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K-moduli, moduli of smoothness, and Bernstein polynomials on a simplex

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1. INTRODUCTION

For a given function f on $I=[0,1]$, the well-known Bernstein polynomial of f is defined as

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{k,n}(x), \quad \text{where } p_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \; n\in\mathbb{N}_o.$$

These polynomials have been studied intensively, cf. [6]. They are the prototype for sequences of positive, linear, polynomial operators, and many aspects of the latter have their origin in the investigations of these polynomials. By now, the approximation behavior of the Bernstein polynomials is well understood, it is characterized by the weighted K-modulus, cf. [1], [5], [7],

$$K_\varphi^2(f;t)=\inf_{g\in C_o^2}\{\|f-g\|_\infty+t^2\|\varphi^2 g''\|_\infty\}, \quad \text{where} \quad \varphi(x)=\sqrt{x(1-x)}, \; t>0,$$

which is known to be equivalent to the modulus of smoothness of Ditzian and Totik [5]. More precisely,

THEOREM A. *Let $f\in C[0,1]$. Then there are constants such that*

$$\|B_nf-f\|_\infty\le \text{const } K_\varphi^2\left(f;\frac{1}{n+1}\right),$$

and, conversely,

$$\sum f\left(\frac{k}{n}\right) p_{k,n}(x)$$

$$K_\varphi^2\left(f; \frac{1}{n}\right) \le \frac{\text{const}}{n+1} \sum_{k=0}^{n} \|B_k f - f\|_\infty .$$

Let S be the simplex defined by

$$S = \{\mathbf{x} \in \mathbb{R}^d : x_i \ge 0, 0 \le i \le d, 1 - |\mathbf{x}| \ge 0\},$$

where, here and in the following, we shall use the standard notation: for $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $|\mathbf{x}| = \sum_{i=1}^{d} x_i$; we shall also write for $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{k} \in \mathbb{N}_o^d$, and $n \in \mathbb{N}_o$

$$\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \quad \mathbf{k}! = k_1!\, k_2! \cdots k_d!, \quad \text{and} \quad \binom{n}{\mathbf{k}} = \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!}.$$

The Bernstein polynomials for a function on the simplex S are defined as follows (cf. [6])

$$B_{n,d}(f; \mathbf{x}) = \sum_{|\mathbf{k}| \le n} f\left(\frac{\mathbf{k}}{n}\right) p_{\mathbf{k},n}(\mathbf{x}),$$

$$\text{where} \quad p_{\mathbf{k},n}(\mathbf{x}) = \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - |\mathbf{x}|)^{n - |\mathbf{k}|}, \ n \in \mathbb{N}_o.$$

Efforts to characterize the approximation behavior of $\{B_{n,d}\}_{n=0}^{\infty}$, $d \ge 2$, have been made by several authors, see [3], [4], [8], and [9]. The results obtained so far seem not to match the elegant theorems for the one dimensional case.

In this paper, we propose a new K-modulus which generalizes the one for one variable in a natural way, even for Lebesgue spaces. We show that this K-modulus is equivalent to a modulus of smoothness and use it to characterize the approximation behavior of the Bernstein polynomials on simplices; more precisely, we prove a strong direct theorem and an inverse theorem of weak type. Saturation is excluded. The K-modulus has also been used in [2] to study the Bernstein-Durrmeyer polynomials on S.

2. WEIGHTED K-MODULUS AND MODULUS OF SMOOTHNESS

Let $L^p(S)$, $1 \le p \le +\infty$, denote the space of (the equivalence class of) Lebesgue measurable functions f on S for which the norm $\|f\|_p^p = \int |f|^p$ is finite. For $\mathbf{x} \in S$, we denote

$$\varphi_i(\mathbf{x}) = \varphi_{ii}(\mathbf{x}) := \sqrt{x_i(1 - |\mathbf{x}|)}, \ 1 \le i \le d; \quad \varphi_{ij}(\mathbf{x}) := \sqrt{x_i x_j}, \ 1 \le i < j \le d,$$

and

$$D_i = D_{ii} := \frac{\partial}{\partial x_i}, \ 1 \le i \le d, \quad D_{ij} := D_i f - D_j f, \ 1 \le i < j \le d,$$

$$D_{ij}^r = D_{ij}(D_{ij}^{r-1}), \quad \text{and} \quad D^{\mathbf{k}} = D_1^{k_1} D_2^{k_2} \cdots D_d^{k_d}, \ \mathbf{k} \in \mathbb{N}_o^d.$$

