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On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds

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Abstract

It is well known that if the tangent bundle TM of a Riemannian manifold (M, g) is endowed with the Sasaki metric g^s, then the flatness property on TM is inherited by the base manifold [Kowalski, J. Reine Angew. Math. 250 (1971) 124–129]. This motivates us to the general question if the flatness and also other simple geometrical properties remain "hereditary" if we replace g^s by the most general Riemannian "g-natural metric" on TM (see [Kowalski and Sekizawa, Bull. Tokyo Gakugei Univ. (4) 40 (1988) 1–29; Abbassi and Sarih, Arch. Math. (Brno), submitted for publication]). In this direction, we prove that if (TM, G) is flat, or locally symmetric, or of constant sectional curvature, or of constant scalar curvature, or an Einstein manifold, respectively, then (M, g) possesses the same property, respectively. We also give explicit examples of g-natural metrics of arbitrary constant scalar curvature on TM.

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Introduction and main results

If (M, g) is an m-dimensional Riemannian manifold, then the Sasaki metric g^s is the most 'natural' metric on its tangent bundle TM depending only on the Riemannian structure on M. Other metrics on TM, naturally constructed from the base metric g, are given in [11]. Indeed, using the concept of "natural operations" and related notions, O. Kowalski and M. Sekizawa have given a full classification of such metrics, supposing that M is oriented. Other presentations of the basic results from [11] (involving also the non-oriented case and something more) can be found in [9] or [12] (see also [1]). We have studied these metrics in [3] and [4] and we have called them *g-natural metrics on TM*.

The Sasaki metric has been extensively studied, but it has been shown in many papers that it presents a kind of rigidity. In [10], Kowalski proved that if the Sasaki metric g^s is locally symmetric, then the base metric g is flat and hence g^s is also flat. In [14], Musso and Tricerri have demonstrated an extreme rigidity of g^s in the following sense: if (TM, g^s) is of constant scalar curvature, then (M, g) is flat. They have proposed the Cheeger–Gromoll metric g_{CG} (which is also a g-natural metric) as nicely fitted to the tangent bundle. Indeed, Sekizawa [24] has shown that the scalar curvature of (TM, g_{CG}) is never constant if the original metric on the base manifold has constant sectional curvature (see also [7]). Furthermore, we have proved that (TM, g_{CG}) is never a space of constant sectional curvature (cf. [2]).

More generally, similar phenomena can be studied for an arbitrary Riemannian g-natural metric G on TM (see Section 1 for the precise definition of a g-natural metric and more details). In this paper, we shall prove that every Riemannian g-natural metric G on TM has the following hereditary properties:

> If (TM, G) is flat, or locally symmetric, or of constant sectional curvature, or of constant scalar curvature, or an Einstein manifold, respectively, then (M, g) possesses the same property, respectively.

We start by presenting some necessary conditions for the flatness of G:

Theorem 0.1. *Let (M, g) be a Riemannian manifold of dimension $m \geqslant 3$ and G be a Riemannian g-natural metric on TM. If (TM, G) is flat then the following consequences hold*:

(i) *G is strongly horizontally homothetic to g,*
(ii) *(M, g) is flat.*

Note that G is *strongly horizontally homothetic* to g if there is a constant $c \geqslant 0$ such that $G_{(x,u)}(X^h, Y^h) = c.g_x(X, Y)$, for all vectors X, $Y \in M_x$, $x \in M$, where the horizontal lifts are taken at a point $(x, u) \in M_x$.

Concerning the property of local symmetry, we can assert:

Theorem 0.2. *Let (M, g) be a Riemannian manifold and G be a Riemannian g-natural metric on TM. If (TM, G) is locally symmetric, then (M, g) is also locally symmetric.*

The following theorem deals with the property of having constant scalar curvature:

Theorem 0.3. *Let (M, g) be a Riemannian manifold of dimension $m \geqslant 3$ and G be a Riemannian g-natural metric on TM. If (TM, G) is of constant sectional curvature (or of constant scalar curvature, respectively), then (M, g) has the same property.*

Theorem 0.3 gives a necessary condition for the existence of Riemannian g-natural metrics of constant sectional (respectively scalar) curvature on TM, but does not guarantee its existence. The Sasaki metric gives an example of such Riemannian g-natural metrics, but only when the constant sectional (respectively scalar) curvature vanishes (in the case where (M, g) is flat).

As concerns Einstein manifolds, we have:

Theorem 0.4. *Let (M, g) be a Riemannian manifold of dimension $m \geqslant 3$ and G be a Riemannian g-natural metric on TM. If (TM, G) is an Einstein manifold, then (M, g) is also an Einstein manifold.*

In [17], Oproiu considered an interesting family of Riemannian metrics on TM, which depends on two arbitrary functions of one variable. In Appendix A to this paper, we shall analyze the construction by Oproiu in the more general context of g-natural metrics and, as an application, we can prove the following (see Theorem A.2 for more detailed formulation and proof):

Theorem 0.5. *Let (M, g) be an m-dimensional space of negative constant sectional curvature, where $m \geqslant 3$. Then there is a 1-parameter family $\mathcal{F}$ of Riemannian g-natural metrics on TM with nonconstant defining functions α_i and β_i such that, for every $G \in \mathcal{F}$, (TM, G) is a space of positive constant scalar curvature. Moreover, for each (M, g) as above, and each prescribed constant $S > 0$, there is a metric $G \in \mathcal{F}$ with the constant scalar curvature S.*

We have dealt, in Theorems 0.1–0.5, with only the necessity conditions, the sufficiency part being very complicated and requiring a separated study for each case. Indeed, Oproiu and its collaborators devoted a series of papers (cf. [15–21,23]) to sort out, inside a broader family of metrics (not only on the tangent bundle but also on tubes in it and on the nonzero tangent bundle), those having a certain property: to be Einstein, or locally symmetric, with the additional condition of being Kähler with respect to a natural almost complex structure. They have used, for this, some quite long and hard computations made by means of the Package "RICCI".

