On the Cohomology of Finite Semigroups*

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Given a semigroup $S$ and an $S$-module $D$, we define cohomology by $H^n(S, D) = \text{Ext}^n_{S}(Z, D)$ as is usual for monoids. The cohomology of semigroups exhibits some distinctively different characteristics from that of groups. In contrast to the simple result that for a finite group $G$ every element of $H^n(G, D)$ for $n > 0$ has order dividing the order of $G$, we find in Corollary 3.5 that we can have a finite semigroup $S$ in which every element is idempotent and an $S$-module $D$ such that $H^n(S, D)$ has no elements of finite order. Yet the cohomology of a finite semigroup seems to depend especially upon the subgroups it contains. In fact, it seems reasonable to conjecture that for $S$ a finite semigroup the cohomology in dimensions higher than some $n = n(S)$ will be determined by the subgroups of $S$. In particular, then we would have the cohomology of a finite combinatorial (i.e., all subgroups are of order one) semigroup eventually vanishing.

In this spirit we consider a semigroup $S$ with completely simple kernel $K(S)$, i.e., $K(S)$ is the unique minimal ideal of $S$. The theorems below for the most part only require the existence of such a kernel in $S$ rather than finiteness and hence are so stated. Theorem 2.3 may now be paraphrased by saying that the cohomology of any finite semigroup is determined by the maximal subgroup of the kernel and by the action of the semigroup on the set of minimal right ideals which is induced by left multiplication on the kernel.

In Section 3 we find that the cohomology of a completely simple semigroup is isomorphic to that of its maximal subgroup in dimensions greater than 2.

In Section 4 we study what happens when the kernel breaks from the semigroup. The results here provide us with a type of inductive principle for studying the vanishing of cohomology. At the end of this section, we provide examples which show that if the cohomology of finite combinatorial

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semigroups eventually vanish, there is no global bound on this (Example 4.7) and which show that even if the kernel has no non-trivial subgroups, a semigroup may have nonzero cohomology in infinitely many dimensions (Example 4.8).

The class of semigroups whose cohomology eventually vanishes is closed under Schreier extensions [6], which together with our results gives a rather large class.

Undefined notation for semigroups follows [4] or [9], for cohomology [3] or [7].

1. Definitions and Preliminaries

For a semigroup $S$ let $S^1$ be $S$ with a twosided identity adjoined if $S$ does not already have one, and let it be $S$ otherwise. Then by a left (right) $S$-module we mean a left (right) $Z(S^1)$-module.

To simplify notation, we shall write $\text{Hom}_S$, $\otimes_S$, $\text{Ext}_S$ for $\text{Hom}_{Z(S^1)}$, $\otimes_{Z(S^1)}$, $\text{Ext}_{Z(S^1)}$.

**DEFINITION.** For a left $S$-module $D$, we define the cohomology of $S$ with coefficients in $D$ by $H^n(S, D) = \text{Ext}_S^n(Z, D)$, $n = 0, 1, ...$, where $Z$ is considered as a trivial left $S$-module.

By following the proof of [5] for algebras, we see that nothing is changed by the adjunction of an identity to $S$ (see [2] or [1]). Hence nothing is lost by replacing $S$ by $S^1$ (or even $[10]$ by $S^1$, which is $S$ with a twosided identity $I$ adjoined even if $S$ already has one) and defining cohomology via Ext.

We know that a finite semigroup $S$ has a unique minimal ideal $K(S)$, the kernel of $S$. Then $K(S)$ is a completely simple semigroup, i.e., it contains no proper twosided ideal and satisfies the minimum conditions on left and right ideals. Letting $G$ be a maximal subgroup of $K(S)$ and $e \in G$ the identity of $G$, we know by the Rees Theorem [4] that there exist sets $A$ and $B$ of idempotents in $K(S)$ such that $aa' = a$ and $bb' = b'$ for all $a, a' \in A$ and all $b, b' \in B$ and such that $K(S)$ can be expressed as the disjoint unions

$$K(S) = \bigcup_{a \in A} aS^1 = \bigcup_{b \in B} S^1b = \bigcup_{a \in A, b \in B} aS^1b$$

where for all $a \in A$ and $b \in B$, $aS^1b$ is a group isomorphic to $G$. Furthermore, we may choose $A$ and $B$ such that $e \in A \cap B$; hence, $ea = e$ and $ae = a$ all $a \in A$, $be = e$ and $eb = b$ all $b \in B$, and $ba \in eS^1e = G$ all $a \in A$ and all $b \in B$. Thus, every element of $K(S)$ has a unique expression of the form $agb$ for $a \in A$, $g \in G$, $b \in B$.

We now get functions $\alpha : S \times A \rightarrow A$, $\eta : S \times A \rightarrow G$, $\beta : B \times S \rightarrow B,$
and \( \gamma : B \times S \to G \) given by \( sa = \alpha(s, a) \eta(s, a) \) and \( bs = \gamma(b, s) \beta(b, s) \). Notice that the associativity of \( S \) gives us \( \alpha(ss', a) = \alpha(s, \alpha(s', a)) \) and \( \eta(ss', a) = \eta(s, \alpha(s', a)) \eta(s', a) \). We now can define an action of \( S \) on \( A \) by \( s \cdot a = \alpha(s, a) \). Throughout this paper "\( \cdot \)" will mean this action. Since \( s(aS^1) = saS^1 = (s \cdot a) S^1 \), we may think of this action as the natural action of \( S \) on the set of minimal right ideals.