With these notations, we define for $1 \le p < \infty$ the weighted Sobolev space

$$W_\Phi^{r,p}(S) = \{f \in L^p(S) : D^{\mathbf{k}} f, |\mathbf{k}| \le r, \mathbf{k} \in \mathbb{N}^d, \text{ are in } L_{\text{loc}}(\mathring{S}),$$
$$\text{and} \quad \varphi_{ij}^r D_{ij}^r \in L^p(S), 1 \le i \le j \le d\},$$

where the derivatives are in the distributional sense. For the space $C(S)$, we write

$$C_\Phi^r(S)=\{f\in C(S)\,|\,f\in C^r(\mathring{S}),\quad \text{and}\quad \varphi_{ij}^r D_{ij}^r f\in C(S), 1\le i\le j\le d\}.$$

The Peetre K-moduli on $L^p(S)$, $1\le p<\infty$, and $C(S)$, $p=\infty$, are defined by

$$K_\Phi^r(f;t^r)_p:=\inf\{\|f-g\|_p+t^r\sum_{1\le i\le j\le d}\|\varphi_{ij}^r D_{ij}^r g\|_p\},\quad t>0,$$

where the infimum is taken over all $g\in W_\Phi^{r,p}(S)$, $1\le p<\infty$, and $g\in C_\Phi^r(S)$, $p=\infty$, respectively.

Let $\mathbf{e}_i\in\mathbb{R}^d$ be the unit vector, $\mathbf{e}_i=(0,\ldots,0,\overset{i}{1},0,\ldots,0)$, and $\mathbf{e}_{ij}=\mathbf{e}_i-\mathbf{e}_j$. For any vector $\mathbf{e}$ in $\mathbb{R}^d$, we write for the r-th symmetric difference of a function f in the direction of $\mathbf{e}$

$$\Delta_{h\mathbf{e}}^r f(\mathbf{x})=\begin{cases}\sum_{i=0}^{r}\binom{r}{k}(-1)^k f\left(\mathbf{x}+\left(\frac{r}{2}-k\right)h\mathbf{e}\right), & \mathbf{x}+\frac{rh}{2}\mathbf{e}\in S,\\ 0, & \text{otherwise.}\end{cases}$$

We then define the modulus of smoothness of $f\in L^p(S)$, $1\le p<+\infty$, and $f\in C(S)$, $p=+\infty$, as

$$\omega_\Phi^r(f;t)_p=\sup_{0<h\le t}\sum_{1\le i<j\le d}\|\Delta_{h\varphi_{ij}\mathbf{e}_{ij}}^r f\|_p,\quad 1\le p\le\infty.$$

We have

THEOREM 1. *There exists a positive constant, dependent only on p and r, such that* $\forall f\in L^p(S)$, $1\le p<\infty$, *and* $\forall f\in C(S)$, $p=\infty$, *respectively,*

$$\frac{1}{\text{const}}\,\omega_\Phi^r(f;t)_p\le K_\Phi^r(f;t^r)_p\le\text{const}\;\omega_\Phi^r(f;t)_p,\quad 0<t<1.$$

REMARK. For $d=1$, our definitions and statements in Theorem 1 coincide with the familiar one dimensional ones, cf. [5, Chapter 2]. A multi-dimensional modulus of smoothness has been defined on polytopes by Ditzian and Totik in [5, Chapter 12]. Their definition, however, seems to be more complicated than ours when restricted to the simplex due to its generality.

PROOF OF THEOREM 1. We shall reduce the proof to the one in one dimension. For $\mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$, we write $\mathbf{x}_1=(x_2,\ldots,x_d)$ and

$$S_1=\{\mathbf{x}_1:\mathbf{x}=(x_1,\mathbf{x}_1)\in S\}.$$

Let $x_1=(1-|\mathbf{x}_1|)z$, $0\le z\le 1$, and $F(z)=F(z,\mathbf{x}_1)=f((1-|\mathbf{x}_1|)z,\mathbf{x}_1)$. Then

$$\varphi_1(\mathbf{x})=(1-|\mathbf{x}_1|)\varphi(z),\quad (D_1^r f)(\mathbf{x})=(1-|\mathbf{x}_1|)^{-r}F^{(r)}(z),$$

and

$$\Delta_{h\varphi_1\mathbf{e}_1}^r f(\mathbf{x})=\Delta_{h\varphi}^r F(z).$$

Consequently, for $1 \le p < +\infty$,

$$\|\Delta^r_{h\varphi_1 \mathbf{e}_1} f\|_p^p = \int_{S_1} d\mathbf{x}_1 \int_0^{1-|\mathbf{x}_1|} |\Delta^r_{h\varphi_1 \mathbf{e}_1} f|^p \, dx_1$$

$$= \int_{S_1} (1-|\mathbf{x}_1|) \int_0^1 |\Delta^r_{h\varphi} F(z)|^p \, dz \, d\mathbf{x}_1 .$$

