Rank 2 Amalgams Related to the McLaughlin Group, and to $O_7(3)$

YOAV SEGEV

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

Communicated by David M. Goldschmidt

Received February 20, 1988

INTRODUCTION

Let $G$ be a group, $I = \{1, \ldots, n\}$, and $\Omega = \{P_1, \ldots, P_n\}$ a set of proper subgroups of $G$. $G$ is called an amalgam of rank $n$ (see [2, p. 61]), if the following hypotheses are satisfied:

(A1) $G = \langle P_1, \ldots, P_n \rangle$, and no proper subset of $\Omega$ generates $G$.

(A2) $B = P_i \cap P_j$, $i \neq j$, is independent of $i, j \in I$.

(A3) No nontrivial normal subgroup of $G$ is contained in $B$.

(A4) $G$ is universal in the sense that any group satisfying hypotheses (A1)–(A3) is a homomorphic image of $G$.

This definition was motivated by the fundamental paper [3] which considers amalgams of rank 2 in which $P_1$ and $P_2$ are finite, and $P_i/O_2(P_i) \simeq S_3$, $i = 1, 2$. As mentioned in [3], these amalgams are closely related to rank 2 BN pairs over $GF(2)$.

In an important paper [2] that followed, it was shown that if $G$ is an amalgam of rank 2 such that the following additional hypotheses are satisfied: There exists a prime $p$ such that:

(B1) $P_1$ and $P_2$ are finite subgroups and $\text{Syl}_p(B) \subseteq \text{Syl}_p(P_i)$, $i = 1, 2$.

(B2) $C_{P_i}(O_p(P_i)) \subseteq O_p(P_i)$, $i = 1, 2$.

(B3) $O_p^p(P_i/O_p(P_i))$ is a Chevalley group of rank 1.

Then the structure of $P_1$ and $P_2$ is completely determined. A group $G$ which satisfies hypotheses (A1)–(A3) and (B1)–(B3) is by definition a weak BN pair of rank 2.

In general for an amalgam of rank $n$ satisfying hypotheses (B1) and (B2)

* Current address: Department of Mathematics, Ben-Gurion University, Beer-Sheva 84105, Israel.
the question arises whether the structure of \( P_1, \ldots, P_n \) can be deduced from the structure of \( P_1/O_\varphi(P_1), \ldots, P_n/O_\varphi(P_n) \).

In [4] the possibilities for the structure of \( P_1 \) and \( P_2 \) are determined under the hypothesis: \( P_1/O_2(P_1) \cong S_5 \) and \( P_2/O_2(P_2) \cong S_3 \). It is shown that this configuration is related to many of the sporadic simple groups.

Before we proceed we mention that if \( G \) is an amalgam of rank \( n \), and \( H \) is a group satisfying hypotheses (A1)-(A3), but not necessarily hypothesis (A4), then \( H \) is called a completion of \( G \).

Our main theorem is:

**Theorem A.** Assume \( G \) is an amalgam of rank 2 satisfying hypotheses (B1) and (B2), and suppose further that \( P_1/O_2(P_1) \cong A_7 \) and \( P_2/O_2(P_2) \cong S_3 \). Then:

\[
\begin{align*}
(1) & \quad P_1 \cong 2^6 \cdot A_7 \quad \text{and} \quad P_2 \cong 2^{1+1+1+2+2+1} \cdot S_3, \quad \text{or} \\
(2) & \quad P_1 \cong 2^4 \cdot A_7 \quad \text{and} \quad P_2 \cong 2^{1+2+2+1} \cdot S_3, \quad \text{or} \\
(3) & \quad P_1 \cong 2^{1+6} \cdot A_7 \quad \text{and} \quad P_2 \cong 2^{2+2+2+1+1+1} \cdot S_3.
\end{align*}
\]

By ~ we mean a description of the chief factors of the group (see Section 4 for more details). We mention that \( O_\varphi(3) \) is a completion of case (1) of Theorem A and the McLaughlin group is a completion of case (2) of Theorem A. We did not find a "nice" completion for case (3) of Theorem A, but an amalgam of case (3) of Theorem A does exist; it is denoted by \( .3^* \) and was discovered by A. Chermak.

1. **Notation and Preliminary Results**

Our notations follow [4]. Let \( G, P_1, \) and \( P_2 \) be as in Theorem A. We let \( \Gamma \) be the graph whose vertex set is the right cosets of \( P_1 \), together with the right cosets of \( P_2 \) in \( G \). Two vertices \( \lambda, \delta \in \Gamma \) are adjacent if their intersection is not empty. The basic properties of \( \Gamma \) are listed in [4, Lemma (1.9)]. For the convenience of the reader we list here the notation to be used throughout this paper:

- \( G \) A group satisfying the hypotheses of Theorem A
- \( G_\lambda \) The stabilizer in \( G \) of \( \lambda \in \Gamma \)
- \( d \) The usual distance function in \( \Gamma \)
- \( A^{(i)}(\lambda) \) \{ \( \delta \in \Gamma \mid d(\lambda, \delta) = i \} \)
- \( A(\lambda) \) \( A^{(1)}(\lambda) \)
- \( G^{(1)}_\lambda \) \( \bigcap_{\delta \in A(\lambda)} G_\delta \)
- \( Q_\lambda \) \( O_2(G_\lambda) \)
Let $(\alpha, \alpha')$ be a pair of vertices such that $d(\alpha, \alpha') = b$. Then

\[ \text{will always denote an arc of minimal length connecting } \alpha \text{ and } \alpha'. \]

Note that $\beta$ always denotes $\alpha + 1$. $\alpha - i$ will denote a vertex in $\Delta^{(i)}(\alpha) \setminus \{\alpha + i\}$, and $\alpha' + i$ will always denote a vertex in $\Delta^{(i)}(\alpha') \setminus (\alpha' - i)$. A critical pair is a pair of vertices $(\alpha, \alpha')$ satisfying $d(\alpha, \alpha') = b$, and $Z_\alpha \nsubseteq Q_{\alpha'}$. Or a pair of vertices $(\beta, \alpha')$ satisfying $d(\beta, \alpha') = b - 1$, $V_\beta \nsubseteq Q_{\alpha'}$. Note $(\alpha, \alpha')$ will always denote a critical pair of the first type, and $(\beta, \alpha')$ will always denote a critical pair of the second type. In the next lemma we summarize some basic results which can be found in [4]:

\[ (1.1). \] Let $\lambda \in \Gamma$, and let $(\alpha, \alpha')$ be a critical pair. Then we have:

1. $Q_\lambda = G^{(1)}_\lambda$.
2. $Z_\alpha \subseteq Z(Q_\lambda)$.
3. $Z_\alpha \nsubseteq Z(G_\lambda)$.
4. If $Z_\beta \subseteq Z(G_\beta)$, then $Z(G_\alpha) = 1$.
5. If $[Z_\alpha, Z_{\alpha'}] = 1$, then $Z_{\alpha'} \nsubseteq Z(G_\alpha)$.

6. Let $\alpha, \beta$ be adjacent vertices in $\Gamma$. Suppose there exists a subgroup $U \leq G_\alpha \cap G_\beta$ such that $N_G(U)_\lambda$ is transitive on $\Delta(\lambda)$ for $\lambda = \alpha, \beta$. Then $U = 1$.

$A_7$ modules over $GF(2)$. All modules here are $A_7$ modules over $GF(2)$. A natural module is a faithful module of dimension 4. The permutation module is the six dimensional permutation module. Note that throughout this paper we are assuming knowledge of the module structure of the above two types of modules. The reader who is unfamiliar with these modules can easily verify properties of these modules, which we require in this paper.

Let $V$ be a module. An involution $t \in A_7$ induces a $j$-transvection on $V$ if $C_V(t)$ has codimension $j$ in $V$. The 1-transvections are called transvections.
(1.2) Let $V$ be a module, then:

(1) No involution in $A$, induces a transvection on $V$.

(2) If an involution $t \in A_7$ induces a $j$-transvection on $V$, for $j \leq 3$, then $V/C_V(A_7)$ is either a natural module or the permutation module.

(3) If there exists an elementary abelian subgroup $A \leq A_7$ of order 4 such that $C_V(A)$ has codimension 2 in $V$, then $V/C_V(A_7)$ is a natural module.

Proof: Part (1) follows from the fact that all involutions in $A_7$ are conjugate, $A_7$ is generated by three involutions, and $A_7$ has no faithful modules of dimension $\leq 3$. For (2) see [1, Lemma(8.6), p. 723]. Finally (3) follows from the fact that $A_7$ is generated by two conjugates of $A$.

The following result is due to U. Meierfrankenfeld.

(1.3) Let $H \cong A_7$, and $S \in \text{Syl}_2(H)$. Let $V$ be a $GF(2) - H$ Module and $U$ an $S$ submodule of $V$ such that $V = \langle U^H \rangle$. Let $t \in H$ be an involution and define $X = \langle U^h \mid \langle t, S^h \rangle = H \rangle$. If $|[X, t]| \leq 2$, then $[V, H] = 1$.

Proof: We start with:

(1) There exists an element $d \in H$ of order 5, and $h \in H$ such that:

(a) $\langle d \rangle \langle t \rangle \simeq D_{10}$

(b) $t \in S^h$, and $\langle S^{hd_i}, t \rangle = H$, for $i = 1, 2, 3, 4$.

Let $d = (12345)$, $t = (25)(34)$, $S^h = \langle (25)(67), (26)(57), (25)(34) \rangle$. Then $d \in \langle S^{hd_i}, t \rangle$ and $\langle S^{hd_i}, t \rangle$ is transitive.

(2) We may assume $[U, S] = 1$.

Assume that:

(*) (1.3) holds with the additional assumption that $[U, S] = 1$.

We now prove (1.3) under the hypothesis (*). Let $k$ be the minimal integer satisfying $[U, S, S, \ldots, S]_{k\text{ times}} = 1$. We show (1.3) by induction on $k$. If $k = 1$, this is (*). Assume $k > 1$ and define $U_1 = [U, S]$, $V_1 = \langle U_1^H \rangle$. It is readily verified that if we replace $V$ by $V_1$, $U$ by $U_1$, and $X$ by $X_1 = \langle U_1^h \mid \langle t, S^h \rangle = H \rangle$ in (1.3), then all the hypotheses of (1.3) are satisfied. By the induction hypothesis $[V_1, H] = 1$. Set now $\tilde{V} = V/V_1$, $\tilde{U} = (U + V_1)/V_1$, and $\tilde{X} = \langle \tilde{U}^h \mid \langle t, S^h \rangle = H \rangle$.

Then if we replace in (1.3) $V$ by $\tilde{V}$, $U$ by $\tilde{U}$, and $X$ by $\tilde{X}$, then all the hypotheses of (1.3) are satisfied and in addition the hypothesis in (*) is satisfied. We may conclude by (*) that $[\tilde{V}, H] = 1$, i.e., $[V, H] \leq V_1$. Thus $[V, H, H] = 1$ and by the three subgroup lemma, $[V, H] = 1$. This completes the proof of (2).
(3) \([U^h, t] = 1\).

Let now \(W = \langle U^h, t \rangle \). Then \(W = U^{ht} U^{htd} \cdots U^{htd^4} = U^h U^{hd} \cdots U^{hd^4}\), so \(W \subseteq U^h X\). It follows that \([W, t] \leq [X, t]\) and \([W, t]\) is abelian, and hence \([W, d]\) is 2. It follows that \(U^h = U^{hd}\) is centralized by \(\langle t, S^{hd} \rangle = H\). Thus \(L' = U\), and \([L', H] = 1\).

(1.4) **Corollary.** Let \(H\) be a group satisfying \(H/O_2(H) \cong A_7\), and let \(S \in \text{Syl}_2(H)\). Let \(W \leq O_2(H)\) be a normal subgroup of \(H\). Assume that \(S\) operates on an elementary abelian subgroup \(U \leq W\) satisfying \(\langle U^h \rangle = W\). Let \(i \in H/O_2(H)\) be an involution and define \(X = \langle U^h | \langle i, S^h \rangle = H \rangle\). If \([X, t]\) is abelian, then \([W, O_2(H)] = 1\).

**Proof.** Apply (1.3) to the factors of the series \(W \triangleright [W, O_2(H)] \triangleright [W, O_2(H), O_2(H)] \triangleright \ldots\).

(1.5) Let \(H \cong A_7\). Assume \(V\) is an \(H\)-module involving a natural module (resp. the permutation module) as a unique noncentral chief factor. Then \(V = U + C_r(H)\), where \(U\) is a natural \(H\)-module (resp. the permutation module).

**Proof.** If \(V\) involves a natural module, the proof here is precisely as in [1, Lemma (8.2), p. 720]. So assume that \(V\) involves the permutation module. It is easily verified that we may assume that \(C_r(H)\) is one dimensional, and \(\bar{V} = V/C_r(H)\) is the permutation module. Let \(v \in \bar{V}\) with \(C_{H}(v) \cong A_6\). Then \([v, C_{H}(v)] \leq C_r(H)\), and thus, \(C_{H}(v) = C_{H}(v) \cong A_6\). It follows that \(V\) is the seven dimensional permutation module for \(H\) and we are done.

For the next result see [5].

(1.6) **Pushing up:** Let \(M\) be a finite group \(S \in \text{Syl}_2(M)\), \(Z = \langle \Omega_1(Z(S)) \rangle^M\), and suppose \(Z \leq C_m(Z) \leq O_2(M)\). Assume further that no characteristic subgroup of \(S\) is normal in \(M\). Then we have: If \(M/O_2(M) \cong S_3\) or \(A_7\), then \([O^2(M), O_2(M)] \) is the natural module for \(M/O_2(M)\).

2. **The case** \(G_x/Q_x \cong A_7\)

**Subcase 1.** \([Z_x, Z_x'] \neq 1\).

(2.1) **We have:**

(1) \(b = 0(2)\).

(2) \(|Z_x : Z_x \cap Q_x| = |Z_{x'} : Z_{x'} \cap Q_x| = 4\).
(3) \( C_{G_z}(Z_\alpha) = Q_\alpha \).

(4) \([Z_\alpha, Z_\alpha'] \neq 1\), for any critical pair \((\lambda, \lambda')\), with \(G_z/Q_z \cong A_\gamma\).

(5) \([Z_\alpha, Z_\alpha'] \triangleleft G_z \cap G_\beta\).

(6) \(Z_\alpha\) involves a natural module as a unique noncentral chief factor for \(G_z/Q_z\).

**Proof.** Assume \(G_z/Q_z \cong S_3\), then \(|Z_\alpha : Z_\alpha \cap Q_z| = 2\), and thus \(Z_\alpha\) induces transvections on \(Z_\alpha\), contradicting (1.2.1), this proves (1). Part (2) follows precisely for the same reason. Part (3) is a consequence of (1.1.3), and (4) is a consequence of (3). Part (5) is obvious, and (6) follows from (2) and (1.2.3).