For any set \( X \) we can define a semigroup \( X^r \) (resp., \( X^l \)) by \( xy = y \) (resp., \( xy = x \)). We see that \( A \) above is of the form \( A^r \) and \( B \) is of the form \( B^r \). If in \( K(S) \) we have \( G = \{ e \} \), then we may write \( K(S) = A^l \times B^r \) with the identifications \( A \leftrightarrow A \times \{ e \} \) and \( B \leftrightarrow \{ e \} \times B \).

Observe that as a left \( S \)-module \( Z(K(S)) = \bigoplus_{b \in B} Z(S^1) b \) and that since \( b^2 = b \), each summand, and hence \( Z(K(S)) \), is \( S \)-projective.

We now need the following two lemmas, the proofs of which are exercises in diagram chasing.

**Lemma 1.1.** Let \( R \) be a ring and let

\[
0 \to A \xrightarrow{\alpha} B \to C \to 0
\]

be a short exact sequence of \( \mathbb{R} \)-modules and let \( D \) be an \( \mathbb{R} \)-module. Let \( X \to A \) and \( Y \to B \) be projective resolutions of \( A \) and \( B \), resp., and let \( f : X \to Y \) be a chain transformation lifting \( \alpha \). Then if \( W \) is the mapping cone \([7]\) of \( f \) where \( W_0 = Y_0 \), \( W_n = Y_n \oplus X_{n-1} \) for \( n \geq 1 \), \( \partial_1(y_1, x_0) = -\delta y_1 + f x_0 \) and \( \partial_n(y_n, x_{n-1}) = (\partial y_n + f x_{n-1}, \partial x_{n-1}) \) for \( n \geq 2 \), then \( W \to C \) is a projective resolution of \( C \). Then the exact sequence \( 0 \to Y \to W \to X \to 0 \) where \( X_n = X_{n-1} \), \( X_0 = 0 \) gives rise to the usual long exact sequence for \( \text{Ext} \).

Also if \( f_n = 0 \) for \( n \geq n_0 \), then since \( \text{Ext}^n_R(B, D) \to \text{Ext}^n_R(A, D) \) is zero, the long exact sequence breaks up into the long exact sequence

\[
0 \to \text{Ext}^0_R(C, D) \to \cdots \to \text{Ext}^{n_0}_R(C, D) \to \text{Ext}^{n_0}_R(B, D) \to 0
\]

and, for \( n \geq n_0 + 1 \), the short exact sequences

\[
0 \to \text{Ext}^{n-1}_R(A, D) \to \text{Ext}^n_R(C, D) \to \text{Ext}^n_R(B, D) \to 0.
\]

Since \( f_n = 0 \) for \( n \geq n_0 \) also implies that the sequence of complexes splits in dimensions beyond \( n_0 \), these short exact sequences are split.

**Lemma 1.2.** Let \( R \) be a ring, \( D \) an \( \mathbb{R} \)-module, and \( 0 \to A \to B \to C \to 0 \) a short exact sequence of \( \mathbb{R} \)-modules. If we have a short exact sequence of complexes

\[
0 \to X \to Y \to V \to 0 \quad \text{over} \quad 0 \to A \to B \to C \to 0 \quad \text{such that} \quad X \quad \text{and} \quad Y
\]

projective complexes and $V$ is acyclic, then $W \to C$, defined as in Lemma 1.1 is a projective resolution of $C$ and we have a long exact sequence

$$0 \to \text{Ext}^0_R(C, D) \to H^0(\text{Hom}_R(Y, D)) \to H^0(\text{Hom}_R(X, D)) \to \cdots$$

$$\cdots \to \text{Ext}^n_R(C, D) \to H^n(\text{Hom}_R(Y, D)) \to H^n(\text{Hom}_R(X, D))$$

$$\to \text{Ext}^{n+1}_R(C, D) \to \cdots$$

arising from the short exact sequence of complexes $0 \to Y \to W \to X \to 0$, as above.

2. The General Theorem

Lemma 2.1. Let $S$ be a semigroup and $N$ a submonoid of $S$ with $e \in N$ the identity of $N$. Furthermore, suppose that $Z(S^1) e$ is a projective right $N$-module, then for any $S$-module $D$,

$$\text{Ext}^n_S(Z(S^1) e \otimes_N Z, D) \approx \text{Ext}^n_N(Z, eD) + \text{Hom}_S(M_N, Z_1).$$

Proof. If $X \to Z$ is an $N$-projective resolution of the trivial module $Z$, then since $Z(S^1) e$ is $S$-projective, $Z(S^1) e \otimes_N X$ is a projective complex over $Z(S^1) e \otimes_N Z$. Since $Z(S^1) e$ is $N$-projective this is in fact a projective resolution. The proof now follows from adjoint associativity. (cf. [3], VI)

Corollary ([1]). If $N$ is a monoid left ideal, then

$$H^n(S, D) \approx H^n(N, eD).$$

Proof. $Z(S^1) e \simeq Z(N)$; hence $Z(S^1) e \otimes_N Z \simeq Z$, and the result follows from the lemma.