From the proof of the relevant inequalities in one variable, see [5, Chapter 2], we obtain

$$\|\Delta^r_{h\varphi_1 \mathbf{e}_1} f\|_p^p \le \text{const} \int_{S_1} (1-|\mathbf{x}_1|) \int_0^1 |F(z)|^p \, dz \, d\mathbf{x}_1$$

$$= \text{const} \int_{S_1} \int_0^{1-|\mathbf{x}_1|} |f(x_1, \mathbf{x}_1)|^p \, dx_1 \, d\mathbf{x}_1 = \text{const} \, \|f\|_p^p,$$

and

$$\|\Delta^r_{h\varphi_1 \mathbf{e}_1} f\|_p^p \le \text{const} \int_{S_1} (1-|\mathbf{x}_1|) h^{rp} \int_0^1 |\varphi(z)^r F^{(r)}(z)|^p \, dz \, d\mathbf{x}_1$$

$$= \text{const} \, h^{rp} \int_{S_1} \int_0^{1-|\mathbf{x}_1|} |(\varphi_1^r D_1^r f)(x_1, \mathbf{x}_1)|^p \, dx_1 \, d\mathbf{x}_1$$

$$= \text{const} \, h^{rp} \|\varphi_1^r D_1^r f\|_p^p.$$

Similarly, by using the transformation $T_i : S \to S$, defined by $x_i \to 1 - |\mathbf{x}|$, and $x_j \to x_j$, $j \ne i$, and taking into account that $D_{ij} f = D_j(f \circ T_i)$, we obtain for $1 \le p < +\infty$ and $1 \le i \le j \le d$,

$$\|\Delta^r_{h\varphi_{ij} \mathbf{e}_{ij}} f\|_p \le \text{const} \begin{cases} \|f\|_p, & f \in L^p, \\ h^{rp} \|\varphi_{ij}^r D_{ij}^r f\|_p, & f \in W_\Phi^{p,r}. \end{cases}$$

The case $p = +\infty$ is easier. Adding up these inequalities, we have proved the first estimate.

To prove the second one, we shall again reduce it to the one dimensional case. First we note that for fixed $\mathbf{x}_1$, there exists a function $G_t \in W_\varphi^{r,p}(I)$, $t > 0$, such that

$$\|F - G_t\|_p^p, \, t^{rp} \|\varphi^r G_t^{(r)}\|_p^p \le \frac{\text{const}}{t} \int_0^t \|\Delta^r_{u\varphi} f\|_p^p \, du,$$

(cf. [5, Chapter 2]). Since the construction of G_t in [5] depends on F continuously, for $F(z) = F(z, \mathbf{x}_1)$ we have $G_t(z) = G_t(z, \mathbf{x}_1)$ as well. We now define $g_t \in W_\Phi^{r,p}(S)$ through

$$g_t(\mathbf{x}) = G_t\left(\frac{x_1}{1-|\mathbf{x}_1|}, \mathbf{x}_1\right), \quad \mathbf{x} \in S.$$

Then

$$\|f - g_t\|_p^p = \int_{S_1} (1-|\mathbf{x}_1|) \int_0^1 |F(z) - G_t(z)|^p \, dz \, d\mathbf{x}_1$$

$$\leq \text{const} \int_{S_1} (1-|\mathbf{x}_1|) \frac{1}{t} \int_0^t \int_0^1 |\Delta^r_{u\varphi} F(z)|^p \, dz \, du \, d\mathbf{x}_1$$

$$= \frac{\text{const}}{t} \int_0^t \int_{S_1} \int_0^{1-|\mathbf{x}_1|} |\Delta^r_{u\varphi_1 \mathbf{e}_1} f(x_1, \mathbf{x}_1)|^p \, dx_1 \, d\mathbf{x}_1 \, du$$