The proof of the next lemma was suggested to me by U. Meierfrankenfeld.

(2.2) \(b > 2\).

**Proof.** Assume \(b = 2\). Set \(E_\alpha = C_{Q_\beta}(L_\alpha)\), and \(\bar{Z}_\alpha = [Z_\alpha, G_\gamma]\). By (2.1.6) and (1.5), \(\bar{Z}_\alpha\) is a natural module for \(G_z/Q_z\). Now \(Q_\alpha \cap Q_\beta \leq Z_\alpha Q_\gamma\), and hence \([Q_\alpha \cap Q_\beta, Z_\alpha'] \leq Z_\alpha\). Since by (1.2.1) \(Z_\alpha'\) does not induce transvections on \(Q_\alpha/Z_\alpha\), we have \([Q_\alpha, O^2(G_\gamma)] = \bar{Z}_\alpha\). Now it is readily verified that the inverse image, \(\Phi\), in \(Q_\alpha\) of the Frattini group \(\Phi(Q_\beta/E_\alpha)\), satisfies \(\Phi \cap \bar{Z}_\alpha = 1\), and hence \(\Phi \leq E_\alpha\). It follows that \(Q_\alpha/E_\alpha\) is elementary abelian, and hence by (1.5), \(Q_\alpha = E_\alpha Z_\alpha\). Consider now \(Z_\alpha Q_\beta/E_\alpha\). Since \(Z_\alpha\) does not induce transvections on \(Z_\alpha\), and \(Z_\alpha\) does not induce transvections on \(Z_\alpha\), it follows that \(Z_\alpha \cup \bar{Z}_\alpha\) are the involutions in \(Z_\alpha Q_\beta/E_\alpha\). It follows that \(Z_\alpha \leq Z_\alpha E_\alpha\), where \(\{\delta\} = \Delta(\beta) \setminus \{\alpha, \alpha'\}\). Hence we have \([Z_\alpha, Z_\alpha'] = [Z_\delta, Z_\delta'] \leq [Z_\alpha E_\alpha, Z_\alpha'] = [Z_\alpha', E_\alpha] \leq E_\alpha\), a contradiction.

(2.3) \(Q_\alpha \triangleleft Q_\beta\).

**Proof.** Assume \(Q_\alpha \triangleleft Q_\beta\). Choose \(\alpha - 1 \in \Delta(\alpha)\) with \(\langle Q_{\alpha - 1}, Z_\alpha \rangle = G_\alpha\). Assume that \(V_{\alpha - 1} \subseteq Q_{\alpha - 2}\), then \(V_{\alpha - 1} \leq Q_{\alpha - 2} \leq Q_{\alpha - 1} \leq G_{\alpha'}\). Thus \([V_{\alpha - 1}, Z_\alpha'] \leq [Z_\alpha Q_\gamma, Z_\alpha'] \leq V_{\alpha - 1}\). Hence \(V_{\alpha - 1} \leq \langle G_{\alpha - 1} \cap G_\gamma, Z_\alpha' \rangle = G_\alpha\), contradicting (1.1.6). Choose \(\alpha - 2 \in \Delta(\alpha - 1)\) with \(Z_{\alpha - 2} \notin Q_{\alpha - 2}\). Consider \(R_0 = [Z_{\alpha - 2}, Z_{\alpha - 2}']\). It is normalized by \(\langle Q_{\alpha - 1}, Z_\alpha' \rangle = G_\alpha\), and by (2.1.5), \(R_0 \triangleleft G_{\alpha - 2} \cap G_{\alpha - 1}\). So \(R_0 \triangleleft \langle G_{\alpha - 2} \cap G_{\alpha - 1}, G_{\alpha - 1} \cap G_\gamma \rangle = G_{\alpha - 1}\), contradicting (1.1.6).

(2.4) \(Z_\beta \triangleleft Z(G_\beta)\).

**Proof.** Assume that \(Z_\beta \notin Z(G_\beta)\). Suppose \(Z_\alpha Z_\beta \triangleleft G_\alpha\). Choose \(x \in Q_\gamma \setminus Q_\beta\), and set \(R_0 = [Z_\alpha Z_\beta, x]\). Then \(R_0 = [Z_\alpha Z_\beta, Q_\gamma]\). So \(R_0 = [Z_\beta, Q_\gamma]\). We have \(R_0 \triangleleft G_\alpha\), \(R_0 \triangleleft Z_\beta\), and hence \(R_0\) is centralized by
Q_\alpha and by Q_\delta, for \delta \in A(\alpha). Thus R_\alpha \leq Z(G_\alpha). Since Z_\alpha = [Z_\alpha, L_\alpha] is the natural module for G_\alpha/Q_\alpha, the above implies that L_\alpha \cap G_\beta \leq Q_\beta. Set L = L_\alpha Q_\beta. Then Q_\beta \in \text{Syl}_2(L), and L satisfies the assumptions of (1.6). By (2.2), V_{\alpha-1} \leq Q_\alpha and hence, [V_{\alpha-1}, L_\alpha] \leq Z_\alpha, so V_{\alpha-1} \lhd G_\alpha, contradicting (1.1.6).

Choose now \alpha - 1 \in A(\alpha), such that \langle G_{\alpha-1} \cap G_\alpha, Z_\alpha \rangle = G_\alpha. Assume first that Z_{\alpha-1} \nleq Q_{\alpha-1}. If Z_{\alpha-2} \nleq Q_{\alpha-2}, for some \alpha - 2 \in A(\alpha - 1), then [Z_{\alpha-2}, Z_{\alpha-2}] \text{ is normalized by } \langle G_{\alpha-2} \cap G_{\alpha-1}, Z_{\alpha-1} \rangle = G_{\alpha-1}. Thus [Z_{\alpha-2}, Z_{\alpha-2}] \lhd [G_{\alpha-1} \cap G_\alpha, Z_\alpha] = G_\alpha, so by (1.1.6), [Z_{\alpha-2}, Z_{\alpha-2}] = 1, a contradiction to (2.1.4). Hence for \alpha - 2 \in A(\alpha - 1), Z_{\alpha-2} \leq Q_{\alpha-2}, so [Z_{\alpha-2}, Z_{\alpha-2}] \leq [Z_{\alpha-1}, Z_{\alpha-1}] \leq Z_{\alpha-1}. Hence Z_{\alpha-2} Z_{\alpha-1} \lhd G_{\alpha-1}, so Z_{\alpha-1} Z_{\alpha-2} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_\alpha, contradicting (2.1.4). Hence for \alpha - 2 \in A(\alpha - 1), Z_{\alpha-2} \leq Q_{\alpha-2}, so [Z_{\alpha-2}, Z_{\alpha-2}] \leq [Z_{\alpha-1}, Z_{\alpha-1}] \leq Z_{\alpha-1}. Hence Z_{\alpha-2} Z_{\alpha-1} \lhd G_{\alpha-1}, so Z_{\alpha-1} Z_{\alpha-2} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_\alpha, contradicting (2.1.4). Hence for \alpha - 2 \in A(\alpha - 1), Z_{\alpha-2} \leq Q_{\alpha-2}, so [Z_{\alpha-2}, Z_{\alpha-2}] \leq [Z_{\alpha-1}, Z_{\alpha-1}] \leq Z_{\alpha-1}. Hence Z_{\alpha-2} Z_{\alpha-1} \lhd G_{\alpha-1}, so Z_{\alpha-1} Z_{\alpha-2} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_\alpha, contradicting (2.1.4). Hence for \alpha - 2 \in A(\alpha - 1), Z_{\alpha-2} \leq Q_{\alpha-2}, so [Z_{\alpha-2}, Z_{\alpha-2}] \leq [Z_{\alpha-1}, Z_{\alpha-1}] \leq Z_{\alpha-1}. Hence Z_{\alpha-2} Z_{\alpha-1} \lhd G_{\alpha-1}, so Z_{\alpha-1} Z_{\alpha-2} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_{\alpha-2}, so Z_{\alpha-1} Z_{\alpha} \lhd G_\alpha, contradicting (2.1.4).

Assume next that Z_{\alpha-1} \leq Q_{\alpha-1}, then Z_{\alpha-1} \leq Z_{\alpha} Q_{\alpha}, so [Z_{\alpha-1}, Z_{\alpha}] \leq Z_{\alpha}, and hence Z_{\alpha-1} Z_{\alpha} \lhd G_{\alpha}. This contradicts the first paragraph of the proof.

(2.5) Notation. For \lambda \in \beta^G we set Y_\lambda = \bigcap_{\delta \in A(\lambda)} Z_\delta.

(2.6) Let \lambda \in \beta^G, and let \delta_1, \delta_2 \in A(\lambda). Then Y_\lambda = Z_{\delta_1} \cap Z_{\delta_2}.

Proof. The proof is straightforward and we omit it.

(2.7) The case G_\alpha/Q_\alpha \simeq A_7, [Z_\alpha, Z_\alpha] \neq 1 does not occur.

Proof. We start with:

(1) Z(G_\alpha) = 1, and Z_\alpha is the natural module for G_\alpha/Q_\alpha.

The first assertion of (1) follows from (2.4) and (1.1.4). The second assertions follow now immediately from (2.1.6) and (1.5). Next we claim:

(2) |Z_\beta| = 2.

This follows immediately from (1), and the action of G_\alpha/Q_\alpha on Z_\alpha.

We show now that there exists \alpha - 1 \in A(\alpha), and \alpha - 2 \in A(\alpha - 1), such that Z_{\alpha-2} \cap Q_\alpha = 1, this is an immediate contradiction since if Z_{\alpha-2} \leq Q_{\alpha-2}, then Z_{\alpha-2} \cap Q_{\alpha-2} \neq 1, while if Z_{\alpha-2} \leq Q_{\alpha-2}, then [Z_{\alpha-2}, Z_{\alpha-2}] \leq Q_{\alpha-2} \text{ (b > 2)}.

First note that as Z_\alpha \leq Q_\beta \cap Q_\alpha \beta, it follows that there exists \alpha - 1 \in A(\alpha) and \alpha - 2 \in A(\alpha - 1) \setminus \{\alpha\} such that \langle Q_{\alpha-2} \cap Q_{\alpha-1}, Z_\alpha Q_\alpha \rangle = G_\alpha. Next we claim that Z_{\alpha-2} \cap Q_\alpha \leq Z_\alpha. Indeed set R_0 = Z_{\alpha-2} \cap Q_{\alpha-2}, and L = \langle Q_{\alpha-2} \cap Q_{\alpha-1}, Z_\alpha \rangle. We claim that:

(*) L \cap G_{\alpha-1} \nleq Q_{\alpha-1}.
Assume first that (*) holds. Then $L$ is transitive on $\Delta(\alpha - 1)$ and $L$ centralizes $R_0 \leq Z_{\alpha - 2}$. Thus $R_0 \leq Z_{\alpha}$. It follows that $R_0$ is centralized by $\langle L, Q_\alpha \rangle = G_{\alpha}$, so by (1), $R_0 = 1$.

It remains to show (*). We first note that:

(3) \[ L_\alpha \cap G_{\alpha - 1} \leq Q_{\alpha - 1} \]

Indeed assume that $L_\alpha \cap G_{\alpha - 1} \leq Q_{\alpha - 1}$. Set $L_0 = L_\alpha Q_{\alpha - 1}$. Then $Q_{\alpha - 1} \in \text{Syl}_2(L_0)$, and hence by (1.1.6), $L_0$ satisfies the hypotheses of (1.6). It follows that $[V_{\alpha - 1}, L_\alpha] \leq Z_{\alpha}$, so $V_{\alpha - 1} \lhd G_{\alpha}$, contradicting (1.1.6).

Next we note that by the choice of $\alpha - 1$ and $\alpha - 2$, $L/O_2(L) \cong A_7$. Assume $Q_{\alpha - 2} \cap Q_{\alpha - 1} \in \text{Syl}_2(L)$. Then if $C$ is a characteristic subgroup of $Q_{\alpha - 2} \cap Q_{\alpha - 1}$ which is normal in $L$, then since $Q_{\alpha - 2} \cap Q_{\alpha - 1} \lhd Q_{\alpha - 1}$, $C \lhd Q_{\alpha - 1}$, so $C \lhd \langle Q_{\alpha - 1}, L \rangle \geq L_{\alpha}$, and $C \lhd \langle Q_{\alpha - 2}, L \cap G_{\alpha - 1} \rangle$. By (3) $\langle Q_{\alpha - 2}, L \cap G_{\alpha - 1} \rangle$ is transitive on $\Delta(\alpha - 1)$ and $C \lhd L$ so as $L$ is transitive on $\Delta(\alpha)$, (1.1.6) implies that $C = 1$. Next set $S = L \cap G_{\alpha - 1}$. Then $S \in \text{Syl}_2(L)$. Assume $S > Q_{\alpha - 1} \cap Q_{\alpha - 2}$. Let $C$ be a characteristic subgroup of $S$ which is normal in $L$. Then $|Q_{\alpha - 1} : N_{Q_{\alpha - 1}}(C)| \leq 2$, and since $\langle N_{Q_{\alpha - 1}}(C), L \rangle$ has index $\leq 4$ in $G_{\alpha}$, $\langle N_{Q_{\alpha - 1}}(C), L \rangle \geq L_{\alpha}$, so $C$ is normalized by $L_\alpha$. Further since $|SQ_{\alpha - 2} : S| \leq 2$, $C$ is normalized by $Q_{\alpha - 2}$. It follows that $C$ is normalized by $\langle Q_{\alpha - 2}, L \cap G_{\alpha - 1} \rangle$, and again $C = 1$. As $V_{\alpha - 1} \leq O_2(L)$ (1.6) implies that $[V_{\alpha - 1}, O^2(L)] \leq Z_{\alpha} \leq V_{\alpha - 1}$, and $V_{\alpha - 1} \lhd G_{\alpha}$, a contradiction. This completes the proof of (*), and the proof of (2.7) is complete.

Subcase 2. \[ [Z_{\alpha}, Z_{\alpha}] = 1. \]

(2.8) (1) \( b \) is odd.

(2) \( Z_\beta \leq Z(G_\beta). \)

(3) \( Q_\alpha \leq Q_\beta. \)

(4) \( Z_{\alpha} \) is either a natural or the permutation module for $G_\alpha/Q_{\alpha}$.