Lemma 2.2. Let $S$ and $N$ be as in Lemma 2.1. Let $M_N$ be the kernel of the $S$-module homomorphism $Z(S^1) e \otimes_N Z \to Z$ given by $s \otimes k \mapsto sk$. Hence, we have a short exact sequence of $S$-modules

$$0 \to M_N \to Z(S^1) e \otimes_N Z \to Z \to 0.$$ 

For any $S$-module $D$ we get a long exact sequence

$$0 \to H^0(S, D) \to H^0(N, eD) \to \text{Hom}_S(M_N, D) \to \cdots \to H^n(S, D)$$

$$\to H^n(N, eD) \to \text{Ext}^n_S(M_N, D) \to H^{n+1}(S, D) \to \cdots.$$
Proof. This is just the usual long exact sequence with $\text{Ext}_S^n(Z(S^1)e \otimes_K Z, D)$ replaced by $H^n(N, eD)$ via Lemma 2.1.

Now suppose we have a semigroup $S$ with completely simple kernel $K(S)$ with $G, e, A, B$ as above. Now by choosing $N = G$ in the preceding lemmas and writing $M = M_G$ we get

**Theorem 2.3.** For a semigroup $S$ with completely simple kernel $K(S)$, let $G$ be a maximal subgroup of $K(S)$ with $e \in G$ the identity of $G$ as above. If $M$ is as above and if $D$ is any $S$-module, then we have a long exact sequence

$$0 \rightarrow H^0(S, D) \rightarrow H^0(G, eD) \rightarrow \text{Hom}_S(M, D) \rightarrow \cdots \rightarrow H^n(S, D) \rightarrow H^n(G, eD) \rightarrow \text{Ext}_S^n(M, D) \rightarrow H^{n+1}(S, D) \rightarrow \cdots.$$

**Proof.** As a right $G$-module we have

$$Z(S^1)e = \bigoplus_{a \in A} Z(aS^1e) \approx \bigoplus_{a \in A} Z(G),$$

which is projective. The theorem now follows from Lemmas 2.1 and 2.2.

We now observe that in accordance with our preliminary remarks every element of $S^1e$ is uniquely of the form $ag$ for $a \in A, g \in G$. Hence we see that $Z(S^1)e \otimes_G Z$ is naturally isomorphic as an $S$-module to $Z(A)$, where the $S$-module structure on $Z(A)$ comes from the action of $S$ on $A$ defined above by $(s, a) \mapsto s \cdot a$.

With this observation we see now that $M$ is isomorphic to

$$Z(\{a - e \mid a \in A - \{e\}\}) \subset Z(A).$$

Whenever convenient we shall express $M$ in this form.

**Corollary.** If $S$ is as in Theorem 2.3 and if $K(S)$ is right simple, then for any $S$-module $D$, $H^n(S, D) \approx H^n(G, eD)$.

**Proof.** $K(S)$ right simple implies $|A| = 1$, and hence $M = \{0\}$.

**Corollary.** If $S$ has a right zero $z$ (i.e., $xz = z$, all $x \in S$), then for any $S$-module $D$, $H^n(S, D) = 0$ for $n > 0$.

**Proof.** $S$ having a right zero implies that $S$ has a completely simple kernel $K(S)$ in which $|A| = 1$ and $G = \{e\}$. (See [4]).
Remark. If \( S \) has a left zero \( z \) (i.e., \( zx = z \), all \( x \in S \)), then \( K(S) \) is completely simple with \( |B| = 1 \) and \( G = \{e\} \). It can be shown that for a trivial \( S \)-module \( D \) we have \( H^n(S, D) = 0 \) for \( n > 0 \). (cf. [2], [8]) We shall see below that such semigroups can have nonzero cohomology groups.

3. Completely Simple and Constant Action Semigroups

Let \( S \) be a semigroup with completely simple kernel \( K(S) \). Let \( G, e, A, B \) be as above. We will say that \( S \) has constant action on \( A \) if for all \( s \in S \) we have \( s \cdot a = s \cdot a' \) for all \( a, a' \in A \). In terms of the function \( \alpha : S \times A \to A \), this says that \( \alpha(s, a) \) depends only on \( s \in S \) and hence we may write \( \alpha(s, a) = \alpha(s) \) for all \( s \in S \), all \( a \in A \).

Clearly if \( S \) is itself completely simple, then it has constant action on \( A \).

Now let \( A = A - \{e\}, B = B - \{e\} \), and let \( \{[a] : a \in A\} \) be a set of symbols to be used below as generators of a free \( S \)-module; by convention \([e] = 0\), if it appears in a computation.

Notice also that if \( S \) has constant action on \( A \) and if \( |A| > 2 \), then \( S \) cannot have an identity. Hence, \( Z(S) \neq Z(S^1) \). Since the case \( |A| = 1 \) has already been settled, we can assume \( |A| \geq 2 \) in what follows.

