mathrm{d}t$$

$$= k^2\int_0^1 t^{k-1}f(\psi_s, \Omega_1^s)\,\mathrm{d}t$$

$$+ \sum_{\substack{p+q+r=h-1\\ p\neq h-1}} f_{p,q,r}(\psi_s, \Omega_1^s, \Omega_0^s, [\omega_1^s - \omega_0, \omega_1^s - \omega_0])\int_0^1 c_{p,q,r}\,\mathrm{d}t,$$

where

$$f_{p,q,r}(\psi_s, \omega_1^s, \Omega_0^s, [\omega_1^s - \omega_0, \omega_1^s - \omega_0])$$

$$= f(\psi_s, \underbrace{\Omega_1^s, \ldots, \Omega_1^s}_{p}, \underbrace{\Omega_0^s, \ldots, \Omega_0^s}_{q}, \underbrace{[\omega_1^s - \omega_0, \omega_1^s - \omega_0], \ldots, [\omega_1^s - \omega_0, \omega_1^s - \omega_0]}_{r})$$

and

$$c_{p,q,r} = k^2 t\left\{t^{p-1}(1-t)^q\left(\frac{1}{2}t(1-t)\right)^r\binom{k-2}{p-1, q, r}\right.$$

$$+ \left(\frac{1-kt}{kt}\right)t^p(1-t)^{q-1}\left(\frac{1}{2}t(1-t)\right)^r\binom{\quad -2}{p, q-1, r}$$

$$\left. + \frac{1}{2}\left(1 - \frac{2k-1}{k}t\right)t^p(1-t)^q\left(\frac{1}{2}t(1-t)\right)^{r-1}\binom{k-2}{p, q, r-1}\right\}.$$

Simplifying this expression we get

$$c_{p,q,r} = \frac{k^2}{2^r(k-1)}\binom{k-1}{p,q,r} t^{p+r}(1-t)^{q+r-1}\left\{(1-t)p + \left(\frac{1-kt}{k}\right)q + \left(1 - \frac{2k-1}{k}t\right)r\right\}.$$

Using the fact that $\int_0^1 t^m(1-t)^n = \frac{m!n!}{(m+n+1)!}$ we finally have

$$\int_0^1 c_{p,q,r}\, \mathrm{d}t = 0.$$

This, of course, covers only the cases where p, q and r are all non-zero. We leave it to the reader to check the other cases. Since

$$\int_0^1 t^{k-1}\, \mathrm{d}t = \frac{1}{k}$$

we are done.

University of California, Berkeley
University of California, Los Angeles

REFERENCES

1. R. Bott: On a topological obstruction to integrability, *Proc. Symposia Pure Math.*, Vol. 16, *Global Analysis*, 127–131. Am. Math. Soc.
2. R. Bott: The Lefschetz formula and exotic characteristic classes, *Proc. Diff. Geometry Conf., Rome, May* 1971.
3. R. Bott: *Lectures on Foliations, Mexico*, 1971. To be published by Springer, Berlin.
4. R. Bott and A. Haefliger: On characteristic classes of Γ-foliations, to appear in *Bull. Am. math. Soc.*
5. S. S. Chern: Geometry of characteristic classes, to appear in *Proc. Can. Math. Congress, Halifax*, 1971.
6. S. S. Chern and J. Simons: Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, *Proc. natn. Acad. Sci., U.S.A.* **68** (1971), 791–794.
7. C. Godbillon and J. Vey: Un invariant des feuilletages de codimension 1, *C.r. Acad. Sci., Paris*, June 1971.
8. S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry*, Vol. II. Interscience, New York, (1969).
9. S. Kobayashi and T. Ochiai: G-structures of order two and transgression operators, *J. diff. Geometry* **6** (1971), 213–230.
10. J. Simons: Characteristic forms and transgression II: Characters associated to a connection (preprint).
11. W. Thurston: Non-cobordant foliations of S^3, *Bull. Am. math. Soc.* **78** (1972), 511–514.
12. D. Lehmann: Rigidité des classes exotiques (preprint).