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A FUNDAMENTAL EFFECT IN COMPUTATIONS ON REAL NUMBERS

Horst LUCKHARDT

Mathematisches Institut der Universität, 6 Frankfurt a. Main, W.-Germany.

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Abstract. For several elementary constructions which often occur in computations on real numbers we prove that they possess unavoidable intensionalities. Our examples give sharp bounds for constructive extensional choices and Skolem functors in theories of order ≥ 2.

This note deals with the well-known fact familiar from experience with exact computations on real numbers that certain decision procedures during the computation are intensional, i.e. they essentially depend on what number generator is used in the computation. Once one's attention is called to this effect, one realizes that it occurs in many computations or constructions. The main result of this paper is: there are computational situations where this is on principle so; intensionality cannot be avoided everywhere. The proof has a number of consequences (see below) of which we only mention here the following two:

(i) To make procedures on real numbers transparent with regard to intensionality, it is necessary to reduce the intensional sources to a minimum and to such elementary ones for which we *know* that they are essentially intensional. Our proof exhibits several of such elementary, essentially intensional situations.

(ii) Working only with continuous functionals one has to handle Skolem functors very carefully: it is correct to introduce a symbol for a provable functionality; but it is in general *false* to operate with this symbol extensionally.

In classical analysis only little knowledge on real numbers is necessary to prove much about them. As is well-known, this is not the case in intuitionistic, recursive or other so-called constructive analysis theories. For example: equality between reals cannot be decided, and there is also no effective direct comparison with respect to their magnitude. Because the latter is very fundamental, one has to replace it by an always practicable approximation: the comparison with two fixed points, say 0 and 1, which we know to be different. So in computing the real u we reach a point where we can decide whether it is greater than 0 or smaller than 1. This justifies

$$u > 0 \vee u < 1.$$

The effect we are looking for appears already in the simple case

$$u \neq 0 \vee u \neq 1. \tag{α}$$

This statement (α) is extensional in u. Like the most existing constructive theories it reflects only the extensional result and not the process that justifies it. But it is the latter that is important if we really apply the construction, because we first have to verify which case of the decision holds true and then to go on even if the exact decision information is absorbed.

Now we show that every algorithm which verifies the *extensional* (α) is *essentially intensional* in that its decision depends on the real number generator that is given. The proof rests on Brouwer's theorem that a real function which is extensional (a condition that is often forgotten!) and defined on a closed interval, is uniformly continuous there. We need only the corollary that an extensional, discretely valued constructive real function φ is constant:

$$\begin{aligned} &\bigwedge \varphi \{ \bigwedge x, y\, [x = y \rightarrow \varphi x = \varphi y\,] \wedge \bigwedge x\, [\varphi x = 0 \vee \varphi x = 1] \\ &\quad \rightarrow \bigwedge x \varphi x = 0 \vee \bigwedge x \varphi x = 1\}. \end{aligned} \tag{β}$$

Observe that (α) and (β) also hold classically for computable extensional real functions φ.

Next we analyse the algorithms Φ standing behind (α). If the quantifier combination $\bigwedge\, \bigvee$ is read constructively then (α) means:

$$\bigwedge u \ \bigvee y \{[u \neq 0 \wedge y = 0] \vee [u \neq 1 \wedge y = 1]\}.$$

First we give the name Φ to this process:

$$(u \neq 0 \wedge \Phi u = 0) \vee (u \neq 1 \wedge \Phi u = 1); \tag{1}$$

and secondly we require this Φ to behave extensional:

$$u = v \rightarrow \Phi u = \Phi v. \tag{2}$$

But this is too much; (α), (β), (1) and (2) involve the following contradiction:

$$\Phi u = 0 \vee \Phi u = 1, \quad \Phi 0 = 1, \quad \Phi 1 = 0, \qquad (1) \tag{3}$$

$$\bigwedge x \Phi x = 0 \vee \bigwedge x \Phi x = 1, \qquad (2), (3), (\beta)$$

$$\Phi 0 = 0 \vee \Phi 1 = 1,$$

$$0 = 1, \qquad (3)$$

$$0 \neq 1. \qquad (\alpha)$$

What we have proved can be formulated in several ways.