**Theorem 3.1.** Let \( S \) be a semigroup with completely simple kernel \( K(S) \) and with constant action on \( A \). Let \( D \) be any \( S \)-module. Then for \( M \) as in Theorem 2.3 and \( n \geq 3 \) we have

\[
\operatorname{Ext}^n_S(M, D) \cong \operatorname{Ext}^{n-1}_{G,eD}(\bigoplus_{a \in A} Z(S)^1[a], D) \cong \prod_{a \in A} \operatorname{Ext}^{n-1}_S(Z(S), D)[a]
\]

Further, notice that the long exact sequence of Theorem 2.3 now breaks up into the exact sequence \( 0 \to H^n(S, D) \to H^n(G, eD) \to \operatorname{Hom}_S(M, D) \to H^{n+1}(S, D) \to H^{n+1}(G, eD) \to \operatorname{Ext}^n_S(M, D) \to H^n(S, D) \to H^n(G, eD) \to 0 \) and the split short exact sequences \( 0 \to \prod_{a \in A} \operatorname{Ext}^{n-2}_S(Z(S), D)[a] \to H^n(S, D) \to H^n(G, eD) \to 0 \) for \( n \geq 3 \).

**Proof.** The first claim follows from considering the short exact sequence

\[
0 \to \bigoplus_{a \in A} Z(S)^1[a] \to \bigoplus_{a \in A} Z(S^1)[a] \to M \to 0
\]

where \([a] \mapsto a - e\). This is exact since \( S \) has constant action on \( A \). Since the middle term is projective, we have the first part of the theorem.

For the second part of the theorem let \( V \mapsto M \) be an \( S \)-projective resolution of \( M \); we may assume that \( V_0 = \bigoplus_{a \in A} Z(S^1)[a] \). Let \( X(G) \mapsto Z \) be the standard \( G \)-projective resolution of \( Z \). (cf. [7].) Then by the proof of
Lemma 2.1 \( Z(S^1) e \otimes_G X(G) \rightarrow Z(S^1) e \otimes_G Z \) is a projective resolution of \( Z(S^1) e \otimes_G Z \). Notice that in the short exact sequence
\[
0 \rightarrow M \rightarrow Z(S^1) e \otimes_G Z \rightarrow Z \rightarrow 0
\]
the map on the left is given by \( a - e \mapsto (a - e) \otimes 1 \). We now define a chain transformation \( f : V \rightarrow Z(S^1) e \otimes_G X_0(G) \) lifting this map and such that \( f_0 = 0 \) for \( n > 2 \).

We define \( f_0 : (s[a]) \mapsto s((a - e) \otimes [ ]) = (sa - se) \otimes [ ] \)
and we define \( f_1 : V_1 \rightarrow Z(S^1) e \otimes_G X_1(G) \) by
\[
f_1(v_1) := \sum_{s \in A, s' \in S} m_{sa}se \otimes [\eta(s, e)^{-1}\eta(s, a)]
\]
where
\[
\hat{\epsilon}v_1 := \sum_{s \in A, s' \in S} m_{sa}[a] \quad \text{with} \quad m_{sa} \in \mathbb{Z}.
\]

(Sequence 3.1.1 guarantees that \( \hat{\epsilon}v_1 \) is of this form.) We see that \( f_1 \) is an \( S \)-homomorphism provided that for \( s, s' \in S \) and \( a \in A \) we have
\[
\eta(s' e^{-1})\eta(s e^{-1}) = \eta(s', a) = \eta(s, a).
\]
But since \( \eta(s', a) = \eta(s, \alpha(s', a)) \eta(s', a) \) and since \( \alpha(s', a) = o(s', e) \) by hypothesis, this follows easily.

The verification that \( f \) is a chain transformation is now direct. For example,
\[
\hat{\epsilon}f_1(v_1) := \sum_{s \in A, s' \in S} m_{sa}se \otimes (\eta(s, e)^{-1}\eta(s, a) - e) \otimes [ ]
\]
\[
= \sum_{s \in A, s' \in S} m_{sa}(se\eta(s, e)^{-1}\eta(s, a) - se) \otimes [ ]
\]
\[
= \sum_{s \in A, s' \in S} m_{sa}(sa - se) \otimes [ ] = f_0\hat{\epsilon}(v_1).
\]

By Lemma 1.1, we now see that the sequence of Theorem 2.3 breaks into the exact sequence
\[
0 \rightarrow H^0(S, D) \rightarrow \cdots \rightarrow H^n(S, D) \rightarrow H^n(G, eD) \rightarrow 0
\]
and the split short exact sequences
\[
0 \rightarrow \text{Ext}^{n-1}_S(M, D) \rightarrow H^n(S, D) \rightarrow H^n(G, eD) \rightarrow 0 \quad \text{for} \quad n \geq 3.
\]
Replacing \( \text{Ext}^{n-1}_S(M, D) \) via the isomorphism in the first part of the theorem completes the proof.
Corollary 3.2. If $S$ is as in the theorem and if we have $ba = e$ for all $a \in A$, all $b \in B$, then the first sequence of the theorem breaks further into the exact sequence $0 \rightarrow H^0(S, D) \rightarrow H^0(G, eD) \rightarrow \text{Hom}_S(M, D) \rightarrow H^1(S, D) \rightarrow H^1(G, eD) \rightarrow 0$ and the split exact sequence $0 \rightarrow \text{Ext}^1_S(M, D) \rightarrow H^2(S, D) \rightarrow H^2(G, eD) \rightarrow 0$.