Proof. Part (1) follows from (1.1.3) and (1.1.5), and (2) follows from (1.1.5). For (3) observe that $Z_\alpha \leq Q_{\alpha - 1'}$, but $Z_\alpha \nleq Q_{\alpha'}$. Next observe that for $\alpha' + 1 \in \Delta(\alpha')$, $|Z_{\alpha' + 1} : Z_{\alpha' + 1} \cap Q_{\alpha'}| \leq 8$, and $Z_{\alpha' + 1} \cap Q_{\alpha'}$ is centralized by $L = \langle Q_{\alpha' + 1}, Z_{\alpha'} \rangle$. If $Q_{\alpha' + 1} \in \text{Syl}_2(L)$, then $L = \langle Q_{\alpha' + 1} \rangle$, and $L \lhd G_{\alpha' + 1}$, further $L$ satisfies the assumptions of (1.6), so $U = [O^2(L), O_2(L)]$ is the natural module for $L/O_2(L)$, but $Z_{\alpha'} \cap U \neq 1$, a contradiction. We deduce that $L \cap G_{\alpha' + 1} \nleq Q_{\alpha' + 1}$, so some element in $L \cap G_{\alpha' + 1}$ induces a $j$-transvection on $Z_{\alpha' + 1}$, for $j \leq 3$. Hence as $Z(G_\alpha) = 1$, (4) follows from (1.2.2).

(2.9) \( b \leq 1. \)
Proof. Assume $b > 1$, then by $(2.8.1)$, $b \geq 3$. We now need several lemmas before proving $(2.9)$. We discuss two main subcases as follows:

Case A. $Z_a$ is the permutation module.

\begin{equation}
(2.10) \quad 8 \leq |Y_b| \leq 16.
\end{equation}

Proof. For $\alpha' + 1 \in A(\alpha')$, $|Z_{\alpha' + 1} : Z_{\alpha' + 1} \cap Q_{\alpha'}| \leq 8$, and $Z_{\alpha' + 1} \cap Q_{\alpha'}$ is centralized by $Z_a$. Thus by $(2.6)$, $|Y_a| = |Y_b| \geq 8$. Hence since $Y_a$ is centralized by $L = \langle Q_{\alpha' + 1}, Z_{\alpha'} \rangle$, and since as before $L \cap G_{\alpha' + 1} \nsubseteq Q_{\alpha' + 1}$, we get an element in $G_{\alpha'} \cap G_{\alpha' + 1} \setminus Q_{\alpha' + 1}$, centralizing $Y_a$. It follows that if $|Y_a| > 16$, we again get a contradiction via $(1.2.1)$. This completes the proof of $(2.10)$.


\begin{equation}
(2.11) \quad \text{Let } \lambda, \delta \in \beta^G, \text{ and assume that } d(\lambda, \delta) = b - 1, \text{ and that } V_{\lambda} \nsubseteq Q_{\delta + 1}, \text{ for some } \delta + 1 \in A(\delta). \text{ Then:}
\end{equation}

\begin{enumerate}
    \item[(a)] $|V_{\lambda} : V_{\lambda} \cap Q_{\delta}| = 2$, and $|V_{\lambda} : V_{\lambda} \cap Q_{\delta + 1}| = 4$.
    \item[(b)] $V_{\lambda} \nsubseteq Q_{\delta + 1}$ for any $\delta + 1 \in A(\delta)$.
    \item[(c)] For any $\lambda - 1 \in A(\lambda), Z_{\lambda - 1} \nsubseteq Q_{\delta}$, and $|Z_{\lambda - 1} : Z_{\lambda - 1} \cap Q_{\delta + 1}| = 4$ for any $\delta + 1 \in A(\delta) \setminus \{\delta - 1\}$.
    \item[(d)] (a), (b), and (c) hold for $\lambda$ in place of $\delta$, and $\delta$, in place of $\lambda$.
\end{enumerate}

Proof. $V_{\lambda}$ centralizes $Y_{\delta}$, and hence by $(1.2.3)$, $|V_{\lambda} \cap Q_{\delta} : V_{\lambda} \cap Q_{\delta + 1}| \leq 2$. Note now that $Z_{\delta + 1} \nsubseteq Q_{\lambda - 1}$, for some $\lambda - 1 \in A(\lambda)$, else $Z_{\delta + 1}$ would be centralized by $V_{\lambda}$, which is impossible. Now if $|V_{\lambda} : V_{\lambda} \cap Q_{\lambda + 1}| \leq 2$, then $|Z_{\lambda - 1} : Z_{\lambda - 1} \cap Q_{\lambda + 1}| \leq 2$, and then either $Z_{\delta + 1}$ induces transvections on $Z_{\lambda - 1}$, or $|Y_{\lambda}| \geq 32$; in either case we get a contradiction. Thus $|V_{\lambda} : V_{\lambda} \cap Q_{\delta + 1}| > 4$. Now (a) and (b) are an easy consequence of the definitions. To prove (c), note that we saw that $Z_{\delta + 1} \nsubseteq Q_{\lambda - 1}$, for some $\lambda - 1 \in A(\lambda)$. Thus $V_{\delta} \nsubseteq Q_{\lambda - 1}$, and by symmetry (a) and (b) hold for $\lambda$ in place of $\delta$, and $\delta$, in place of $\lambda$. Hence for some $\delta + 1 \in A(\delta)$, $Z_{\lambda + 1} \nsubseteq Q_{\lambda}$, and in particular $Z_{\delta + 1} \nsubseteq Q_{\lambda - 1}$, for any $\lambda - 1 \in A(\lambda) \setminus \{\lambda + 1\}$. Thus as above we must have $|Z_{\lambda - 1} : Z_{\lambda - 1} \cap Q_{\delta + 1}| = 4$. If $Z_{\delta - 1} \nsubseteq Q_{\delta}$, then we are done, else $Z_{\lambda - 1} \nsubseteq Q_{\delta}$, and then by (a), $|Z_{\lambda - 1} : Z_{\lambda - 1} \cap Q_{\delta + 1}| = 2$, impossible. This proves (c). Part (d) follows by symmetry.

\begin{equation}
(2.12) \quad \text{Choose } z - 1 \in A(\alpha) \text{ such that } \langle G_{\alpha} \cap G_{z - 1}, V_{\alpha' + 1} \rangle = G_{\alpha} \text{ (observe that by (2.11) such a choice is possible). Then } V_{z - 1} \nsubseteq Q_{z - 1}.
\end{equation}

Proof. Assume that $V_{x - 1} \nsubseteq Q_{x - 1}$. Note that by $(2.11)$ our assumption implies $\langle G_{x} \cap G_{x - 1}, Z_{x' + 1} \cap Q_{\beta} \rangle = G_{x}$, for $x' + 1 \in A(x') \setminus \{x' - 1\}$. Next as $V_{x - 1} \nsubseteq Q_{x - 1}, V_{x - 1}$ centralizes $Y_{x}$ and as $C_{G_{x - 1}}(Y_{x}) = (Z_{x} \cap Q_{2})Q_{x + 1}$,
We thus have that \( [I, -1, Z_{\alpha - 1} \cap Q_\beta] \leq [Z_{\alpha - 1}, Z_{\alpha - 1} \cap Q_\beta] \). Thus \( V_{\alpha - 1} \) is normal in \( \langle G_{\alpha - 1} \cap G_{\alpha}, Z_{\alpha - 1} \cap Q_\beta \rangle = G_{\alpha} \), a contradiction.

(2.13) \( b \leq 3 \).

**Proof.** Assume \( b > 3 \). Choose \( \alpha - 1 \in A(\alpha) \) with \( \langle G_{\alpha - 1} \cap G_{\alpha}, Z_{\alpha - 1} \cap Q_\beta \rangle = G_{\alpha} \), for \( \alpha' + 1 \in A(\alpha') \backslash \{\alpha' - 1\} \). Then by (2.12), \( V_{\alpha - 1} \not\leq Q_{\alpha - 1} \). Hence by (2.11), \( Z_{\alpha - 1} \cap Q_{\alpha - 1} \not\leq Q_{\alpha - 2} \), for \( \alpha - 2 \in A(\alpha - 1) \backslash \{\alpha\} \). Set \( R_0 = [Z_{\alpha - 1} \cap Q_{\alpha - 1}, Z_{\alpha - 2}] \), \( R_0 \not\leq V_{\alpha - 2} \), and hence \( R_0 \) is centralized by \( Z_{\alpha - 1} \cap Q_\beta \).

Further it is readily verified that \( R_0 \cap Z_{\alpha - 1} \neq 1 \). Thus \( R_0 \cap Z_{\alpha - 1} \) is centralized by \( \langle G_{\alpha - 1} \cap G_{\alpha}, Z_{\alpha - 1} \cap Q_\beta \rangle = G_{\alpha} \), contradicting \( Z(G_{\alpha}) = 1 \).

(2.14) \( b \neq 3 \).

**Proof.** Assume \( b = 3 \). Note that for \( \alpha' + 1 \in A(\alpha') \backslash \{\alpha' - 1\} \), \( (Z_{\alpha' + 1} \cap Q_\beta) Q_{\alpha' + 1} / Q_{\alpha} \) is the central involution of \( G_{\alpha} \cap G_\beta / Q_\alpha \), since \( (Z_{\alpha' + 1} \cap Q_\beta) \) centralizes \( Y_\beta \). Set \( H = \langle Q_\alpha, Z_{\alpha' + 1} \rangle \). Then \( H \) is transitive on \( A(\beta) \), and \( H \) centralizes \( Y_\beta \). It follows that \( [Z_{\alpha' + 1} \cap Q_\beta, Z_\alpha] = [t, Z_{\alpha - 1}] \), where \( tQ_{\alpha' - 1} / Q_{\alpha - 1} \) is the central involution in \( G_\beta \cap G_{\alpha' - 1} / Q_{\alpha' - 1} \). Further we have \( [Z_{\alpha' + 1} \cap Q_\beta, Z_\alpha \cap Q_{\alpha'}] \neq 1 \), so we may conclude:

\[(*) \quad \text{If } tQ_{\alpha' - 1} / Q_{\alpha' - 1} \text{ is the central involution of } G_\delta \cap G_{\alpha' - 1} / Q_{\alpha' - 1}, \text{ for } \delta = \beta, \alpha', \text{ respectively, then } [t_\beta, Z_{\alpha' - 1}] \cap [t_\alpha', Z_{\alpha' - 1}] \neq 1.\]

On the other hand if we choose \( \alpha - 1 \in A(\alpha) \), as in (2.12), we get by (2.12) that \( V_{\alpha - 1} \not\leq Q_{\alpha - 1} \), and by (2.11), that \( V_{\alpha - 1} \not\leq Q_\beta \). Thus choosing in (*) \( \alpha - 1, \alpha, \beta \) in place of \( \beta, \alpha', \beta' \) respectively, we get a contradiction to \( \langle G_{\alpha - 1} \cap G_{\alpha}, Z_{\alpha - 1} \cap Q_\beta \rangle = G_{\alpha} \).

Case A2. \( |Y_\beta| = 8 \).

We start with:

(2.15) Let \( \lambda, \delta \in \beta^G \), and assume that \( d(\lambda, \delta) = b - 1 \), and that \( V_{\lambda} \not\leq Q_{\delta + 1} \), for some \( \delta + 1 \in A(\delta) \). Then we have:

(a) \( |V_{\lambda} : V_{\lambda} \cap Q_\delta| = 2 \), and \( |V_{\lambda} : V_{\lambda} \cap Q_{\delta + 1}| = 8 \).
(b) \( V_{\lambda} \not\leq Q_{\delta + 1} \) for any \( \delta + 1 \in A(\delta) \backslash \{\delta - 1\} \).
(c) For any \( \lambda - 1 \in A(\lambda) \backslash \{\lambda + 1\}, Z_{\lambda - 1} \not\leq Q_\delta \), and \( |Z_{\lambda - 1} \cap Q_\delta : Z_{\lambda - 1} \cap Q_{\delta + 1}| = 4 \), for any \( \delta + 1 \in A(\delta) \backslash \{\delta - 1\} \).
(d) (a), (b), (c) hold for \( \delta \) in place of \( \lambda \) and \( \lambda \) in place of \( \delta \).

**Proof.** To prove (a) and (b) it suffices to show that \( |V_{\lambda} : V_{\lambda} \cap Q_{\delta + 1}| = 8 \). Assume that \( |V_{\lambda} : V_{\lambda} \cap Q_{\delta + 1}| \leq 4 \). Note now that \( Z_{\delta + 1} \not\leq Q_{\lambda - 1} \), for some \( \lambda - 1 \in A(\lambda) \), else \( Z_{\delta + 1} \), would be centralized by
$V_\lambda$, which is clearly impossible. Thus we have $|Z_{\lambda-1} : Z_{\lambda-1} \cap Q_{\delta+1}| \leq 4$, and $Z_{\lambda-1} \cap Q_{\delta+1}$ is centralized by $Z_{\delta+1}$. As $|Y_\lambda| = 8$, it follows by (2.6) that $Z_{\delta+1} \leq Q_{2\lambda}$. Further by (1.2.3), $|Z_{\delta+1} : Z_{\delta+1} \cap Q_{\lambda-1}| = 2$. Note now that $Z_{\lambda-1} \leq Q_{\delta+1}$, and thus either $Z_{\lambda-1}$ induces transvections on $Z_{\delta+1}$, or $|Y_\delta| \geq 32$, in both cases we get a contradiction. Note now that we had actually shown that for $\lambda - 1$, $|Z_{\lambda-1} : Z_{\lambda-1} \cap Q_{\delta+1}| = 8$. It easily follows now that $|Z_{\delta+1} : Z_{\delta+1} \cap Q_{\lambda-1}| = 8$, and then (c) follows immediately. Part (d) follows by symmetry.

(2.16) Choose $\alpha - 1 \in A(\alpha)$ such that $\langle G_\alpha \cap G_{\alpha-1}, V_{\alpha'} \cap Q_\beta \rangle = G_\alpha$ (observe that by (2.15) such a choice is possible). Then $V_{\alpha-1} \leq Q_{\alpha'-1}$.

**Proof.** The proof here is precisely as in (2.12).

(2.17) $b \leq 1$.

**Proof.** Assume $b > 1$. Then $b \geq 3$. Choose $\alpha - 1 \in A(\alpha)$, as in (2.16). By (2.16), $V_{\alpha-1} \leq Q_{\alpha'-1}$, and by (2.15), $V_{\alpha-1} \leq Q_{\alpha'-2}$. Set $R_0 = [Z_{\alpha'-1} \cap Q_{\alpha'-1}, Z_{\alpha'-2} \cap Q_{\alpha'-2}]$ for $\alpha - 2 \in A(\alpha - 1) \setminus \{\alpha\}$. It is readily verified that $R_0 \cap Z_{\alpha'-1} \neq 1$, and $R_0 \leq Z_{\alpha'-1}$. Thus $R_0$ is centralized by $\langle G_{\alpha-1} \cap G_{\alpha}, Z_{\alpha'+1} \cap Q_\beta \rangle = G_\alpha$, for $\alpha' + 1 \in A(\alpha') \setminus \{\alpha' - 1\}$, contradicting $Z(G_\alpha) = 1$.