Theorem 1. *All everywhere defined algorithms Φ which verify the extensional (α) have the following intensional properties:*

(a) $\neg \bigwedge x, y(x = y \rightarrow \Phi x = \Phi y)$, *which is constructively equivalent to* $\neg\neg \bigvee x, y(x = y \wedge \Phi x \neq \Phi y)$.

(b) $\neg\neg\{\Phi$ *is many-valued in the reals*$\}$.

(c) Φ *is continuous in the intensional number generators.*

(d) Φ *cannot be described by an extensional relation* R *in an unique way (because* $\bigwedge u \bigvee ! yR$ *only defines extensional operations).*

A second reading of our proof is:

Theorem 2. *The axiom of choice AC on real numbers and extensionality* $\mathrm{Ext}(\Phi^2)$ *for* 0, 1*-valued real functions* Φ^2 *(of type* 2*) are intuitionistically incompatible.*

The passage from (α) to (1) and (2) can also be viewed as an application of the syntactic introduction rule for a Skolem functor Φ. Therefore a third reading of our proof is:

Theorem 3. *The introduction of Skolem functors is in general for classical and intuitionistic theories of order* $\geqslant 2$ *(especially for those dealing with continuous functionals) not only not conservative but actually false.*

The aim of the Hilbert program among other things is to show that this does not happen with formalized classical mathematics.

As is stressed in the introduction, for practical applications we have to state what elementary situations are essentially intensional. From our considerations we can exhibit several such elementary constructions.

Theorem 4. *The following procedures are essentially intensional:*

(a) *All decision procedures for* $u \neq 0 \vee u \neq 1$.

(b) *All decision procedures for* $u \neq 0 \vee u < 1$, $u > 0 \vee u \neq 1$, $u > 0 \vee u < 1$.

(c) *All procedures* (b) *and* (a) *if we compute the real u only by canonical real number generators* ([2] p. 41–42).

(d) *All procedures leading from the usual or the canonical real number generators to the coefficients of a coinciding (canonical) real number generator.*

Proof. (a) was explicitly shown above; (b) is a consequence of (a). Ad (c): The proof for (a), (b) also works if only canonical number generators are considered. (d) results from (a)–(c) and the fact that there is no intensionality problem with the decision procedures (a) and (b) if for each real we dispose of exactly one (canonical) real number generator with a known rate of convergence. □

We close with the question: Are there further intensionality effects in computations on real numbers?

A referee pointed out that Theorem 4(d) and 2 were first obtained by Ashvinikumar [1]; his paper contains a proof for:

"There exists no effective extensional method which picks out of each constructive real number (species of coinciding number generators) one specific number generator".

The present paper, which was written independently, proves the intensionality effect for the stronger case of well-known discrete 0, 1-decisions on real numbers; these branching tests actually occur at many places in computational practice, so that it cannot be avoided that many programs behave quite differently on the same real number.

The intensionality effect also appears in intuitionistic type theory on level 3 as was recently shown by Troelstra [3]. For real numbers and extensionality with respect to equality between real numbers by Theorem 2 the effect occurs already on level 2 (if one counts the type in which the objects considered can be represented in the full type hierarchy); type 2 is obviously minimal.

References

[1] Ashvinikumar, The intuitionist contradictority of certain classical set-theoretic results, *Nieuw Arch. Wisk.* **17** (3) (1969) 218–223.
[2] A. Heyting, *Intuitionism* (North-Holland, Amsterdam 1971).
[3] A.S. Troelstra, A note on non-extensional operations in connection with continuity and recursiveness, Report 77–02 Mathematisch Instituut Amsterdam (1977).