Proof. Notice that for $s \in S$ and $a \in A$ we have $\gamma(s, a) = esa$ and $\gamma(s, e) = ese$. But since $es \in K(S)$ we know $es = esa$ for some $b \in B$. Hence $\gamma(s, a) = esa = esa = ese = \gamma(s, e)$. Thus $\gamma(s, e) \gamma(s, a) = e$. This says that in the proof of the theorem we have $f_1 = 0$. Now Lemma 1.1 gives the corollary just as it did the theorem.

Corollary 3.3. If $S = K(S)$ is a completely simple semigroup, then for any $S$-module $D$ and $n \geq 3$ we have $H^n(S, D) \cong H^n(G, eD)$.

Proof. Since $S = K(S)$, then $Z(S) = Z(K(S))$, and the latter is projective, by our preliminary remarks.

Corollary 3.4. If $S$ is a finite regular semigroup having constant action on $A$, let $n(S)$ be the length of the longest proper chain of ideals in $S$. Then $H^n(S, D) \cong H^n(G, eD)$ for $n \geq n(S) + 2$.

Proof. By a result in [8] $\text{Ext}^n(Z(S), D) = 0$ for $n \geq n(S)$.

Let us remark at this point that if $S$ is a semigroup without identity then it may easily happen that $Z(S)$ is not a projective $S$-module. For example let $S = \{a_1, a_2\} \cup \{b_1, b_2\}$ where $a_i a_j = b_j a_i = b_j$. Then the $S$-module homomorphism $Z(S') a_1 \oplus Z(S') a_2 \rightarrow Z(S)$ given by $(x, y) \mapsto x + y$ does not split.

Corollary 3.5. If $S$ is completely simple with $G = \{e\}$, that is, $S = A^* \times B^*$, and if $D$ is any $S$-module, then $H^n(S, D) = 0$ for $n \geq 3$.

$$H^1(S, D) = \left( \prod_{a \in A} D^0 \right) / \{( (a, e) - (e, e) d)_{a \in A} : d \in D \},$$

and

$$H^2(S, D) = \left( \prod_{a \in A, b \in B} D' \right) / \{( ((e, b) - (e, e) d)_{a \in A, b \in B} : d \in D \}$$

where

$$D^0 = \{ d \in D : (e, b) d = 0 \text{ all } b \in B \}$$

and $D' = (e, e) D$ (thus $D' = (e, b) D$, all $b \in B$).
Also, if \( D = D' \), in particular, if \( D \) is a trivial \( S \)-module, then

\[
H^2(S, D) = \prod_{a \in A} \prod_{b \in B} D.
\]

**Proof.** By applying Corollary 3.2 and by noticing that 3.1.1 gives a projective resolution of \( M \) and that \( \text{Hom}_S(Z(S^1)(e, e), D) \approx D' \), we can establish the corollary.

4. **Semigroups in Which the Kernel Breaks off**

If \( S \) is a semigroup with completely simple kernel \( K(S) \), we say that the kernel breaks from \( S \) if \( T = S - K(S) \) is a subsemigroup of \( S \). To simplify notation in what follows let us write \( P = Z(K(S)) \). Then \( P \) is always a projective (left or right) \( S \)-module.

**Theorem 4.1.** If \( S \) is a semigroup with completely simple kernel \( K(S) \) which breaks from \( S \), if \( T = S - K(S) \), if \( D \) is any left \( S \)-module, and if \( M \) is as in Theorem 2.3, then we have a long exact sequence

\[
0 \to \text{Hom}_S(M, D) \to \text{Hom}_T(M, D) \to \text{Hom}_T(M, \text{Hom}_S(P, D))
\]

\[
\to \text{Ext}^1_T(M, D) \to \cdots \to \text{Ext}^n_T(M, D) \to \text{Ext}^n_T(M, \text{Hom}_S(P, D)) \to \cdots
\]

where \( S \)-modules are considered as \( T \)-modules via the natural homomorphism \( Z(T^1) \to Z(S^1) \).

**Proof.** We have a natural short exact sequence of abelian groups

\[
0 \to P \to Z(S^1) \to Z(T^1) \to 0.
\]

This allows us to define a natural left \( S \)-module structure on \( Z(T^1) \) so that we may consider this sequence as a short exact sequence of left-right \( S \)-\( T \)-bimodules.