$$= \frac{\text{const}}{t} \int_0^t \|\Delta^r_{u\varphi_1 \mathbf{e}_1} f\|_p^p \, du,$$

and

$$t^{rp} \|\varphi_1^r D_1^r g_t\|_p^p = t^{rp} \int_{S_1} (1-|\mathbf{x}_1|) \int_0^1 |\varphi(z) G_t^{(r)}(z)|^p \, dz \, d\mathbf{x}_1$$

$$\leq \text{const} \int_{S_1} (1-|\mathbf{x}_1|) \frac{1}{t} \int_0^t \int_0^1 |\Delta^r_{u\varphi} F(z)|^p \, dz \, du \, d\mathbf{x}_1$$

$$\leq \frac{\text{const}}{t} \int_0^t \|\Delta^r_{u\varphi_1 \mathbf{e}_1} f\|_p^p \, du.$$

Similarly, we can prove that for each i, j, $1 \leq i \leq j \leq d$, there are functions $g_t \in W_\Phi^{r,p}(S), t > 0$, such that

$$\|f - g_t\|_p^p, \; t^{rp} \|\varphi_{ij}^r D_{ij}^r g_t\|_p^p \leq \frac{\text{const}}{t} \int_0^t \|\Delta^r_{u\varphi_{ij}\mathbf{e}_{ij}} f\|_p^p \, du.$$

Adding up these inequalities, and we have proved the second estimate. ■

3. THE APPROXIMATION BEHAVIOR OF $B_{n,d}(f)$

The following theorem gives a characterization of the approximation behavior of $\{B_{n,d}(f)\}_{n=0}^\infty$ for a function $f \in C(S)$ by means of a K-modulus and/or a modulus of smoothness.

THEOREM 2. *If $f \in C(S)$, then there is a positive constant such that*

$$\|f - B_{n,d} f\|_\infty \leq \text{const} \left[\omega_\Phi^2(f; 1/(n+1))_\infty + \frac{\|f\|_\infty}{n+1} \right],$$

and, conversely,

$$\omega_\Phi^2(f; 1/(n+1)) \leq \frac{\text{const}}{n+1} \sum_{k=0}^n \|B_{k,d}(f) - f\|_\infty.$$

COROLLARY. *For $f \in C(S)$, $0 < \alpha < 2$,*

$$\|f - B_{n,d} f\|_\infty = O(n^{-\alpha/2}) \text{ if and only if } \omega_\Phi^2(f; 1/n) = O(n^{-\alpha}).$$

Our proof is based on an induction argument. We shall prove the case $d=2$ in detail and outline the proof for an arbitrary dimension. This way, we hope, the idea of the proof will become evident, not obscured behind a heavy notation. Some ideas from [3] and [9] are used in the proof.

In the following, we shall write $\|\cdot\|$ instead of $\|\cdot\|_\infty$, and sometimes even $\|f\|_{u \in I} := \max_{u \in I} |f|$ for multivariable functions $f(u, x, y, \dots)$ to indicate that the maximum is taken w.r.t. u only. The "const" will always stand for a positive constant, independent of f and n, its value may be different at different occurrences.

PROOF OF THEOREM 2. First we prove the case $d=2$. For $\mathbf{x}\in\mathbb{R}^2$, we write $\mathbf{x}=(x,y)$,

$$\varphi_1(x,y)=\sqrt{x(1-x-y)},\quad \varphi_2(x,y)=\sqrt{y(1-x-y)},\quad \varphi_3(x,y)=\sqrt{xy},$$

and $D_3f=D_{12}f:=D_1f-D_2f$.

By standard arguments, the direct part of the theorem follows from the estimates

$$\text{(1)}\qquad \|B_{n,2}f-f\|\le\text{const}\begin{cases}\|f\|, & f\in C(S),\\ [\sum_{i=1}^{3}\|\varphi_i^2D_i^2f\|+\|f\|]/n, & f\in C_\varphi^2(S).\end{cases}$$

The first estimate for all f in $C(S)$ is evident as $B_{n,2}$ is a positive, linear contraction on S. We shall prove the second one by reducing it to the one dimensional inequality

$$\text{(2)}\qquad \|B_nf-f\|\le\text{const}\,\|\varphi^2f''\|/n.$$

We need the following formula of $B_{n,2}f$ which can be easily checked (cf. [9]),

$$\text{(3)}\qquad B_{n,2}(f;x,y)=\sum_{k=0}^{n}p_{kn}(x)\sum_{j=0}^{n-k}f\left(\frac{k}{n},\frac{j}{n}\right)p_{j,n-k}\left(\frac{y}{1-x}\right).$$