**Case B.** $Z_\alpha$ is the natural module.

(2.18) $b \leq 3$.

**Proof.** Assume $b > 3$. The proof consists of several steps as follows:

1. for $L = \langle Q_{\alpha''} \rangle$ we have $L \cap G_\alpha \leq Q_\alpha$.

Assume $L \cap G_\alpha \leq Q_\alpha$, then $L$ satisfies the hypotheses of (1.6), so $U = [O^2(L), O_2(L)]$ is the natural module for $L/O_2(L) \cong S_3$, but $Z_{\beta} \leq U$, a contradiction.

2. $Z_{\alpha'+1} \cap Q_\beta \leq Q_\alpha$, for $\alpha' + 1 \in A(\alpha') \setminus \{\alpha' - 1\}$.

Assume $Z_{\alpha'+1} \cap Q_\beta \leq Q_\alpha$. Set $L = \langle Q_{\alpha'+1}, Z_\beta \rangle$. By (1), $L \cap G_{\alpha'+1} \leq Q_{\alpha'+1}$, and $L$ centralizes $Z_{\alpha'+1} \cap Q_\beta$. Since $|Z_{\alpha'+1} : Z_{\alpha'+1} \cap Q_\beta| \leq 2$, we get a contradiction.

3. $Z_\alpha \cap Q_\alpha' \leq Q_{\alpha'+1}$.

This follows immediately from (2). Set $R_0 = [Z_\alpha \cap Q_\alpha', Z_{\alpha'+1}]$.

4. $R_0 = Y_{\alpha'}, |Y_{\alpha'}| = 4$, and $R_0 \leq V_\beta$.

Clearly $|R_0| = 4$ and $R_0 \leq V_\beta$, so it remains to show that $R_0 = Y_{\alpha'}$. We have
that $R_0 \leq Z_{x'+1}$, and $R_0$ is centralized by $Z_x$; hence by (2.6), $R_0 \leq Y_{x'}$, and then $R_0 = Y_{x'}$.

(5) $Z_{x'+1} \nsubseteq Q_\beta$.

Assume that $Z_{x'+1} \subseteq Q_\beta$. Then since $Z_{x'+1}$ operates quadratically on $Z_x$, $[[Z_x, Z_{x'+1}]] = 4$. But $R_0 = [Z_x \cap Q_\alpha, Z_{x'+1}] \leq [Z_x, Z_{x'+1}]$ and $|R_0| = 4$. Hence we get $[Z_x, Z_{x'+1}] = R_0 \leq Z_{x'+1}$, so $Z_{x'+1} \triangleleft G_{x'}$, a contradiction.

(6) $Y_{x'} \neq Y_\beta$.

Assume $Y_\beta = Y_{x'} = R_0$. Then $R_0 = Y_\beta \leq Z_x \cap Q_{x'}$; so $Z_x \cap Q_{x'}$ is normalized by $Z_{x'+1}$; thus by (5), $|Y_\beta| = 8$, impossible.

(7) $W_{x'} \nsubseteq Q_{x' - 2}$.

Otherwise by (4), $W_{x'} \leq C_{G_{x'}}(Y_{x'})$, and $W_{x'} \leq G_{x'}$. Hence $W_{x'} \leq Z_{x'}(W_{x'} \cap Q_{x'})$ and then $[W_{x'}, Z_{x'+1}] \leq [Z_x, Z_{x'+1}[W_{x'} \cap Q_{x'}, Z_{x'+1}] \leq Y_{x'}Y_\beta \leq W_{x'}$; and $W_{x'} \triangleleft G_\beta$, a contradiction.

(8) $Y_{x'} = Y_{x'-2}$.

$Y_{x'-2} = C_{Z_{x'-1}}(W_{x'} \cap Q_{x'-2}) = Y_{x'}$.

We can now get the final contradiction. Iterating the above steps with $x - 2$, $x' - 2$ in place of $x$, $x'$, for suitable choice of $x - 2 \in \Delta(2)(x)$ (namely such that $Z_{x-2} \nsubseteq Q_{x'-2}$), we obtain $Y_{x'-4} = Y_{x'-7}$, and proceeding in the above manner we get: $Y_{x'} = Y_{x'-2} = Y_{x'-4} = \cdots = Y_\beta$. This contradicts (6).

The proof of (2.18) is complete.

(2.19) $b \neq 3$.

Proof: Assume $b = 3$. Note first that the proof of (1)-(5) of (2.18) relied only on the fact that $b > 1$. Hence we have:

(1) The assertions of (1)-(5) of (2.18) hold here too.

We thus have $Z_{x'+1} \cap Q_\beta \nsubseteq Q_\alpha$, for $x' + 1 \in \Delta(x') \setminus \{x' + 1\}$, $Z_x \cap Q_\alpha \nsubseteq Q_{x'+1}$, and $|Y_\beta| = 4$. Further we have $[Z_x \cap Q_{x'}, Z_{x'+1}] = Y_{x'}$, $[Z_{x'+1} \cap Q_\beta, Z_x] = Y_{x'}$, and $[Z_x \cap Q_{x'}, Z_{x'+1} \cap Q_\beta] \neq 1$, and hence $Y_{x'} \subseteq Y_\beta \neq 1$. It follows that:

(2) If $x - 1 \in \Delta(x)$ satisfies $Y_{x-1} \cap Y_\beta = 1$, then $V_{x-1} \subseteq Q_\beta$.

We use now the same argument that appears in [4, Lemma (4.3) (6), p. 105], to find a vertex $x - 1 \in \Delta(x)$ such that $Y_{x-1} \cap Y_\beta = 1$ and $V_{x-1} \nsubseteq Q_\beta$. This will be an immediate contradiction to (2).

For $\delta \in \beta$, set $Q^*_\delta = C_{Q_\delta}(Y_\delta)$. Note that:

(3) $Y_\beta$ is the unique two dimensional submodule of $Z_x$ normalized by $G_x \cap G_\beta$. The module structure of $Z_x$ implies that:
Furthermore using (3) it is easily verified that we can choose \( \alpha - 1 \in \mathcal{A}(\alpha) \) such that \( Y_{x-1} \cap Y_\beta = 1 \) and \( \langle G_{x-1} \cap G_\alpha, G_\alpha \cap G_\beta \rangle = G_\alpha \). Note that by (2) we have \( V_{x-1} \leq Q_\beta \). By (1) we have that \( V_\beta \leq Q_{x-1} \). Note that \( V_{x-1} \leq Q^*_\beta \) and \( V_\beta \leq Q^*_x \). Hence by (4) we have \( [V_{x-1}, V_\beta] \leq Y_{x-1} \cap Y_\beta = 1 \). Note further that \( C_\beta \leq Q_{x+1} \leq G_\alpha \), and \( C_\beta \cap Q_\alpha \leq Q^*_\alpha \). Thus \( [C_\beta, V_x'] \leq [Z_x, V_x'] \leq [Z_x, V_x'] = [Z_x, V_x'] Y_\beta \leq V_\beta \). We have shown:

(5) \( [C_\beta, V_x'] \leq V_\beta \).

Let \( d \) be an element of order 3 in \( \langle Q_\alpha, V_x \rangle \). Then by (5) we have:

(6) \( C_\beta = C_{C_\beta}(d) V_\beta \) and hence \( C_\beta' = C_{C_\beta}(d)' \).

Let \( \Omega = \{ V_\mu | \mu \in \mathcal{A}(\alpha) \) and \( [V_\mu, V_\beta] = 1 \} \). Set \( \bar{W}_x = \langle V_\mu | V_\mu \in \Omega \rangle \). Note that by (6):

(7) \( [V_{x-1}, V_\mu] \leq C_{C_\beta}(d) \), for every \( V_\mu \in \Omega \).

Choose now \( V_\mu \in \Omega \), and set \( \bar{R} = [V_{x-1}, V_\mu] \) and assume that \( \bar{R} \neq 1 \). Assume first that \( V_\mu \leq Q_{x-1} \). As \( \alpha - 1 \in \beta^G \), (4) implies that \( \bar{R} \leq Y_{x-1} \). As \( Y_{x-1} \cap Y_\beta = 1 \), we have that \( |\bar{R} Y_\beta| \geq 8 \). But by (7) \( \bar{R} \) is centralized by \( d \). Furthermore \( \bar{R} Y_\beta \leq Z_\alpha \) and \( d \) acts transitively on \( \mathcal{A}(\beta) \). This implies that \( \bar{R} \leq Y_\beta \), contradicting \( |Y_\beta| = 4 \).

Assume next that \( V_\mu \leq Q_{x-1} \). Then using (1) we get that \( \bar{R} \geq Y_{x-1} \). We get a contradiction as in the previous paragraph of the proof.

We have shown that \( \Omega = \{ V_\mu | \mu \in \mathcal{A}(\alpha) \) and \( [V_\mu, V_{x-1}'] = 1 \} \). Hence \( \bar{W}_x \) is normalized by \( \langle G_{x-1} \cap G_\alpha, G_\alpha \cap G_\beta \rangle = G_\alpha \). Note now that \( \bar{W}_x \leq C_\beta \), and hence by (5), \( \bar{W}_x \) is normalized by \( \langle Q_\alpha, V_x \rangle \). But \( \langle Q_\alpha, V_x \rangle \) acts transitively on \( \mathcal{A}(\beta) \). This contradicts (1.1.6). The proof of (2.19) is complete.

Proof of (2.9). The assertion of (2.9) follows from (2.13), (2.14), (2.17), (2.18), and (2.19).

3. The Case \( G_\alpha / Q_\alpha \approx S_3 \)

Subcase 1. \( |Z_x, Z_x'| \neq 1 \).

Here we know already (see (2.1.1)) that \( b \equiv 0(2) \).

(3.1) \( b = 2 \).

Proof. Assume that \( b > 2 \). We need now several lemmas before proving (3.1):

(3.2) \( Z_{x-1} \leq Z(G_{x-1}) \).
Proof. Assume $Z_{x-1} \not\subseteq Z(G_{x-1})$. If $Z_{x-1} \not\subseteq Q_{x-1}$, then by (1.1.5), $[Z_{x-1}, Z_{x'-1}] \neq 1$, impossible by (2.7). If $Z_{x-1} \subseteq Q_{x'-1}$, then $Z_{x-1} \cap Q_{x'}$ is centralized by $\langle Q_{x-1}, Z_{x'} \rangle = L$. If $Q_{x-1} \in \text{Syl}_2(L)$, then $L$ satisfies the assumptions of (1.6), so by (1.6), $[V_\beta, O^2(L)] \subseteq Z_2$, hence $V_\beta \triangleleft G_2$, a contradiction. Hence there is an element in $G_{x-1} \setminus Q_{x-1}$ which either centralizes or induces a transvection on $Z_{x-1}$, impossible.

\((3.3)\)

1. $Z(G_x) = 1$.
2. $|Z_x| = 4$.
4. $[Z_x, Z_{x'}] = Z_\beta$.
5. $Q_\beta \not\subseteq Q_x$.

Proof. Part (1) follows from (1.1.4). Part (2) follows immediately from (1). For (3) observe that $Z_\beta \subseteq Z_x$, but $Z_\beta \neq Z_x$. For (4) note that $[Z_x, Z_{x'}]$ is centralized by $Z_{x'} Q_\beta = G_x \cap G_\beta$. Part (5) is obvious since $Z_{x'} \not\subseteq Q_\beta$, but $Z_x \subseteq Q_x$.

\((3.4)\)

$C_{V_\beta Z_\beta}(G_\beta) = Q_\beta$.

Proof. $[Z_x, Q_\beta]$ is normal in $G_x \cap G_\beta$ and has order 2 (note $Q_\beta \not\subseteq Q_x$), thus $[Z_x, Q_\beta] = Z_\beta$. So $Z_\beta = [Z_x^{G_\beta}, Q_\beta] = [V_\beta, Q_\beta]$. Next if $C_{V_\beta Z_\beta}(G_\beta) > Q_\beta$, then $C_{V_\beta Z_\beta}(G_\beta) = G_\beta$, and then $Z_\beta \triangleleft G_\beta$, a contradiction.

\((3.5)\) Notation. Here we denote $Y_x = \bigcap_{\delta \in \Delta(x)} V_\delta$.

\((3.6)\)

$Y_x = V_{\delta_1} \cap V_{\delta_2}$ for any pair of vertices $\delta_1, \delta_2 \in \Delta(x)$.

Proof. This follows immediately from (3.4).

Proof of (3.1). Case 1. $V_{x-1} \not\subseteq Q_{x'-1}$.

In this case we have $Z_{x-2} \not\subseteq Q_{x'-2}$, for some $x-2 \in \Delta(x-1)$. By (3.3.4), $[Z_{x-2}, Z_{x'-2}] = Z_{x-1}$, and thus $Z_{x-1}$ is centralized by $\langle G_{x-1} \cap G_x, Z_{x'} \rangle = G_x$, a contradiction.

Case 2. $V_{x-1} \subseteq Q_{x'-1}$.

We have $[V_{x-1}, Z_{x'}] \subseteq [Z_x Q_{x'}, Z_{x'}] \subseteq V_{x-1}$, and $V_{x-1}$ is normalized by $\langle G_{x-1} \cap G_x, Z_{x'} \rangle = G_x$, a contradiction.

Case 3. $V_{x-1} \not\subseteq G_{x'-1}$, but $V_{x-1} \not\subseteq Q_{x'-1}$.

Consider $V_{x-1} \cap Q_{x'-1}$. It is normalized by $L = \langle Q_{x-1}, Z_{x'} \rangle$. If $Q_{x-1} \in \text{Syl}_2(L)$, then $L$ satisfies the assumptions of (1.6), and hence $[V_\beta, O^2(L)] \subseteq Z_2$, and hence $V_\beta \triangleleft G_2$, a contradiction. Thus by (3.6), $V_{x-1} \cap Q_{x'-1} \subseteq Y_x$. Now if $|V_{x-1} : V_{x-1} \cap Q_{x'-1}| = 2$, then $|V_{x-1} : Y_{x'-2}| = 2$, and as $V_{x-1}$ centralizes $Y_{x'-2}$, we see that $V_{x-1}$ induces transvections on $V_{x-1}/Z_{x'-1}$, a contradiction. It follows that $|V_{x-1} : V_{x-1} \cap Q_{x'-1}| =$.
Now (3.3.3) together with (1.5) implies: \( |[V_{x'-1}, V_{x' -1}]| \leq 8 \). Note now that \( V_{x'-1} \cap Q_{x'-1} \) centralizes \( V_{x'-1} \), else we get \( [V_{x'-1}, V_{x' -1} \cap Q_{x'-1}] = Z_{x'-1} \), and again \( Z_{x'-1} \) is in the center of \( G_{z} \), impossible. Thus \( |[V_{x'-1}, V_{x' -1} \cap Q_{x'-1}]| \geq 4 \), and \( V_{x'-1} \cap Q_{x} \neq Q_{x'-1} \). Let now \( R_{0} = [V_{x'-1}, V_{x' -1}] \). We have \( |R_{0}| = 2, 4 \) or \( 8 \). If \( |R_{0}| = 2 \), then \( V_{x'-1} \cap Q_{x} \) induces transvections on \( V_{x'-1} \), impossible. If \( |R_{0}| = 8 \), then \( V_{x'-1} \cap R_{0} = [V_{x'-1}, V_{x' -1}] \), so \( V_{x'-1} \triangleleft \langle G_{y-1} \cap G_{z}, V_{x' -1} \rangle = G_{z} \), a contradiction.