Let \( X \to M \) be a \( T \)-projective resolution of \( M \). Then since \( Z(T^1) \) is right \( T \)-projective we find that

\[
0 \to P \otimes_T X \to Z(S^1) \otimes_T X \to Z(T^1) \otimes_T X \to 0
\]

is a short exact sequence of complexes over the short exact sequence of \( S \)-modules

\[
0 \to P \otimes_T M \to Z(S^1) \otimes_T M \to Z(T^1) \otimes_T M \to 0.
\]

Since \( P \) and \( Z(S^1) \) are \( S \)-projective, \( P \otimes_T X \) and \( Z(S^1) \otimes_T X \) are projective complexes, while \( Z(T^1) \otimes_T X \approx X \) is acyclic over \( Z(T^1) \otimes_T M \). Observing that for all \( s \in K(S) \) we have \( s \cdot (a - a') = 0 \) for \( a, a' \in A \), so that \( s \cdot m = 0 \) for all \( m \in M \), we see that we get a natural \( S \)-module isomorphism \( Z(T^1) \otimes_T M \approx M \).
Now we may apply Lemma 1.2 and get a long exact sequence
\[ 0 \to \text{Ext}^0_S(M, D) \to H^0 \left( \text{Hom}_S \left( Z(S^1) \otimes_T X, D \right) \right) \to H^0 \left( \text{Hom}_S \left( P \otimes_T X, D \right) \right) \to \text{Ext}^1_S(M, D) \to \cdots. \]

By adjoint associativity we have
\[ H^n \left( \text{Hom}_S \left( Z(S^1) \otimes_T X, D \right) \right) \cong H^n \left( \text{Hom}_S(X, \text{Hom}_S(Z(S^1), D)) \right) \]
\[ \cong H^n \left( \text{Hom}_S(X, D) \right) \cong \text{Ext}^n_S(M, D). \]
Likewise \( H^n(\text{Hom}_S(P \otimes_T X, D)) \cong \text{Ext}^n_S(M, \text{Hom}_S(P, D)). \) Thus we get the long exact sequence of the theorem.

**Theorem 4.2.** Suppose that \( S \) is as in Theorem 4.1 and that \( D \) is an \( S \)-module. Suppose further that for all \( t \in T = S - K(S) \) we have \( \eta(t, a) = \eta(b, t) = e \) for all \( a \in A \) and all \( b \in B \). Then the long exact sequence of Theorem 2.3 breaks up into the exact sequence
\[ 0 \to H^0(S, D) \to H^0(G, eD) \to \cdots \to \text{Ext}^3_S(M, D) \to H^2(S, D) \to H^2(G, eD) \to 0 \]
and the split short exact sequences.
\[ 0 \to \text{Ext}^{n-1}_S(M, D) \to H^n(S, D) \to H^n(G, eD) \to 0 \quad \text{for} \quad n \geq 3. \]

**Proof.** Notice that the new conditions may be stated as \( ta \in A \) for all \( t \in T \), all \( a \in A \), and \( bt \in B \) for all \( t \in T \), all \( b \in B \). In particular, \( t \cdot a = ta \) for \( t \in T \) and \( a \in A \). We notice that for \( t \in T \), \( a \in A \), and \( s \in S \),
\[ \eta(st, a) = \eta(s, t \cdot a) \quad \eta(t, a) = \eta(s, ta) \quad e = \eta(s, e). \]
Further notice that if \( s \in K(S) \) and \( t \in T \), then \( s = sb \) for some \( b \in B \) and \( bt \in B \) by hypothesis; hence \( ste = sbte = se \), which gives
\[ \eta(s, te) = este = ese = \eta(s, e). \]

Let us consider the \( S \)-projective resolution of \( M \) that can be constructed by means of Lemmas 1.1 and 1.2 as implied in the proof of Theorem 4.1. Then as in Theorem 3.1 we define a chain transformation \( f \) from this complex to \( Z(S^1) \otimes_G X(G) \) lifting the map \( M \to Z(S^1) \otimes_G Z \) and such that \( f_n = 0 \) for \( n \geq 2 \). Let \( f_0 : Z(S^1) \otimes_T X_0 \to Z(S^1) \otimes_G X_0(G) \) be defined by
\[ f_0(s \otimes x_0) = sf_0(1 \otimes x_0) = s \left( \sum_{a \in A} m_a(a - e) \otimes [ \ ] \right) \]
\[ = \sum_{a \in A} m_a(sa - se) \otimes [ \ ] \]
where $e(1 \otimes x_0) = \sum_{a \in \mathcal{A}} m_a(a - e) \in M$, $m_a \in Z$. This is a well-defined $S$-module homomorphism.

We define $f_1 : (Z(S^1) \otimes_T X_1) \oplus (P \otimes_T X_0) \rightarrow Z(S^1) \otimes_G X_1(G)$ by $f_1(s \otimes x_1, p \otimes x_0) = \sum_{a \in \mathcal{A}} m_a \otimes [\eta(p, e)^{-1} \eta(p, a)]$ where $p \in K(S)$ and where $e(1 \otimes x_0) = \sum_{a \in \mathcal{A}} m_a(a - e) \in M$. We see that $f_1$ is an $S$-module homomorphism if it is well-defined; hence we must check that $f_1(0, p \otimes x_0) = f_1(0, pt \otimes x_0)$. Since

$$e(1 \otimes tx_0) = t e(1 \otimes x_0) = \sum_{a \in \mathcal{A}} m_a(ta - te)$$

$$= \sum_{a \in \mathcal{A}} m_a(ta - e) - \sum_{a \in \mathcal{A}} m_a(te - e),$$

we have

$$f_1(0, p \otimes x_0) = \sum_{a \in \mathcal{A}} m_a \otimes [\eta(p, e)^{-1} \eta(p, ta)]$$

$$- \sum_{a \in \mathcal{A}} m_a \otimes [\eta(p, e)^{-1} \eta(p, te)].$$

But $f_1(0, pt \otimes x_0) = \sum_{a \in \mathcal{A}} m_a \otimes [\eta(pt, e)^{-1} \eta(pt, a)]$. Since by the early part of this proof $pte = pe$, $\eta(p, te) = \eta(p, e)$, and $\eta(pt, a) = \eta(p, ta)$, we have $f_1$ well-defined.