Now let

$$E_1=\{(x,y):x\ge\tfrac14\},\quad E_2=\{(x,y):y\ge\tfrac14\},\quad E_3=\{(x,y):1-x-y\ge\tfrac14\}.$$

Let ψ_i, $1\le i\le 3$, be a partition of unity on S satisfying the following conditions:

$$\psi_i\in C^\infty(S),\ \psi_i\ge 0,\ \sum_{i=1}^{3}\psi_i(x,y)=1\text{ on }S,\quad\text{and}\quad \operatorname{supp}\psi_i\subset E_i.$$

We shall prove

$$\text{(4)}\qquad \|B_{n,2}(f\psi_i)-f\psi_i\|\le\frac{\text{const}}{n}[\sum_{j=1}^{3}\|\varphi_j^2D_j^2f\|+\|f\|],\quad 1\le i\le 3,$$

which implies the second inequality of (1) as

$$B_{n,2}f=\sum_{i=1}^{3}B_{n,2}(f\psi_i),\quad f=\sum_{i=1}^{3}f\psi_i\text{ on }S.$$

First we consider the term $f\psi_3$. Since $\psi_3(x,y)=0$ on the complement of E_3, we can assume w.l.o.g. that

$$\text{(5)}\qquad f(x,y)=0\text{ on }E_3^*:=\{(x,y):x\ge 0,y\ge 0,x+y\ge\tfrac45\}.$$

From formula (3) it follows that

$$B_{n,2}(f\psi_3;x,y)-f\psi_3(x,y)$$
$$=\sum_{k=0}^{n} p_{kn}(x)\left[\sum_{j=0}^{n-k}(f\psi_3)\left(\frac{k}{n},\frac{j}{n}\right)p_{j,n-k}\left(\frac{y}{1-x}\right)-(f\psi_3)\left(\frac{k}{n},\left(1-\frac{k}{n}\right)\frac{y}{1-x}\right)\right]$$
$$+\left[\sum_{k=0}^{n} p_{kn}(x)(f\psi_3)\left(\frac{k}{n},\left(1-\frac{k}{n}\right)\frac{y}{1-x}\right)-f\psi_3(x,y)\right]=:J+L.$$

Let $g_k(u):=f(k/n,(1-k/n)u)$, $0\le u\le 1$, and $z:=y/(1-x)$. Then for $(x,y)\in S$ and fixed $x,z\in[0,1]$

$$J=\sum_{k=0}^{n} p_{kn}(x)[B_{n-k}(g_k\psi_3,z)-(g_k\psi_3)(z)].$$

Recall that $\psi_3=0$ on the complement of E_3, giving

$$(f\psi_3)\left(\frac{k}{n},\frac{j}{n}\right)=0 \text{ for } \frac{k}{n}\le\frac{3}{4}. \tag{6}$$

Thus the sum in J can be taken on $0\le k\le 3n/4$ only. From (2) it follows that

$$|B_{n-k}(g_k\psi_3;z)-g_k\psi_3(z)|\le\frac{\text{const}}{n-k}\|\varphi^2(g_k\psi_3)''\|_{z\in I}$$
$$\le\frac{\text{const}}{n}\|\varphi^2(g_k\psi_3)''\|_{z\in I},\quad k\le\frac{3n}{4}.$$

By (5) and (6), $g_k(z)=0$ for $4/5\le z\le 1$, in particular, $g_k(1)=0$ and $g_k'(1)=0$. Thus

$$\int_z^1\varphi^2(u)g_k''(u)\,du=-\varphi^2(z)g_k'(z)+(1-2z)g_k(z)-2\int_z^1 g_k(u)\,du,$$

from which it follows that

$$\|\varphi^2 g_k'\|_{z\in I}\le 3\|g_k\|_{z\in I}+\|\varphi^2 g_k''\|_{z\in I}.$$