Thus \( |R_{0}| = 4 \). Assume that \( R_{0} \geq Z_{x'-1} \); then it is easy to check that as \( |[V_{x'-1}, V_{x' -1} \cap Q_{x'-1}]| = 4 \), there will be an involution \( t \in V_{x'-1} \setminus V_{x'-1} \cap Q_{x'-1} \), satisfying \( [V_{x'-1} \cap Q_{x}, t] \leq Z_{x'-1} \), and hence \( t \) acts as a transvection on \( V_{x'-1}/Z_{x'-1} \), a contradiction. Now by (3.3.4), \( [Z_{x'}, Z_{x'}] = Z_{x'-1} \). Hence \( V_{x'-1} \geq R_{0} Z_{x'-1} = [V_{x'-1}, V_{x' -1}] \), and as before we get that \( V_{x'-1} \triangleleft G_{z} \), a contradiction. This completes the proof of (3.1).

Subcase 2. \( [Z_{x'}, Z_{x'}] = 1 \).

(3.7) (1) \( b \) is odd.

(2) \( Z_{B} \triangleleft Z(G_{b}) \).

Proof. Part (1) follows from (1.1.3), and (1.1.5). Then (2) is immediate.

(3.8) \( Q_{B} \triangleleft Q_{z} \).

Proof. Assume \( Q_{B} \leq Q_{z} \). Then \( V_{B} \leq Z(Q_{B}) \), and hence \( V_{z} \leq Z(Q_{z}) \). But \( |[V_{z}, V_{z} \cap Q_{z}]| \leq 8 \), and \( V_{z} \cap Q_{z} \) is centralized by \( Z_{z} \). Now (1.2.2) implies that \( V_{z} \) involves either a natural module or the permutation module as a unique noncentral chief factor. By (1.5), \( U = [V_{z'}, G_{z}] \) is either a natural or the permutation module, but \( Z_{z} \cap U \neq 1 \), a contradiction.

(3.9) (1) \( Z_{z} = Z_{z'-1} \times Z_{B} \).

(2) \( [x, Z_{z}] = Z_{B} \), for \( x \in G_{z} \cap G_{b} \setminus Q_{z} \).

Proof. First observe that \( Z_{z'-1} \cap Z_{B} \triangleleft Z(\langle Q_{z'-1}, Q_{z}, Q_{B} \rangle) = Z(G_{z}) = 1 \). Next for \( x \in G_{z} \cap G_{b} \setminus Q_{z} \), \( x \) acts as an involution on \( Z_{z} \), and hence \( [x, Z_{z}] \leq Z_{B} \). Hence \( [Z_{z}, Z_{B}] \leq Z_{B} \) and \( [Z_{z}, Q_{z'-1}] \leq Z_{z'-1} \); hence \( Z_{z'-1} Z_{B} \) is normalized by \( Q_{z'-1} \) and \( Q_{B} \), and clearly \( Z_{z'-1} Z_{B} \) is normalized by \( Q_{z} \). Thus \( Z_{z'-1} Z_{B} \triangleleft \langle Q_{z'-1}, Q_{z}, Q_{B} \rangle = G_{z} \). This proves (1). For (2) observe that \( x \) acts as an involution on \( Z_{z} \), and \( C_{Z_{z}}(x) = Z_{B} \).

(3.10) \( C_{V_{B}/Z_{B}}(G_{B}) = Q_{B} \).

Proof. By (3.9.2), \( [Z_{z'}, Q_{B}] = Q_{B} \), so \( [Z_{B}^{G_{z'}}, Q_{B}] = Z_{B} = [V_{B}, Q_{z}] \). Now if \( C_{V_{B}/Z_{B}}(G_{B}) > Q_{B} \), then \( C_{V_{B}/Z_{B}}(G_{B}) = G_{B} \), and then \( Z_{z} \triangleleft G_{B} \), a contradiction.
(3.11) \( V_{x'} \leq Q_\beta \).

**Proof.** Assume \( V_{x'} \leq Q_\beta \), then \( V_{x'} \leq Q_x \), or \( |V_{x'} : V_x \cap Q_x| = 2 \). In the first case we get that \( V_{x'} \) is centralized by \( Z_x \). But \( Z_x \not\leq Q_x \); this contradicts (3.10). In the second case we get that involutions in \( Z_x \) act as transvections on \( V_{x'} / Z_{x'} \), contradicting (1.2.1).

(3.12) \( b > 1 \).

**Proof.** Assume \( b = 1 \), then \( V_\beta \) contains an element \( d \) of order 3, and by (3.10), \( [V_\beta, Q_\beta] \leq Z_\beta \). It follows that \( [d, Q_\beta] = 1 \), but \( G_\beta \) is 2-constrained, a contradiction.

The next three lemmas are technical lemmas which we use later:

(3.13) (1) \( L_\beta Q_x = G_\beta \).

(2) Assume \( \delta \in \Delta(\beta) \) satisfies \( \langle G_\delta \cap G_\beta, V_{x'} \rangle = G_\beta \). Then \( \langle Q_\delta, V_{x'} \rangle = G_\beta \).

**Proof.** Set \( L = L_\beta Q_x \). If \( L \neq G_\beta \), then \( Q_x \in \text{Syl}_2(L) \), and hence \( L \) satisfies the hypotheses of (1.6). It follows that \( U = [O^2(L), O_2(L)] \) is the natural module for \( L/O_2(L) \), but \( Z_\beta \cap U \neq 1 \), a contradiction. Part (2) follows immediately from (1).

(3.14) For \( \{\lambda, \lambda_1\} = \{x', \beta\} \) if \( |V_{\lambda} : V_{\lambda} \cap Q_{\lambda_1}| = 2 \), or if \( |V_{\lambda} : V_{\lambda} \cap Q_{\lambda_1}| = |V_{\lambda_1} : V_{\lambda_1} \cap Q_{\lambda}| = 4 \), and \( \tilde{P}_x \) is the permutation module, then \( [V_{\lambda} \cap Q_{\lambda_1}, V_{\lambda_1}] = Z_{\lambda_1} \), and in particular \( Z_{\lambda_1} \leq V_{\lambda} \).

**Proof.** This follows easily from (1.2.1), (1.2.3), and (3.9.2).

(3.15) Assume there exists an involution \( t \in (G_\beta \cap G_{x+2}) \setminus Q_\beta \) such that \( X = \langle W_\delta, \delta \in \Delta(\beta), \langle Q_\delta, t \rangle = G_\beta \rangle \) we have \( |[X, t]V_\beta/V_\beta| \leq 2 \). Then \( b \leq 3 \).

**Proof.** Assume \( b > 3 \). Then \( W_\alpha \) is elementary abelian. We now use (1.4). Set \( H = G_\beta/V_\beta \), \( W = W_\beta/V_\beta \), and \( U = W_\delta/V_\beta \), for \( \delta \in \Delta(\beta) \) satisfying \( \langle Q_\delta, t \rangle = G_\beta \). Then in the notation of (1.4), the assumptions of (1.4) are satisfied for \( H, W, U, \) and \( t \), so by (1.4), \( [W_\beta, L_\beta] \leq V_\beta \). Hence \( W_\alpha \leq G_\beta \), a contradiction.

(3.16) (1) Let \( \lambda \in \beta^G \). Then \( \tilde{V}_{x'} \) is either a natural or the permutation module for \( G_{x'/Q_{x'}} \).

(2) \( Z_\alpha \cap D_\beta = Z_\beta \), and in particular if \( \tilde{V}_{\beta} \) is a natural module, then \( |Z_\beta| = |Z_\beta|^2 = 4 \).

**Proof.** Choose \( x' + 1 \in \Delta(x') \) such that \( Z_{x'+1} \leq Q_\beta \). Let \( t \in Z_{x'+1} \setminus Q_\beta \).
Then $\iota$ centralizes $V_\beta \cap Q_{x^2 + 1}$, and hence $|[[\overline{V_\beta}, \iota]]| \leq 8$, so (1) follows from (1.2.2). For the first part of (2) note that by (3.13.1) there exist $x \in L_\beta \cap G_{x^2} \setminus Q_x$, and as we saw already $C_{Z_x}(x) = Z_\beta$. The second part of (2) follows immediately from the module structure of $\overline{V_\beta}$.

(3.17) $b > 3$.

**Proof.** Assume $b = 3$. Suppose $V_\beta/D_\beta$ is a natural module.

(1) $|V_\beta| = 32$ and $|V_\beta \cap V_{x^2}| = 8$.

First note that $[V_\beta, V_{x^2}] \leq V_\beta \cap V_{x^2}$, and since $V_\beta$ does not induce transvections on $V_{x^2}/Z_{x^2}$, we have $V_\beta \cap V_{x^2} \neq Z_{x^2}$. Now $[V_\beta \cap V_{x^2}, Q_{x^2 + 2}] \neq 1$, else we get that $|\Omega_1(Z(G_\beta \cap G_{x^2 + 2}))| > 2$. It follows that $Z_{x^2 + 2} \leq [V_\beta \cap V_{x^2}, Q_{x^2 + 2}] \leq [V_\beta, G_{x^2}]$, and hence $V_\beta = [V_\beta, G_{x^2}]$, and by (1.5), $|V_\beta| = 32$. If $|V_\beta \cap V_{x^2}| = 16$, then $V_\beta$ induces transvections on $V_{x^2}$. This completes the proof of (1).

(2) $Q_{x^2}/C_{x^2}$ is the dual of $V_{x^2}/Z_{x^2}$, for $G_{x^2}/Q_{x^2}$.

This can be readily verified and we omit the details.

(3) $[C_{x^2}, L_{x^2}] \leq V_{x^2}$.

Since $Z_{x^2 + 2} \leq V_{x^2}$, $C_r \leq Q_{x^2 + 2}$, and $|[C_{x^2}, V_\beta]/[V_{x^2}, V_\beta]| \leq 2$, now (3) follows as $V_\beta$ does not induce transvections in $C_{x^2}/V_{x^2}$.

(4) $|V_\beta \cap Q_{x^2}| = 16$, and $V_\beta \cap Q_{x^2} \nleq C_{x^2}$.

Let $W \leq V_\beta$ with $|W| = 16$, $W > V_\beta \cap V_{x^2}$, and $[W, G_\beta \cap G_{x^2 + 2}] \leq V_\beta \cap V_{x^2}$. Then $[W, Q_{x^2} \cap Q_{x^2 + 2}] \leq V_\beta \cap V_{x^2}$, and if $W \nleq Q_{x^2}$, then $W$ induces transvections in $Q_{x^2}/C_{x^2}$, contradicting (2). Hence $W \leq Q_{x^2}$, and the first part of (4) is proved. The second part follows since $V_{x^2}$ does not induce transvections on $V_{x^2}$.

Set now $V = \langle (V_\beta \cap Q_{x^2})^{G_{x^2 + 2}} \rangle$. Then as $V_\beta \cap Q_{x^2} \nsim G_\beta \cap G_{x^2 + 2}$, it follows that $V = (V_\beta \cap Q_{x^2})(V_{x^2} \cap Q_{x^2})(V_\beta \cap Q_{x^2})^x$, for some $x \in Q_{x^2}$, and hence $|V| \leq 2^b$. Set now $Q = \langle V^{G_{x^2}} \rangle$ and $U = [V, Q_{x^2}]/V_{x^2}$. Since $[V, Q_{x^2}] \leq C_{x^2} \cap V$, we have $|U : V_{x^2}| \leq 2$. Note further that by (3), $U \nleq G_{x^2}$. We have shown:

(5) $U \nleq G_{x^2}$, and $|U : V_{x^2}| \leq 2$.

Now if $Q \leq Q_{x^2 + 2}$, then $[Q, V_\beta] \leq V_\beta \cap Q_{x^2}$, and $V_\beta$ induces transvections on $Q/C_Q(V_{x^2})$, impossible; hence

(6) $Q \nleq Q_{x^2 + 2}$.

Assume first that $U = V_{x^2}$. Then it follows that $V = (V_\beta \cap Q_{x^2})(V_{x^2} \cap Q_\beta)$. Hence $|V| = 32$. Then using (2) and (3) we find an element $d \in G_{x^2}$ of order
3, such that \( d \) normalizes \( V \), \( d \) centralizes \( Z_{x+2} \), and \( d \) does not centralize \( V/Z_{x+2} \). Also if \( d_1 \in G_{x+2} \) is of order 3, then \( d_1 \) does not centralize both \( Z_{x+2} \) and \( V/Z_{x+2} \). We note now that as \( d \) centralizes \( Z_{x+2} \) and \( d_1 \) normalizes but does not centralize \( Z_{x+2} \), the image of \( \langle d, d_1 \rangle \) in \( N_G(V)/C_G(V) \) has order 9. Furthermore, a moment of thought about the action of \( d \) and \( d_1 \) on \( V \) yields that the image of \( \langle d, d_1 \rangle \) in \( N_G(V/Z_{x+2})/C_G(V/Z_{x+1}) \) has order 9. It follows that \( N_G(V/Z_{x+2})/C_G(V/Z_{x+1}) \leq L_3(2) \), and has order divisible by 9, a contradiction.

Assume next that \( U \neq V_x \). Then \( |V/U| = 2 \), and since \( V_\beta \) does not induce transvections on \( Q/U \), \([Q \cap Q_{x+2}, V_\beta]U = VU \). It follows that \([Q/U, G_x] = Q/U \), and hence \( |Q/U| = 16 \). But then \( Q/V_x \) is elementary abelian and \([V, Q] \leq V_x \). Now (6) implies \([V, Q_x] \leq V_x \), contradicting \( U \neq V_x \).

Suppose next that \( V_\beta/D_\beta \) is the permutation module.

(7) \(|V_\beta : V_\beta \cap Q_x| = |V_x : V_x \cap Q_\beta| = 2|.