To see that $f$ is the desired chain transformation is now an easy direct computation. For example, we see that

$$f_0 \partial_1(s \otimes x_1, p \otimes x_0) = f_0(-s \otimes \partial x_1 + p \otimes x_0)$$

$$= p \left( \sum_{a \in \mathcal{A}} m_a(a - e) \otimes [ \ ] \right) = \sum_{a \in \mathcal{A}} m_a(p(a - e)) \otimes [ \ ]$$

where

$$e(1 \otimes x_0) = \sum_{a \in \mathcal{A}} m_a(a - e)$$

and

$$\partial f_1(s \otimes x_1, p \otimes x_0) = \partial \left( \sum_{a \in \mathcal{A}} m_a \otimes [\eta(p, e)^{-1} \eta(p, a)] \right)$$

$$= \sum_{a \in \mathcal{A}} m_a \otimes ((\eta(p, e)^{-1} \eta(p, a) - e)[ \ ]$$

$$- \sum_{a \in \mathcal{A}} m_a (pe \eta(p, e)^{-1} \eta(p, a) - pe) \otimes [ \ ]$$

$$= \sum_{a \in \mathcal{A}} m_a (pa - pe) \otimes [ \ ].$$

Thus $f_0 \partial_1 = \partial f_1$.

Now the theorem follows from Lemma 1.1.
COROLLARY 4.3. Suppose $S$ is as in Theorem 4.1 and that $D$ is an $S$-module. Suppose further that for all $t \in T = S - K(S)$ we have $ta = a$ for all $a \in A$. Then Theorem 4.2 applies to $S$. Moreover, in Theorem 4.1 we may replace the groups $\text{Ext}^n(M, -)$ by $\prod_{a \in A} H^n(T, -)[a]$.

Proof. Since $ta = a$ for all $t \in T$ and all $a \in A$, we have $\gamma(t, a) = e$. Now $\gamma(b, t) = bte = be = e$ for all $t \in T$ and all $b \in B$. Hence Theorem 4.2 holds.

Moreover, the hypothesis of the corollary implies that as a $T$-module $M = \bigoplus_{a \in A} Z(a - e) \approx \bigoplus_{a \in A} Z[q]$, where $Z$ is the trivial $T$-module. The last statement of the corollary follows easily now.

COROLLARY 4.4. Let $S$ be as in Theorem 4.2. If in addition to the hypotheses of Theorem 4.2 we have $ba = e$ for all $a \in A$ and all $b \in B$, then the initial exact sequence of the theorem breaks up further into the exact sequence $0 \rightarrow H^0(S, D) \rightarrow \cdots \rightarrow H^1(S, D) \rightarrow H^1(G, eD) \rightarrow 0$ and the split short exact sequence $0 \rightarrow \text{Ext}^1(M, D) \rightarrow H^2(S, D) \rightarrow H^2(G, eD) \rightarrow 0$.

Proof. As in Corollary 3.2 we can show that $\eta(p, e)^{-1} \eta(p, a) = e$ for all $p \in K(S)$ and $a \in A$. Hence $f_1 = 0$ in the proof of Theorem 4.2, and Lemma 1.1 completes the proof.

COROLLARY 4.5. Let $S$ be as in Corollary 4.3. Suppose $T$ is such that $H^n(T, D') = 0$ for all $T$-modules $D'$ for $n \geq p$. Then for $n \geq p + 2$, $H^n(S, D) \approx H^n(G, eD)$ for all $S$-modules $D$.

Proof. Clear.

Corollaries 4.3 and 4.5 give us a sort of induction method for certain semigroups. Let us apply them now to two examples.

Example 4.6. Suppose $S$ is a finite band (i.e., $s^2 = s$, all $s \in S$) such that i) the usual ordering of $\mathscr{J}$-classes ([12], p. 150) is linear and given by $J_k \succ J_{k-1} \succ \cdots \succ J_1 = K(S)$ and ii) such that for $j \in J_q$ and $j' \in J_q$ with $1 \leq q \leq p \leq k$, we have $jj' = j'$. Then for any $S$-module $D$, $H^n(S, D) = 0$ for $n \geq 2k + 1$.

Proof. Since $S$ is a band each $\mathscr{J}$-class is of the form $A^j \times B^j$. By condition i) if $S_q = J_k \cup \cdots \cup J_{k-q}$ for $0 \leq q \leq k - 1$, then each $S_q$ is a subsemigroup of $S$. Now $K(S_q) = J_{k-q}$, so that ii) allows us to apply Corollary 4.3. By Corollary 3.5 we know that for any $S_q$-module $D$, $H^n(S_q, D) = 0$ for $n \geq 3$. Now if we assume $H^n(S_q, D') = 0$ for all $S_q$-modules $D'$ for $n \geq 3 + 2q$, then Corollary 4.5 shows that $H^n(S_{q+1}, D) = 0$ for all $S_{q+1}$-modules $D$ for $n \geq 3 + 2(q + 1)$. Hence by induction, since $S = S_{k-1}$, we have the result.
Example 4.7. For every \( k \geq 0 \) there is a semigroup \( S \) like that of Example 4.6 and an \( S \)-module \( D \) such that \( H^n(S, D) \neq 0 \). In particular, for every \( k \geq 0 \) there is some finite, combinatorial semigroup \( S \) and left \( S \)-module \( D \) such that \( H^n(S, D) \neq 0 \).