However, by definition,

$$\varphi^2(z)g_k''(z)=(\varphi_2^2D_2^2f)\left(\frac{k}{n},\left(1-\frac{k}{n}\right)z\right),$$

and consequently,

$$\|\varphi^2 g_k''\|_{z\in I}\le\|\varphi_2^2D_2^2f\|.$$

Since $\|g_k\|_I\le\|f\|$, and since $\psi\in C^\infty$ is independent of n and f, we have proved that

$$\|\varphi^2(g_k\psi_3)''\|\le\text{const}\,[\|f\|+\|\varphi_2^2D_2^2f\|].$$

Therefore,

$$\|J\|\le\frac{\text{const}}{n}[\|f\|+\|\varphi_2^2D_2^2f\|].$$

To estimate the second term L, we define

$$h(u)=h(x,y,u):=f\left(u,(1-u)\frac{y}{1-x}\right)$$

for each fixed $(x,y)\in S$, and rewrite L as

$$L=B_n(h\psi_3;x)-h\psi_3(x).$$

We now apply (2) again, and conclude that

$$|L(x,y)|\leq\frac{\text{const}}{n}\|\varphi^2(h\psi_3)''\|_{u\in I}.$$

By (5) it is easy to see that $h(u)=0$ if $4/5\leq u\leq 1$, thus as for the estimate of g_k, we have

$$\|\varphi^2(h\psi_3)''\|_{u\in I}\leq\text{const}\,[\|h\|_{u\in I}+\|\varphi^2h''\|_{u\in I}].$$

Since by definition,

$$\varphi^2(u)h''(u)=\left[\varphi_1^2D_1^2f+\varphi_3^2D_3^2f-\frac{u}{1-u}(\varphi_2^2D_2^2f)\right]\left(u,(1-u)\frac{y}{1-x}\right),$$

we have using $h(u)=0$, $4/5\leq u\leq 1$, again,

$$\|\varphi^2h''\|_{u\in I}=\max_{0\leq u\leq 4/5}|\varphi^2(u)h''(u)|$$

$$\leq\|\varphi_1^2D_1^2f\|+\|\varphi_3^2D_3^2f\|+4\|\varphi_2^2D_2^2f\|.$$

Since $\|h\|_{u\in I}\leq\|f\|$, we proved

$$\|L\|\leq\frac{\text{const}}{n}[\|f\|+\sum_{j=1}^{3}\|\varphi_j^2D_j^2f\|],$$

giving (4) for $i=3$. The other two cases can be reduced to this one.

Let

$$f^*(x,y):=f(1-x-y,y),\quad \psi_1^*:=\psi_1(1-x-y,y)$$

on S. Since $(x,y)\in E_3$ if and only if $(1-x-y,y)\in E_1$, we have

$$\|B_{n,2}(f\psi_1)-f\psi_1\|=\|B_{n,2}(f^*\psi_1^*)-f^*\psi_1^*\|.$$

Clearly, ψ_1^* takes the role of ψ_3, we therefore have

$$\|B_{n,2}(f\psi_1)-f\psi_1\|\leq\frac{\text{const}}{n}[\|f^*\|+\sum_{j=1}^{3}\|\varphi_j^2D_j^2f^*\|].$$

However, $\|f^*\|=\|f\|$, and it can be easily checked that

$$(\varphi_1^2D_1^2f^*)(x,y)=(\varphi_1^2D_1^2f)(1-x-y,y),$$

$$(\varphi_2^2D_2^2f^*)(x,y)=(\varphi_3^2D_3^2f)(1-x-y,y),$$

and

$$(\varphi_3^2D_3^2f^*)(x,y)=-(\varphi_2^2D_2^2f)(1-x-y,y),$$

giving

$$\sum_{j=1}^{3} \|\varphi_j^2 D_j^2 f^*\| = \sum_{j=1}^{3} \|\varphi_j^2 D_j^2 f\|,$$

which proves (4) for $i=1$. Similarly, we can prove (4) for $i=2$. The direct part of the theorem is proved.

To prove the inverse part, we first remark that by [7, Lemma 2.2] all we need to prove are the following inequalities

$$\text{(7)} \qquad \|\varphi_i^2 D_i^2 B_{n,2} f\| \le \begin{cases} \text{const } n\|f\|, & f \in C(S), \\ \|\varphi_i^2 D_i^2 f\| + \dfrac{1}{n}\|D_i^2 f\|, & f \in C^2(S), \end{cases} \qquad 1 \le i \le 3,$$

and

$$\text{(8)} \qquad \|D_i^2 B_{n,2} f\| \le \begin{cases} \text{const } n^2\|f\|, & f \in C(S), \\ \|D_i^2 f\|, & f \in C^2(S), \end{cases} \qquad 1 \le i \le 3.$$

Indeed, once these inequalities are proved, it follows immediately as in [7] that

$$\|f - B_{m,2} f\| + n^{-1}\|\varphi_i^2 D_i^2 B_{m,2} f\| \le \frac{\text{const}}{n} \sum_{k=1}^{n} \|f - B_{n,2} f\|, \quad 1 \le i \le 3,$$

where m is the integer, $n/2 \le m \le n$, such that

$$\|B_{m,2} f - f\| \le \|B_{k,2} f - f\|, \quad n/2 \le k \le n,$$

proving the inverse part.