If not then we may assume that \(|V_\beta : V_\beta \cap Q_x| = 4| But then since \([Q_\beta, V_\beta] = Z_{p, Q} \leq Q_x \), the structure of the Sylow 2 subgroup of \( A_7 \) implies \( Q_\beta \cap Q_{x+2} \leq V_\beta Q_x \), then \([Q_\beta \cap Q_{x+2}, V_x] \leq [V_\beta Q_x, V_x] \leq [V_\beta, V_x] Z_x \leq V_\beta \); thus, since \( V_x \) does not centralize \( Q_\beta/V_\beta \), we conclude that \([Q_\beta, L_\beta] \leq V_\beta \). It follows that \( Q_\beta \cap Q_{x+2} \leq G_\beta \). But then \([V_x \cap Q_\beta, V_\beta] = 1 \), contradicting (3.14)

Now \([V_{x'}, V_\beta \cap Q_{x'}] = Z_x \), and the module structure of \( V_\beta \), together with (3.16.2) imply:

(8) \(|Z_x| = 2|.

We have \( Z_{x+2} \leq [V_\beta, V_x] \leq [V_\beta, G_\beta] \), and it follows that \( V_\beta \leq [V_\beta, G_\beta] \), so by (1.5):

(9) \(|V_\beta| = 2^7|.

It follows that:

(10) \(Q_\beta/C_\beta \) is the dual of \( V_\beta/Z_\beta \), for \( G_\beta/Q_\beta \).

Now \([V_\beta \cap Q_x, V_{x'} \cap Q_\beta] \leq Z_{p, Q} \cap Z_x = 1| \); also there exists \( x' + 1 \in A(x') \) with \( Z_{x'+1} \not\subseteq Q_\beta \), hence \( V_{x'} = (V_{x'} \cap Q_\beta) Z_{x'+1} \), and \( V_\beta \cap Q_{x'} \cap Q_{x'+1} \) centralizes \( V_{x'} \). It follows that:

(11) \(|V_\beta \cap C_{x'}| = 2^5|.

Note now that \( C_{x'} \leq Q_{x+2} \), and \([C_{x'}, V_{x'}] \) is centralized by \( V_{x'} \). It follows that \([C_{x'}, V_\beta] : [V_{x'}, V_\beta] = 2 \), and since \( V_\beta \) does not induce transvections in \( C_{x'}/V_{x'} \), we have:

(12) \([C_{x'}, L_{x'}] \leq V_{x'} \).
Next we observe that (12) implies that \((V_\beta \cap C_{x'}) V_{x'} \triangleleft G_{x'}\). Set \(Q = \langle (V_\beta \cap Q_{x'}) V_{x'} \rangle\). Then \(Q \subseteq C_{x'}\), and as in the proof of (12) it is readily verified that:

(13) \(Q \not\subseteq Q_{x+2}\).

But \([Q, (V_\beta \cap C_{x'}) V_{x'}] \leq Z_{x'}\), and in particular \([Q, (V_\beta \cap C_{x'})] \leq Z_{x'} \leq V_\beta \cap C_{x'}\). Let now \(x \in Q \cap Q_{x+2} \leq G_{x+2} \setminus Q_{x+2}\). Then \(V_\beta \cap C_{x'} = (V_\beta \cap C_{x'})^x = V_{x'} \cap C_{x'}\). It follows that: \(V_\beta \cap C_{x'} = V_\beta \cap V_{x'} \cap C_{x'} = V_{x+2} \cap C_{x'}\) (see (3.6)) = \(V_\beta \cap V_{x'} \cap C_{x'} = V_\beta \cap V_{x'}\). We conclude that:

(14) \(V_\beta \cap C_{x'} = V_\beta \cap V_{x'}\).

Now if \(|Q_{x'} \cap Q_{x+2} : Q_{x'} \cap Q_{x}| = 8\), then \(V_{x'} Q_\beta / Q_\beta\) is the central involution in \((Q_{x'} \cap Q_{x+2}) Q_{\beta}/Q_{\beta}\), and \([Q_{x'} \cap Q_{x+2}, V_\beta \cap V_{x'}] = Z_{x'}\). The module structure of \(V_\beta\) together with (8), (11), and (14) supplies a contradiction. Hence, \(|Q_{x'} : Q_{x'} \cap Q_{\beta}| \leq 8\), and \([Q_{x'} \cap Q_{\beta}, V_{x'} \cap V_{x'}] = 1\); this contradicts (11), (14), and (10).

The next lemma is a technical result we require later:

(3.18) If \(V_\beta\) is a natural module then \(|V_{x'}/V_{x'} \cap Q_{\beta}| = |V_\beta/ V_\beta \cap Q_{x'}|\).

Proof. We may assume without loss that \(|V_{x'}/V_{x'} \cap Q_{\beta}| = 4\), while \(|V_\beta/V_\beta \cap Q_{x'}| = 2\). By considering the action of \(V_{x'}\) on \(V_\beta\), we easily conclude that \(C_{V_\beta}(V_{x'}) = C_{V_\beta}(x)\), for each \(x \in V_{x'} \setminus Q_{\beta}\). Thus to get a contradiction, it suffices to show that for every \(y \in V_\beta \cap Q_{x'}\) we have: \(C_{V_\beta}(y) \setminus Q_{\beta} \neq \emptyset\), since this would imply \(V_\beta \cap Q_{x'}\) is centralized by \(V_{x'}\), and thus \(V_{x'}\) induces transvections on \(V_\beta\). But \([V_{x'}, V_\beta \cap Q_{x'}] = Z_{x'}\), and hence for each \(y \in V_\beta \cap Q_{x'}\), \(C_{V_\beta}(y)\) has codim at most 1 in \(V_{x'}\). As \(|V_{x'}/V_{x'} \cap Q_{\beta}| = 4\), we are done.

The next lemma is a technical lemma which we require twice in the following. Its proof appears in [4 Lemma (3.5)(6), p. 96], in a slightly different context:

(3.19) Let \(\beta_1 \in \beta^G\). Let \((\beta_1, \lambda, \delta)\) be an arc of length 2. Set \(L = L_{\beta_1}\) and \(Q = [Q_{\beta_1}, L]\). Let \(E \triangleleft Q_{\beta_1}\) be an elementary abelian subgroup. Assume that:

(a) \(V_{\beta_1}\) is a natural module for \(G_{\beta_1}/Q_{\beta_1}\).
(b) \(E \triangleleft G_{\beta_1}\).
(c) \(V_{\beta_1} \subseteq E\).
(d) \([E, L] \subseteq V_{\beta_1}\).

Then we have:

(i) \(Q \not\subseteq Q_{\lambda}\).
(ii) \(V_{\delta} \cap E \subseteq V_{\delta} \cap V_{\beta_1}\).
To prove (i), note first that \([Q, L] = Q\). Assume that \(Q \leq Q_\lambda\).
Then \([Q_\lambda \cap Q_{\beta_1}, L] \leq Q_\lambda \cap Q_{\beta_1}\) and hence \([V_{\beta_1}, Q_\lambda \cap Q_{\beta_1}] = 1\). Thus \([V_{\beta_1}, Q] = 1\). Thus \([Q_{\beta_1}, L, V_{\beta_1}] = 1\). By (3.10), \([V_{\beta_1}, Q_{\beta_1}, L] = 1\). The three subgroup lemma implies that \([L, V_{\beta_1}, Q_{\beta_1}] = 1\). Now (1.5) implies that \([L, V_{\beta_1}]\) is a natural module for \(G_{\beta_1}/Q_{\beta_1}\). But clearly, \(Z_{\beta_1} \leq [L, V_{\beta_1}]\) contradicting (3.2). Hence (i) is proved.

We proceed to prove (ii). We claim that:

1. either \(|V_{\beta_1} \cap V_{\delta}| = 8\) or \(V_{\beta_1} \cap V_{\delta} = Z_\lambda\).

Assume that \(V_{\beta_1} \cap V_{\delta} \neq Z_\lambda\). Note that \(V_{\beta_1} \cap V_{\delta} \leq G_\lambda\). It follows that \(Z_\lambda \leq [V_{\beta_1} \cap V_{\delta}, Q] \leq [V_{\beta_1}, G_{\beta_1}]\), so \(V_{\beta_1} = [V_{\beta_1}, G_{\beta_1}]\) and hence \(|V_{\beta_1}| - 2^5\). This implies \(|V_{\beta_1} \cap V_{\delta}| = 8\).

Set \(B = V_{\delta} \cap E\). Since \([Q, L] = Q\), the three subgroup lemma together with assumption (d), implies that \([B, Q] \leq V_{\beta_1}\) and \([B, Q \cap Q_{\beta_1}] \leq V_{\beta_1} \cap V_{\beta_1}\).

Pick \(b \in B\). By (1) we have, \(|[bV_{\beta_1}, Q]/Z_{\beta_1}| \leq 8\). Thus, \([bV_{\beta_1}, Q]\) cannot contain a noncentral chief factor for \(L\). Hence \([bV_{\beta_1}, Q] \leq C_{V_{\beta_1}}(L)\). It follows that \([B, Q] \leq C_{V_{\beta_1}}(L)\). Now the three subgroup lemma implies that \([B, Q] \leq Z_{\beta_1}\). Let \(t \in Q \setminus Q_{\beta_1}\) (\(t\) exists by (i)). Then \(t\) normalizes \(B\) and hence \(B = V_{\beta_1} \cap V_{\alpha}\). This completes the proof of (3.19).

(3.20) \(b > 5\).

Proof. Assume that \(b = 5\). We wish to show that if \((\beta, \alpha')\) is a critical pair, then, in all possible cases, there exists an involution \(t \in (G_{\alpha + 2} \cap G_{\beta}) \setminus Q_{\beta}\) such that:

1. If \(X = \langle W_\delta | \delta \in \Delta(\beta), \langle Q_\delta, t \rangle = G_{\beta}\rangle\), then \(|[X, t] V_{\beta}/V_{\beta}| \leq 2\).

This contradicts (3.15).

Assume first that for all critical pairs \((\beta, \alpha')\), \(|V_{\beta} : V_{\beta} \cap Q_{\alpha'}| = |V_{\alpha'} : V_{\alpha'} \cap Q_{\beta}| = 2\). Then by (3.14), \(Z_{\beta} \leq V_{\alpha'}\), and \(Z_{\alpha'} \leq V_{\beta}\), for all critical pairs \((\beta, \alpha')\). Choose now a critical pair \((\beta, \alpha')\). Fix an involution \(t \in V_{\alpha'} \setminus Q_{\beta}\). Let \(\delta \in \Delta(\beta)\) satisfy \(\langle Q_\delta, t \rangle = G_{\beta}\) (see (3.13.2)). Assume \(W_{\delta} \leq Q_{\alpha'} - 2\), then by the above \(Z_{\delta - 1} \leq V_{\alpha'} - 2\), for some \(\delta - 1 \in \Delta(\delta)\), hence \(Z_{\delta - 1}\) is centralized by \(\langle Q_\delta, t \rangle = G_{\beta}\), a contradiction. It follows that \(W_{\delta} \leq Q_{\alpha'} - 2\).

Assume now that \([W_{\delta}, V_{\alpha'} - 2] \neq 1\). Then \([V_{\delta - 1}, V_{\alpha'} - 2] \neq 1\), for some \(\delta - 1 \in \Delta(\delta)\). We note now that \(V_{\alpha'} - 2 \leq Q_{\alpha'} - 1\), and it follows that \([V_{\delta - 1}, V_{\alpha'} - 2] = Z_{\delta - 1} = Z_{\alpha'} - 2\). We conclude that \(Z_{\delta - 1}\) is centralized by \(\langle Q_\delta, t \rangle = G_{\beta}\), a contradiction. Thus \([W_{\delta}, V_{\alpha'} - 2] = 1\), and in particular \([W_{\delta}, Z_{\alpha'} - 1] = 1\), and hence \(W_{\delta} \leq Q_{\alpha'} - 1\).

Now since \(Z_{\alpha'} \leq V_{\beta}\), it follows that \(|[W_{\delta}, V_{\alpha'}]| = 2\). Further since \(W_{\beta}\) centralizes \([V_{\beta}, V_{\alpha'}]\), we have \(|W_{\beta} \cap Q_{\alpha'} - 1 : W_{\beta} \cap Q_{\alpha'}| \leq 4\). Thus in this case (1) holds.
Assume next that there exists a critical pair \((\beta, \alpha')\) such that \(|V_\beta : V_\beta \cap Q_\alpha| = 4\). Then as \(b = 5\), we have that for \(\lambda \in A^2(\alpha' - 2)\), \(|W_{\alpha' - 2} : W_{\alpha' - 2} \cap Q_\lambda| \geq 4\), and further since \(V_\beta\) acts quadratically on \(V_{\alpha'}\), \(W_{\alpha' - 2}\) contains an elementary abelian subgroup of order 4 not contained in \(Q_\lambda\), and acting quadratically on \(V_{\lambda'}\). Hence:

\[ (2) \] For any \(\lambda \in \beta\), and \(\lambda' \in A^2(\alpha')\) we have that \(W_{\lambda'}\) contains an elementary abelian subgroup of order 4 not contained in \(Q_{\lambda'}\), and acting quadratically on \(V_{\lambda'}\).

Assume first that \(V_\beta / D_\beta\) is the permutation module. Choose a critical pair \((\beta, \alpha')\). We claim that there exists an involution \(t \in W_{\alpha' - 2} \setminus Q_\beta\) such that:

\[ (3) \] If \(\delta \in A(\beta)\) satisfies \(\langle Q_\delta, t \rangle = G_\beta\), then \([W_{\delta - 1}, Z_{\alpha' - 2}] \neq 1\) for \(\delta - 1 \in A(\delta) \setminus \{\beta\}\).

Indeed since by (2), \(W_{\alpha' - 2} Q_\beta / Q_\beta \leq Z(G_{\alpha' - 2} \cap G_{\beta} / Q_\beta)\) we can choose \(t \in W_{\alpha' - 2} \setminus Q_\beta\), such that \([t, Q_{\alpha' - 2}] \leq Q_\beta\). Now since \(\delta - 1 \in \beta\) and \(\beta \in A^2(\delta - 1)\), (2) implies that \(W_{\delta - 1}\) contains an elementary abelian subgroup of order 4 not contained in \(Q_\beta\) and acting quadratically on \(V_{\beta}\). By the module structure of \(V_\beta\) and by the choice of \(t\) we have:

\[ (*) \langle W_{\delta - 1}, t \rangle \] does not centralize a vector in \(V_\beta\).