**Proof.** Suppose in Example 4.6 that each \( f_p = A_p \) for some finite set \( A_p \) for \( 1 \leq p \leq k \). Then letting \( T = f_1 \cup \cdots \cup f_k \) and letting \( A_1 = A \), we find that \( at \cdot a \) for all \( a \in A \) and all \( t \in T \), by associativity. Now as an abelian group, \( \text{Hom}_S(Z(A^t), D) \approx eD \) where \( e \in A^t \) is some chosen element, and these are trivial \( T \)-modules. Hence since \( T \) has left zeros, we have \( H^n(T, eD) = 0 \) for \( n > 0 \) by the remark at the end of Section 2.

Thus, noticing that here \( P : Z(A^t) \) we apply Corollaries 4.3 and 4.4 to find that \( H^n(S, D) \approx \text{Ext}^{n-1}_S(M, D) \) for \( n \geq 2 \) and that we have

\[
\text{Ext}^1_S(M, D) \rightarrow \prod_{a \in A} H^1(T, D) \rightarrow 0
\]

is exact and \( \text{Ext}^n_S(M, D) \approx \prod_{a \in A} H^n(T, D) \) for \( n \geq 2 \).

For \( k = 1 \), we let \( S = A^t \) and \( D = Z(A^t) \) and see by an application of Corollary 3.5 that \( H^1(S, D) \neq 0 \).

For \( k > 1 \) we go by induction. Suppose there is a \( T \)-module \( D \) such that \( H^{k-1}(T, D) \neq 0 \). Then \( D \) becomes an \( S \)-module by setting \( ad \cdot a \) for all \( d \in D \), all \( a \in A^t := A_1^t \cdot K(S) \). Thus since \( H^{k-1}(T, D) \neq 0 \), we have \( 0 \neq \text{Ext}^{k-1}_S(M, D) \approx H^k(S, D) \), and we are done.

**Example 4.8.** Let \( H \) be a finite group and \( K \) a subgroup of \( H \). For \( h \in H \) let \( h : hK \). Letting \( H/K \) be a copy of the set of cosets we form the semigroup \( S = H \cup (H/K) \), disjoint union, with \( h_1 h_2 \ldots h_{n-2} h_2 h_1 \ldots h_2 \) for \( h_1 \in H \) and \( h_2 \in (H/K)^t \). By Theorem 2.3 for \( n \geq 2 \), \( H^n(S, D) = \text{Ext}^{n-1}_S(M, D) \) for any \( S \)-module \( D \). We observe now that if \( 1 \in H \) is the identity element of \( H \), then here \( \text{Hom}_S(P, D) \approx \tilde{D} \) is a trivial \( \tilde{H} \)-module. Now by Theorem 4.1 (in which now \( T := H \)) we get

\[
\text{Ext}^0_H(M, \tilde{D}) \rightarrow H^0(S, D) \rightarrow \cdots
\]

\[
\text{Ext}^{n-1}_H(M, D) \rightarrow \text{Ext}^{n-1}_H(M, \tilde{D}) \rightarrow H^{n+1}(S, D) \rightarrow \text{Ext}^n_H(M, D) \rightarrow \cdots
\]

Suppose now we assume that \( K = \{1\} \), that is that \( S = H \cup \tilde{H} \). Then it is easy to see that for \( n \geq 2 \), \( \text{Ext}^{n-1}_H(M, D) \approx H^n(H, D) \). Hence in this case the sequence of the previous paragraph gives

\[
H^q(H, \tilde{D}) \rightarrow H^q(S, D) \rightarrow H^q(H, D) \rightarrow \cdots
\]

\[
H^{n-1}(H, \tilde{D}) \rightarrow H^n(S, D) \rightarrow H^n(H, D) \rightarrow H^n(H, \tilde{D}) \rightarrow H^{n+1}(S, D) \rightarrow \cdots
\]
Now if we further assume here that $H$ is a finite cyclic group and that the $S$-module $D$ is torsion free as an abelian group, we see that since $\overline{ID}$ is a trivial $H$-module $H^n(H, \overline{ID}) = 0$ for all even $n > 0$. Hence by choosing $D$ such that $H^n(H, D) \neq 0$ for all even $n > 0$ we have $H^n(S, D) \neq 0$ for all even $n > 0$.

This last example illustrates how subgroups of a semigroup besides that in the kernel can affect the cohomology rather drastically.

Using semigroups of the type in this example one can construct counterexamples to a conjecture of Rhodes, alluded to in [II], that semigroup homomorphisms which are one to one on subgroups will preserve cohomology [8]. In fact the induced maps may fail to be one to one or may fail to be onto. Whether such maps admit some reasonable cohomological characterization is an open question.

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References