The proof of the estimates (7) and (8) is again reduced to the corresponding one dimensional ones, which are

$$\text{(9)} \qquad \begin{cases} \|\varphi^2 B_n'' f\| \le \begin{cases} \text{const } n\|f\|, & f \in C, \\ \|\varphi^2 f''\| + \dfrac{1}{n}\|f''\|, & f \in C^2, \end{cases} & \text{and} \\ \|B_n'' f\| \le \begin{cases} \text{const } n^2\|f\|, & f \in C, \\ \|f''\|, & f \in C^2. \end{cases} \end{cases}$$

We prove (7) and (8) for $i=2$ first. As in the proof of the direct part, setting $g_k(u) = f(k/n, (1-k/n)u)$ and $z = y/(1-x)$ we obtain from (3) that

$$B_{n,2} f(x, y) = \sum_{k=0}^{n} p_{kn}(x) B_{n-k}(g_k, z).$$

Since

$$\varphi_2^2(x, y) D_2^2 B_{n-k}(g_k, z) = \varphi^2(z) B_{n-k}''(g_k, z),$$

it follows from (9) that

$$|(\varphi_2^2 D_2^2 B_{n,2} f)(x, y)| \le \sum_{k=0}^{n} p_{kn}(x) \begin{cases} \text{const } (n-k)\|g_k\|_{z \in I}, \\ \|\varphi^2 g_k''\|_{z \in I} + \dfrac{1}{n-k}\|g_k''\|_{z \in I}. \end{cases}$$

However, it can be easily checked that

$$|g_k''(z)| = \left|\left(1-\frac{k}{n}\right)^2 D_2^2 f\left(\frac{k}{n}, \left(1-\frac{k}{n}\right)z\right)\right| \leq \frac{(n-k)^2}{n^2}\|D_2^2 f\|,$$

and

$$|(\varphi^2 g_k'')(z)| = \left|(\varphi_2^2 D_2^2 f)\left(\frac{k}{n}, \left(1-\frac{k}{n}\right)z\right)\right| \leq \|\varphi_2^2 D_2^2 f\|.$$

Therefore, we proved (7) for $i=2$. Notice that

$$(D_2^2 B_{n,2} f)(x,y) = \sum_{k=0}^{n} p_{kn}(x) B_{n-k}''(g_k, z)/(1-x)^2,$$

and

$$\sum_{k=0}^{n} p_{kn}(x)(n-k)^2 = n^2(1-x)^2,$$

and consequently, from (9) that

$$|(D_2^2 B_{n,2} f)(x,y)| \leq \sum_{k=0}^{n} p_{kn}(x) \frac{1}{(1-x)^2} \begin{cases} \text{const}\,(n-k)^2\|g_k\|, \\ \|g_k''\|, \end{cases}$$

$$\leq \begin{cases} \text{const}\, n^2\|f\|, \\ \|D_2^2 f\|, \end{cases}$$

giving (8) for $i=2$. To prove (7) and (8) for $i=1$ and 3, we can either use an analogous representation of the Bernstein polynomials as in (3) or use the transformations

$$f^*(x,y) = f(1-x-y, y) \quad \text{and} \quad f_*(x,y) = f(x, 1-x-y)$$

on S, as in the proof of the direct part. With these remarks we completed the proof for $d=2$.

We now outline the proof for the d-dimensional case which follows from an induction argument. Assuming the theorem to be true for dimension $d-1$, we shall prove that it is true for dimension d, too. This can be done exactly as in the proof of the theorem for the two dimensional case above.