Now by (3.9.1), \(Z_{\alpha' - 2} = Z_{\beta} \times Z_{\alpha' - 2}\). Hence by (3.16.2), \(Z_{\alpha' - 2} \leq D_\beta\). Furthermore, \([t, Z_{\alpha' - 2}] = 1\). Hence if \([W_{\delta - 1}, Z_{\alpha' - 2}] = 1\), then \([\langle W_{\delta - 1}, t\rangle, Z_{\alpha' - 2}] = 1\) contradicting (*) This completes the proof of (3).

Fix an involution \(t \in W_{\alpha' - 2} \setminus Q_\beta\) such that (3) holds for \(t\). Pick a \(\delta \in A(\beta)\) such that \(\langle Q_\delta, t \rangle = G_\beta\). Assume \(W_\delta \leq Q_{\alpha' - 2}\), then there exists \(\delta - 1 \in A(\delta)\) such that \(V_{\delta - 1} \leq Q_{\alpha' - 2}\). Now since by (3), \(Z_{\alpha' - 2} \leq V_{\delta - 1}\) (else \([W_{\delta - 1}, Z_{\alpha' - 2}] = 1\)), (3.14) implies that \(Z_{\delta - 1} \leq V_{\alpha' - 2}\), so \(Z_{\delta - 1}\) is centralized by \(\langle Q_\delta, t \rangle = G_\beta\), a contradiction. It follows that \(W_\delta \leq Q_{\alpha' - 2}\), and then also \(W_\delta \leq Q_{\alpha' - 1}\), else \(Z_{\delta - 1} \leq Z(G_\delta)\). Now by (3.14) we may assume that \(Z_{\alpha'} \leq V_{\beta}\), thus as in the first paragraph of the proof we get (1).

Assume next that \(V_\beta / D_\beta\) is a natural module. Assume further that:

\[ (**) \] There exists a critical pair \((\beta, \alpha')\) such that for some involution \(t \in W_\beta\), \(\langle Q_{\alpha' - 1}, t \rangle = G_{\alpha' - 2}\).

Note next that (**) implies that \(Z_{\alpha'} \leq V_{\beta}\); hence by (3.14) and (3.18) we must have \(|V_\beta : V_\beta \cap Q_{\alpha'}| = |V_{\alpha'} : V_{\alpha'} \cap Q_{\beta}| = 4\). Set \(R = [V_\beta, V_{\alpha'}]\), then \(RV_{\alpha' - 2} \leq \langle Q_{\alpha' + 2}, Q_{\alpha' - 1} \rangle = G_{\alpha' - 2}\). Set \(E = RV_{\alpha' - 2}\). Then the hypotheses of (3.19) are easily verified (taking, in the notation of (3.19), \(\beta_1 = \alpha' - 2\), \(\lambda = \alpha' - 1\), and \(\delta = \alpha'\)) and hence by (3.19), \(E \cap V_{\alpha'} \leq V_{\alpha' - 2}\). In particular we get that \(R \leq V_{\alpha' - 2}\). We note now that as \(b > 4\), \([W_{\alpha' - 2}, V_{\alpha' - 2}] = 1\), and hence \(W_{\alpha' - 2} \leq V_{\alpha'} Q_{\beta}\). It follows that \([W_{\alpha' - 2}, V_\beta] \leq V_{\alpha' - 2}\), and hence, \([W_{\alpha' - 2}, W_{\alpha' - 2}] \leq V_{\alpha' - 2}\). We conclude that:
(4) For $\beta_1 \in \beta^G$ we have $[W_{\beta_1}, W_{\beta_1}] \leq V_{\beta_1}$.

Further we have that $R \leq Z_{x + 2}$, and it follows that $[V_\beta \cap V_{x - 2}, Q_{x + 2}] \neq 1$. Hence $Z_{x + 2} \leq [V_\beta, G_\beta]$, so $V_\beta = [V_\beta, G_\beta]$, and it follows that:

(5) $|V_\beta| = 2^k$.

Choose $\delta \in A(\beta)$, and $\lambda \in A(\delta)$, such that $\langle W_\lambda, V_{x'} \rangle Q_\beta = G_\beta$. Set $R_1 = [V_\lambda, V_{x - 2}]$. Then by (4), $R_1 \leq V_\lambda \cap V_\beta$, and $R_1$ is centralized by $\langle W_\lambda, V_{x'} \rangle$. Assume $R_1 \neq 1$. If $R_1 = Z_\lambda$, then $Z_\lambda$ is centralized by $\langle Q_\delta, V_{x'} \rangle = G_\beta$, a contradiction. Else if $1 \leq R_1 \leq 2$, and $R_1 Z_{x - 2}$ is centralized by $G_\beta$, contradicting (5).

Let $\mu \in A^{12}(\beta)$ with $[V_\mu, V_{x - 2}] = 1$. Set $R_2 = [V_\mu, V_\lambda]$. Then by (4), $R_2 \leq V_\lambda \cap V_\mu$. Further $V_\lambda$, and $V_\mu$ centralize $V_{x - 2}$, and hence $V_\lambda \cap V_\mu \leq Q_{x - 1} \leq G_{x'}$. Also since $W_\beta$ centralize $R = [V_\beta, V_{x'}]$, $W_\beta \cap G_{x'}$ is abelian and hence $R_2 \leq Q_{x'}$. Now if $[R_2, V_{x'}] = Z_{x'}$, then $Z_{x'} \leq V_\beta$, which is false, hence $[R_2, V_{x'}] = 1$, and $R_2$ is centralized by $\langle W_\lambda, V_{x'} \rangle$, and as above we get $R_2 = 1$. We conclude that $\langle V_\mu | \mu \in A^{12}(\beta) \rangle$ and $[V_\mu, V_{x - 2}] = 1$. Then $K = \langle V_\mu | \mu \in A^{12}(\beta) \rangle$ and $[V_\mu, V_{x'}] = 1$. Then $K$ is normalized by $\langle Q_{x + 2}, Q_\delta \rangle = G_\beta$, this is a contradiction as $[W_\beta, V_{x - 2}] \neq 1$. Note now that the contradiction just obtained was obtained under hypothesis (**), and that if we would interchange in (**) the role of $\beta$ and $\alpha'$ we would obtain a similar contradiction. Hence we conclude:

(6) If $(\beta, \alpha')$ is a critical pair and $t \in W_{x'}$, then $\langle Q_{x + 2}, t \rangle \not\subseteq G_{x - 2}$.

Choose now a critical pair $(\beta, \alpha')$. Let $t \in V_{x'} \setminus Q_\beta$. Let $\delta \in A(\beta)$ with $\langle Q_\delta, t \rangle = G_\beta$. Then by (6), $W_\delta \leq Q_{x - 2}$, and then $W_\delta \leq Q_{x - 1} \leq G_{x'}$. It follows that $[W_\delta, V_{x'}] \leq [V_\beta, V_{x'}] Z_{x'}$. Hence (1) holds in this case too. This completes the proof of (3.20).

(3.21) $\overline{V}_\beta$ is not the permutation module.

Proof. Assume that $\overline{V}_\beta$ is the permutation module. We divide the proof into two cases as follows:

Case 1. There exists a critical pair $(\beta, \alpha')$, such that $Z_{x'} \not\subseteq W_{\beta}$.

By (3.14) we may assume:

(1) $|V_\beta : V_\beta \cap Q_{x'}| = 4$, $|V_{x'} : V_{x'} \cap Q_\beta| = 2$, $[V_\beta \cap Q_{x'} \cap V_{x'}] = 1$, and $[V_{x'} \cap Q_\beta, V_\beta] = Z_\beta$.

Choose $\delta \in A(\beta)$ with $\langle Q_\delta, V_{x'} \rangle = G_\beta$ (see (3.13.2)). Choose also $\delta - 1 \in A(\delta)$. We wish to show that:

(2) $V_{\delta - 1} \cap Q_{x - 1}$ has codim at most 2 in $V_{\delta - 1}$.
If \(|V_{\delta-1} : V_{\delta-1} \cap Q_{x'-2}| \leq 2\), this is obvious. So assume that
\(|V_{\delta-1} : V_{\delta-1} \cap Q_{x'-2}| = 4\), then by (3.14) (taking, in the notation of (3.14),
\(|\delta - 1, x' - 2|\) in place of \(|\beta, x'|\)), \([V_{x'-2} \cap Q_{\delta-1}, V_{\delta-1}] = Z_{\delta-1}\), it follows that
\(Z_{\delta-1}\) is centralized by \(\langle Q_\delta, V_{x'} \rangle = G_\beta\), a contradiction. Note now that:

(3) \(V_{x'-2} \cap V_{x'} = \bigcap_{\lambda \in \Delta(x'-1)} V_{\lambda}\).

Now \(V_{\delta-1} \cap Q_{x'-1} \leq V_\beta Q_{x'}\), and further \([V_{\delta-1} \cap Q_{x'}, V_{x'}] = 1\), else \(Z_{x'} \leq W_\beta\). We conclude that:
\([V_{\delta-1} \cap Q_{x'-1}, V_{x'} \cap Q_\beta] \leq Z_\beta \leq V_{\delta-1} \cap Q_{x'-1}\).
So \(V_{\delta-1} \cap Q_{x'-1}\) is normalized by \(V_{x'} \cap Q_\beta\). Observe now that \(V_{x'} \cap Q_\beta \leq Q_{x'}\), else \(Z_{x'} \leq Q_{x'}\), and then \([Z_{x'} \cap V_{x'}] = 1\), as \(Z_{x'} \leq W_\beta\). But then \(Z_{x'}\) is centralized by \(\langle Q_\delta, V_{x'} \rangle = G_\beta\), a contradiction. Now (2) and (3) (with \(\delta - 1\) in place of \(x'\) and \(\delta\) in place of \(x'-1\)) implies that \(V_{x'-1} \cap V_\beta\) has codim at most 2 in \(V_\beta\), and hence \(V_{x'} \cap V_{x'-1}\) has codim at most 2 in \(V_{x'}\). We see that \(V_\beta / V_\beta \cap Q_{x'}\) is an elementary 4-group centralizing a subspace of codim at most 2 in \(V_{x'}\), contradicting (1.2.3).

**Case 2.** For all critical pairs \((\beta, x')\) we have \(Z_{x'} \leq W_\beta\) and \(Z_\beta \leq W_{x'}\).

By (3.14) we may assume that \(Z_{x'} \leq V_\beta\). Fix an involution \(t \in V_{x'} \setminus Q_\beta\). Choose \(\delta \in \Delta(\beta)\) with \(\langle Q_\delta, t \rangle = G_\beta\). Assume first that for some \(\delta - 1 \in \Delta(\delta)\), \(V_{\delta-1} \leq Q_{x'}\). Then \(Z_{\delta-1} \leq W_{x'-2}\) and in particular \(Z_{\delta-1}\) is centralized by \(\langle Q_\delta, t \rangle = G_\beta\), a contradiction. Thus \(V_{\delta-1} \leq Q_{x'-2}\), and as \(Z_{x'} \leq W_\beta\), \(V_{\delta-1} \leq Q_{x'-1}\). But now the hypotheses of (3.15) hold for \(t\), a contradiction.

(3.22) \(\bar{V}_\beta\) is not a natural module and hence the case \(G_\beta / Q_{x} \simeq S_3\), \([Z_{x}, Z_{x'}] = 1\) do not occur.

**Proof.** Assume \(\bar{V}_\beta\) is a natural module. We first show:

(1) \(V_{x'-1} \cap V_\beta \cap D_{\beta} = Z_\beta\).

Set \(V = V_{x'-1} \cap V_\beta\). Observe first that \([V, Q_\alpha] \leq Z_\lambda \leq Z_{x}\), for each \(\lambda \in \Delta(a)\).
Set \(H = \langle Q_\beta^G \rangle\). By the above \([V, H] \leq Z_{x}\). It follows that \((V \cap D_{\beta}) Z_{x} \leq G_{x}\).
Hence also \([V \cap D_{\beta}, Q_\alpha] \simeq G_\alpha\). Furthermore, \([V \cap D_{\beta}, Q_\alpha] \leq D_{\beta}\), and hence \([V \cap D_{\beta}, Q_\alpha] \) is normalized by \(L_\beta\). Now (1.1.6) implies that \([V \cap D_{\beta}, Q_\alpha] = 1\). Thus \(V \cap D_{\beta} \leq Z_{x}\), and then \(V \cap D_{\beta} = Z_\beta\). We divide the proof now into two main cases as follows:

**Case 1.** There exists a critical pair \((\beta, x')\) such that \(Z_{x} \leq W_\beta\).
Note that here we must have \([V_{\beta} \cap Q_{x'}, V_{x'}] = 1\), else \(Z_{x} \leq W_\beta\). Thus by (3.18), \(V_\beta \cap Q_{x'}\) has codim 2 in \(V_\beta\). Choose now \(\delta \in \Delta(\beta)\) with \(\langle Q_\delta, V_{x'} \rangle = G_\beta\).

**Subcase A.** \(W_{\delta} \leq Q_{x'-2}\). Choose \(\delta - 1 \in \Delta(\delta)\), and assume that
\[ [V_{\delta-1}, V_{x-2}] \neq 1. \] Then by (3.11), \( V_{x-2} \leq Q_{\delta-1} \), and hence \( Z_{\delta-1} = Z_{x-2} = [V_{\delta-1}, V_{x-2}] \). But then \( Z_{\delta-1} \) is centralized by \( \langle Q_{\delta}, V_{x} \rangle = G_{\beta} \), impossible. Thus \( W_{\delta} \) centralizes \( Z_{x-1} \), and hence, \( W_{\delta} \leq Q_{x-1} \). Now we have \( W_{\delta} \leq V_{\beta} Q_{x} \), and \( [W_{\delta} \cap Q_{x}, V_{x}] = 1 \), else \( Z_{x} \leq W_{\beta} \). Thus \( [W_{\delta}, V_{x}] = [V_{\beta}, V_{x}] \leq V_{\beta} \leq W_{\delta} \), and \( W_{\delta} \not< \langle Q_{\delta}, V_{x} \rangle = G_{\beta} \), a contradiction.

Subcase B. \( W_{\delta} \not< Q_{x-2} \). Choose \( \delta - 1 \in A(\delta) \), note that \( Z_{\delta-1} \not< W_{x-2} \), else \( Z_{\delta-1} \) would be centralized by \( \langle Q_{\delta}, V_{x} \rangle = G_{\beta} \), impossible. Thus after perhaps renaming the vertices we may assume that \( \langle W_{\beta}, Q_{x-1} \rangle = G_{x-2} \). Further observe that we are free to choose notation so that if \( x \in G_{x-2} \) satisfies \( \{(\alpha' - 4)^x, (\alpha' - 3)^x\} = \{\alpha', \alpha' - 1\} \), then \( \langle W_{\beta}, (W_{\beta})^x \rangle \cap Q_{2}(\langle W_{\beta}, (W_{\beta})^x \rangle) \simeq A_7 \). Set \( R = [V_{\beta}, V_{x}] \). Note now that \( RZ_{x} \) is normal in \( G_{x-1} \cap G_{x} \), and \( [W_{\beta}, RZ_{x}] \not< V_{x-2} \). Thus we get that \( RV_{x-2} \not< \langle Q_{x-1}, W_{\beta} \rangle = G_{x-2} \). Set \( E = RV_{x-2} \). Then the hypotheses of (3.19) are easily verified (taking, in the notation of (3.19), \( \beta_1 = \alpha' - 2, \lambda = \alpha' - 1 \) and \( \delta = \alpha' \)) and hence by (3.19), \( E \cap V_{x} \leq V_{x-2} \). In particular we get that \( R \leq V_{x-2} \). Now (1) implies that \( C_{R}(L_{x-2}) = Z_{x-2} \). But \( R \) is centralized by \( \langle W_{\beta}, (W_{\beta})^x \rangle \), a contradiction.

Case 2. For all critical pairs \((\beta, \alpha')\), \( Z_{x-1} \leq W_{\beta} \) and \( Z_{\beta} \leq W_{x} \). Choose \( \delta \in A(\beta) \) with \( \langle Q_{\delta}, t \rangle = G_{\beta} \), for some \( t \in V_{x} \). Assume first that \( W_{\delta} \not< Q_{x-2} \). Then for some \( \delta - 1 \in A(\delta) \setminus \{\beta\} \), \( \{\delta - 1, \alpha' - 2\} \) is a critical pair and hence \( Z_{\delta-1} \leq W_{x-2} \). This implies that \( Z_{\delta-1} \) is centralized by \( \langle Q_{\delta}, t \rangle = G_{\beta} \), a contradiction. Thus, \( W_{\delta} \leq Q_{x-2} \).

Choose \( \delta - 1 \in A(\delta) \), and assume that \( [V_{\delta-1}, V_{x-2}] \neq 1 \). Then by (3.11), \( V_{x-2} \leq Q_{\delta-1} \), and hence \( Z_{\delta-1} = Z_{x-2} = [V_{\delta-1}, V_{x-2}] \). But then \( Z_{\delta-1} \) is centralized by \( \langle Q_{\delta}, V_{x} \rangle = G_{\beta} \), impossible. It follows that \( W_{\delta} \) centralizes \( Z_{x-1} \), and hence, \( W_{\delta} \leq Q_{x-1} \).

If \( |V_{\beta} : V_{\beta} \cap Q_{x}| = 4 \), then \( [W_{\delta}, V_{x}] \leq V_{\beta} Z_{x} \). If \( |V_{\beta} : V_{\beta} \cap Q_{x}| = 2 \), then by (3.14), \( Z_{x} \leq V_{\beta} \), and it follows that in any case \( t \) satisfies the hypotheses of (3.15), a contradiction.

4. THE STRUCTURE OF \( G_{x} \) AND \( G_{\beta} \)

The results of the earlier chapters imply:

(4.1) Either one of the following possibilities occur:

1. \( b = 1, G_{x} / Q_{x} \simeq A_{7} \), and \( Z_{x} \) is the \( A_{7} \) permutation module.
2. \( b = 1, G_{x} / Q_{x} \simeq A_{7} \), and \( Z_{x} \) is a natural \( A_{7} \) module.
3. \( b = 2, G_{x} / Q_{x} \simeq S_{3} \), and \( V_{\beta} \) is the \( A_{7} \) permutation module.
(4.2) Assume that $b = 1$, and that $Z_x$ is the permutation module, then we have:

(a) $G_x \cong 2^6 \cdot A_7$.
(b) $G_\beta \cong 2^{1+1+1+2+2+1} \cdot S_3$.

Proof Let $\alpha + 2 \in A(\beta) \backslash \{\alpha\}$. The proof consists of several steps as follows:

(1) $Z_x \cap Q_\beta \not\leq Q_{x+2}$.

Set $R = Z_x \cap Q_\beta$. Assume $R \leq Q_{\alpha+2}$. Then $R \leq \langle G_x \cap G_\beta, \ Z_{\alpha+2} \rangle = G_\beta$. Further we have $[Z_x, Q_\beta] \leq R$, and hence $[Z_{x+2}, Q_\beta] \leq R$. Set $H = \langle Z_x, Z_{\alpha+2} \rangle$. Then $H$ contains a 3-element, $[H, Q_\beta] \leq R$, and $H$ centralizes $R$. Thus $[O^2(H), Q_\beta] = 1$, a contradiction since $G_\beta$ is 2 constrained.

(2) $Q_x$ is elementary abelian.

We have $\Phi(Q_x) \leq Q_\beta$, if $\Phi(Q_x) \neq 1$, then $\Phi(Q_x) \cap Z_x \neq 1$, and then $Z_x \leq Q_\beta$, a contradiction.

(3) $Q_x = Z_x$.

Note first that by (2.6), $Z_x \cap Q_{x+2} \leq Z_{x+2}$. If $|Z_x \cap Q_{x+2}| = 16$, then as $Q_x \cap Q_\beta$ centralizes $Z_x \cap Q_{x+2}$, it follows that $Q_x \cap Q_\beta \leq (Z_x \cap Q_\beta)Q_{x+2}$. Thus in any case, $Q_x \cap Q_\beta \leq (Z_x \cap Q_\beta)Q_{x+2}$. Hence we have: $[Z_{x+2} \cap Q_\beta, Q_x \cap Q_\beta] \leq Z_x$. As $Z_{x+2}$ does not induce transvections on $Q_x/Z_x$, we have that $[L_x, Q_x] \leq Z_x$. But then the only noncentral chief factor in $Q_x$ is $Z_x$, so (2) and (1.5) together with $Z(G_x) = 1$ finishes the proof.

Assume now that $|Z_x \cap Z_{x+2}| = 16$. Then $Z_x \cap Z_{x+2} = C_{Z_{x+2}}(Z_x \cap Q_\beta)$, and hence $[Z_x \cap Q_\beta, Z_{x+2}] \leq Z_x \cap Z_{x+2} \leq Z_x \cap Q_\beta$. By (2.6), $Z_x \cap Q_\beta \leq Z_{x+2}$, contradicting (1). We conclude:

(4) $|Z_x \cap Z_{x+2}| = 8$.

Note now that:

(5) $Z_x \cap Q_\beta$ does not operate quadratically on $Z_{x+2}$
else we would have $[Z_x \cap Q_\beta, Z_{x+2}] = Z_x \cap Z_{x+2} \leq Z_x \cap Q_\beta$. Furthermore, the module structure of $Z_{x+2}$ implies:

(6) $|Z_\beta| = 4$.

Let now $U_\delta \leq Z_\delta$ with $|U_\delta| = 8$, $U_\delta \geq Z_\beta$, and $[U_\delta, Q_\beta] \leq Z_\beta$, for $\delta \in \{\alpha, \alpha + 2\}$. Set $U_1 = U_\delta U_{x+2}$. Set also $U_2 = (Z_x \cap Q_\beta)(Z_{x+2} \cap Q_\beta)$. Then $Z_x \cap Z_{x+2} \leq (Z_x \cap Z_{x+2})U_1 \leq U_2 \leq Q_\beta \leq G_\beta$, is a chief series exhibiting (b).
(4.3) Assume that $b = 1$, and that $Z_x$ is a natural module, then we have:

(a) $G_x \sim 2^4 \cdot A_7$

(b) $G_\beta \sim 2^{1+2+2+1} \cdot S_3$.

**Proof.** Let $x + 2 \in \mathcal{A}(\beta) \setminus \{a\}$. We draw the attention of the reader to the proof of (4.2). As in the proof of steps (1) and (3) in this proof we have:

(1) $Z_x \cap Q_\beta \leq Q_{x+2}$, and $Q_x = Z_x$.

Hence (a) follows immediately. For (b), assume that $|Z_x \cap Z_{x+2}| = 4$, then $|Z_x \cap Q_\beta : Z_x \cap Q_{x+2}| = 2$. But then $[Z_x \cap Q_\beta, Z_{x+2}] = Z_x \cap Z_{x+2}$, and $Z_{x+2}$ normalizes $Z_x \cap Q_\beta$, it follows that $Z_x \cap Q_\beta \leq Z_x \cap Z_{x+2}$, a contradiction. We have proved:

(2) $|Z_x \cap Z_{x+2}| = 2$, so $Z_x \cap Z_{x+2} = Z_x$.

Further since $[Z_x \cap Q_\beta, Z_{x+2} \cap Q_\beta] = Z_\beta$, $Z_{x+2} \cap Q_\beta$ does not operate quadratically on $Z_x$, and hence: $[Z_x, Z_{x+2} \cap Q_\beta] = Z_x \cap Q_\beta$; so if we set $U = (Z_x \cap Q_\beta)(Z_{x+2} \cap Q_\beta)$, then

(3) $U \trianglelefteq G_\beta$, $|U| = 32$, and $[U, G_\beta] = U$.

It is easy to check now that the series $1 \trianglelefteq Z_\beta \trianglelefteq [U, Q_\beta] \trianglelefteq U \trianglelefteq Q_\beta \trianglelefteq G_\beta$, is a chief series proving that (b) holds.

(4.4) Suppose $b = 2$. Then we have:

(a) $G_\beta \sim 2^1 \cdot A_7$.

(b) $G_\sim 2^{2+2+} \cdot S_3$.

**Proof.** We start with:

(1) $Z_\beta \leq Z(G_\beta)$.

Assume $Z_\beta \leq Z(G_\beta)$. Then by (2.7), $Z_{x-1} \leq Q_\beta$. Set $L = \langle Q_{x-1}, Z_x \rangle$. Note that $Z_{x-1} \cap Q_x$ is centralized by $L$ and since no transvections are induced on $Z_{x-1}$, we must have $L \cap G_{x-1} = Q_{x-1}$. It follows that $Q_{x-1} \in \text{Syl}_2(L)$. We can now use [6, Lemma (3.5), p. 16], to deduce $V_{x-1} \leq H$, a contradiction. As an immediate consequence we have:

(2) $|Z_\beta| = |Z_\beta^2| = 4$.

Set $Z_\beta = \langle t \rangle$. Note first that $(x, y) = i$, where $[x, y] = t'$, $x, y \in V_\beta$, defines a symplectic form on $V_\beta/Z_\beta$ preserved by $G_\beta/Q_\beta$. Now clearly $|V_\beta : V_\beta \cap Q_{x-1}| \leq 8$, and since $[V_\beta \cap Q_{x-1}, V_{x-1}] \leq Z_{x-1}$, we have $V_\beta \cap Q_{x-1} = V_\beta \cap V_{x-1}$. Also $V_\beta \cap V_{x-1} \trianglelefteq G_{x-1}$, so $V_\beta \cap V_{x-1}$ is elementary abelian, else $Q_x \geq \Phi(V_\beta \cap V_{x-1}) \Rightarrow Z_x$. Since $A_7 \leq \text{Sp}(4, 2)$, $|V_\beta \cap Q_x : V_\beta \cap V_{x-1}| = 4$. Now $[Q_\beta \cap Q_x, V_\beta \cap Q_x] \leq Z_\beta \leq Q_{x-1}$, and it follows that $Q_\beta \cap Q_x \leq (V_\beta \cap Q_x)(Q_\beta \cap Q_{x-1})$; thus $Q_\beta = (Q_\beta \cap Q_{x-1}) V_\beta$. Now
$Q_{\beta} \cap Q_{x-1} \simeq G_x$, and hence it is elementary abelian. It follows that $Q_{\beta}/Z_{\beta}$ is elementary abelian. Hence as above we have a symplectic form on $Q_{\beta}/Z_{\beta}$, and since $|Q_{\beta}:Q_{\beta} \cap Q_{x-1}| = 8$, $(Q_{\beta} \cap Q_{x-1})/Z_{\beta}$ is a maximal totally isotropic subspace of $Q_{\beta}/Z_{\beta}$. It follows that $Z(Q_{\beta}) \leq Q_{\beta} \cap Q_{x-1}$, hence $Z(Q_{\beta})$ is elementary abelian, and it follows that $Z(Q_{\beta}) = Z_{\beta}$. Hence we have $|Q_{\beta}| = 2^7$, and $Q_{\beta}/Z_{\beta}$ is the permutation module for $G_{\beta}/Q_{\beta}$. This completes the proof of (a).

For (b) note that the above implies:

(3) $|V_{\beta} \cap V_{x-1}| = 2^4$, and $V_{\beta} \cap V_{x-1}$ is a maximal totally isotropic subspace of $V_{\beta}/Z_{\beta}$. Also $|V_{\beta} \cap Q_{x}:V_{\beta} \cap V_{x-1}| = 4$.

By symmetry, (3) holds for $\alpha - 1$ in place of $\beta$, and $\beta$ in place of $\alpha - 1$. Note now that (3) implies that for $v \in V_{\beta} \setminus V_{x-1}$, $[v, V_{\beta} \cap V_{x-1}] = Z_{\beta}$. It follows that for $v \in (V_{\beta} \cap Q_{x}) \setminus V_{x-1}$, $[v, V_{x-1}] \geq Z_{\beta}$. The module structure of $V_{x-1}$ implies that:

(4) $V_{\beta} \cap Q_{x}$ does not operate quadratically on $V_{x-1}/Z_{x-1}$.

We may further deduce that:

(5) $[V_{\beta} \cap Q_{x}, V_{x-1}] \cap V_{\beta} = Z_{x}$.

Else for some $v \in (V_{\beta} \cap Q_{x}) \setminus V_{x-1}$, we would have $[v, V_{x-1}] \leq V_{\beta} \cap V_{x-1}$, and then $V_{x-1}$ would normalize $\langle v \rangle(V_{\beta} \cap V_{x-1})$. Set now $U_1 = [V_{x-1} \cap Q_{x}, V_{\beta}]$, $U_2 = [V_{\beta} \cap Q_{x}, V_{x-1}]$, and $U = U_1 U_2$. Then $U \leq G_x$, $|U_1| = |U_2| = 2^4$, and by (5), $|U| = 2^6$. It is easy to verify now that:

(6) $U$ involves three noncentral chief factors for $\langle V_{x-1}, V_{\beta} \rangle$.

Note further that by (5), $U \cap V_{x-1} \cap V_{\beta} = Z_{x}$, and so $|U(V_{x-1} \cap V_{\beta})| = 2^8$. As $|Q_{\beta}| = 2^7$, (b) follows.

ACKNOWLEDGMENTS

I am very grateful to B. Stellmacher for suggesting the problem to me. I am also very thankful to U. Meierfrankenfeld and B. Stellmacher for many discussions and many useful remarks.

REFERENCES