First, similar to (3), we have

$$B_{n,d}(f;\mathbf{x}) = \sum_{k=0}^{n} p_{k,n}(x_1) \sum_{|\mathbf{j}| \leq n-k} f\left(\frac{k}{n}, \frac{\mathbf{j}}{n}\right) p_{\mathbf{j},n-k}\left(\frac{\mathbf{y}}{1-x_1}\right),$$

where for $\mathbf{x} \in \mathbb{R}^d$; we write $\mathbf{x} = (x_1, \mathbf{y})$, $\mathbf{y} \in \mathbb{R}^{d-1}$, and $\mathbf{j} \in \mathbb{N}_o^{d-1}$. Corresponding to E_i and ψ_i, $1 \leq i \leq 3$, for the two dimensional case, we now define

$$E_i = \left\{\mathbf{x} \in \mathbb{R}^d : x_i \geq \frac{1}{2d}\right\}, \quad 1 \leq i \leq d, \quad E_{d+1} = \left\{\mathbf{x} : 1 - |\mathbf{x}| \geq \frac{1}{2d}\right\},$$

and the partition of unity

$$\psi_i \in C^\infty, \quad \psi_i \geq 0, \quad \sum_{i=1}^{d+1} \psi_i = 1 \text{ on } S, \quad \text{and} \quad \operatorname{supp} \psi_i \subset E_i, \quad 1 \leq i \leq d+1.$$

If we consider $B_{n,d}(f\psi_{d+1};\mathbf{x})$ first, we can assume w.l.o.g.

$$f(\mathbf{x})=0, \text{ for } 1-|\mathbf{x}|\le \frac{1}{4d}, \quad \mathbf{x}\in \mathbb{R}^d.$$

As before, $B_{n,d}(f\psi_{d+1})-f\psi_{d+1}$ can be written into two terms, J and L. With $g_k(\mathbf{u})=f(k/n,(1-k/n)\mathbf{u})$, $\mathbf{u}\in\mathbb{R}^{d-1}$, and $\mathbf{z}=\mathbf{y}(1-x_1)$, $\mathbf{y}\in\mathbb{R}^{d-1}$, we have

$$J=\sum_{k=0}^{n} p_{kn}(x_1)[B_{n-k,d-1}(g_k\psi_{d+1},\mathbf{z})-(g_k\psi_{d+1})(\mathbf{z})],$$

which can be estimated by using the inductional assumption. The inequalities

$$\|\varphi_{ij}^2 D_{ij}(g_k\psi_{d+1})\|\le \text{const } [\|g_k\|+\|\varphi_{ij}^2 D_{ij}^2 g_k\|], \quad 1\le i\le j\le d-1,$$

which are needed in the proof, can be obtained by reducing them to the one variable inequalities in the proper direction. Also with

$$h(u)=h(u,\mathbf{x}):=f\left(u,(1-u)\frac{\mathbf{y}}{1-x_1}\right), \quad u\in\mathbb{R}^1, \quad \mathbf{x}=(x_1,\mathbf{y})\in\mathbb{R}^d,$$

we have

$$L=B_n(h\psi_{d+1},x_1)-(h\psi_{d+1})(x_1),$$

which can be estimated almost exactly as in the two dimensional case. For the other terms $f\psi_i$, $1\le i\le d$, we use the transformations

$$f_i(\mathbf{x})=f(x_1,\dots,x_{i-1},1-|\mathbf{x}|,x_{i+1},\dots,x_d)$$

on S. The inverse part is easier, and follows almost identically from the proof of the two dimensional case. ■

REFERENCES

1. Berens, H. and G.G. Lorentz – Inverse theorems for Bernstein polynomials, Indiana Univ. Math. J. **21**, 693–708 (1972).
2. Berens, H., H.J. Schmid, and Y. Xu – Bernstein-Durrmeyer polynomials on a simplex, J. Approx. Theory, in print.
3. Ditzian, Z. – Inverse theorems for multidimensional Bernstein operators, Pacific J. Math. **121**, 293–319 (1986).
4. Ditzian, Z. – Best polynomial approximation and Bernstein polynomial approximation on a simplex, Proc. Kon. Nederl. Akad. v. Wetensch. **Ser. A 92**, 243–256 (1989).
5. Ditzian, Z. and V. Totik – Moduli of Smoothness, SSCM **Vol. 9**, Springer Verlag, Berlin 1987.
6. Lorentz, G.G. – Bernstein Polynomials, 2nd ed., Chelsea, New York 1986.
7. Van Wickeren, E. – Steckin-Marchaud-type inequalities in connection with Bernstein polynomials, Constr. Approx. **2**, 331–337 (1986).
8. Wu, Z. – Approximation by Bernstein polynomials on a simplex, private communication, February 1990.
9. Zhou, D.X. – A simple proof of the inverse theorem for multidimensional Bernstein operators, manuscript, November 1989.