Now, for the general case of Riemannian g-natural metrics on TM, the sufficiency problem or, in other words, the problem of classification of such metrics having one or another property becomes more complicated, and it could be more interesting to use the machinery developed in this work to derive nice examples and counterexamples of several kinds of Riemannian spaces, possibly equipped with additional structures or, alternatively, in restricting ourselves to some special subfamilies of the family of Riemannian g-natural metrics. Several examples of this do already exist in complex and quaternionic theory (cf. [25–27]).

On the other hand, all the formulas and machinery and also the derived geometrical results can be considered as a prototype for generalizations to other bundles over manifolds. This was performed successfully for the case of the Sasaki metric (cotangent, frame and Grassmann bundles) and also for the case of the Oproiu metrics (for the cotangent bundle in [22]).

1. Preliminaries and g-natural metrics

1.1. Basic formulas on tangent bundles

Let ∇ be the Levi-Civita connection of g. Then the tangent space of TM at any point $(x,u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If $(x,u) \in TM$ is given then, for any vector $X \in M_x$, there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$, where $p : TM \to M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x,u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to $(x,u) \in TM$ is a vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e. $(df)(x,u) = uf$). Note that the map $X \to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between the vector spaces M_x and $V_{(x,u)}$. Obviously, each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

If φ is a smooth function on M, then

$$X^h(\varphi \circ p) = (X\varphi) \circ p \quad \text{and} \quad X^v(\varphi \circ p) = 0 \tag{1.1}$$

hold for every vector field X on M.

A system of local coordinates $\{(U; x^i, i = 1, \ldots, m)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U); x^i, u^i, i = 1, \ldots, m)\}$. Let $X = \sum_i X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then, the horizontal lift X^h and the vertical lift X^v of X are given, with respect to the induced coordinates, by:

$$X^h = \sum X^i \frac{\partial}{\partial x^i} - \sum \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i}, \quad \text{and} \tag{1.2}$$

$$X^v = \sum X^i \frac{\partial}{\partial u^i}, \tag{1.3}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of g.

Now, let r be the norm of a vector u. Then, for any function f of $\mathbb{R}$ to $\mathbb{R}$, we get

$$X^h_{(x,u)}\big(f(r^2)\big) = 0, \tag{1.4}$$

$$X^v_{(x,u)}\big(f(r^2)\big) = 2f'(r^2) g_x(X_x, u). \tag{1.5}$$

Let X, Y and Z be any vector fields on M. If F_Y is the function on TM defined by $F_Y(x,u) = g_x(Y_x, u)$, for all $(x,u) \in TM$, then we have

$$X^h_{(x,u)}(F_Y) = g_x\big((\nabla_X Y)_x, u\big) = F_{\nabla_X Y}(x,u), \tag{1.6}$$

$$X^v_{(x,u)}(F_Y) = g_x(X, Y), \tag{1.7}$$

$$X^h_{(x,u)}\big(g(Y,Z) \circ p\big) = X_x\big(g(Y,Z)\big), \tag{1.8}$$

$$X^v_{(x,u)}\big(g(Y,Z) \circ p\big) = 0. \tag{1.9}$$

The formulas (1.4)–(1.7) follow from (1.1) and

$$X^h u^i = -\sum X^\lambda u^\mu \Gamma^i_{\lambda\mu} \quad \text{and} \quad X^v u^i = X^i, \tag{1.10}$$

and the relations (1.8) and (1.9) follow easily from (1.1).

Next, we shall introduce some notations which will be used describing vectors getting from lifted vectors by basic operations on *TM*. Let T be a tensor field of type $(1,s)$ on M. If $X_1, X_2, \dots, X_{s-1} \in M_x$, then $h\{T(X_1,\dots,u,\dots,X_{s-1})\}$ (respectively $v\{T(X_1,\dots,u,\dots,X_{s-1})\}$) is a horizontal (respectively vertical) vector at (x,u) which is introduced by the formula

$$h\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\} = \sum u^\lambda \left(T\left(X_1,\dots,\left(\frac{\partial}{\partial x^\lambda}\right)_x,\dots,X_{s-1}\right)\right)^h$$

$$\text{(respectively } v\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\} = \sum u^\lambda \left(T\left(X_1,\dots,\left(\frac{\partial}{\partial x^\lambda}\right)_x,\dots,X_{s-1}\right)\right)^v\text{)}.$$

In particular, if T is the identity tensor of type $(1,1)$, then we obtain the geodesic flow vector field at (x,u), $\xi_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})^h_{(x,u)}$, and the canonical vertical vector at (x,u), $\mathcal{U}_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})^v_{(x,u)}$.

Moreover $h\{T(X_1,\dots,u,\dots,u,\dots,X_{s-t})\}$ and $v\{T(X_1,\dots,u,\dots,u,\dots,X_{s-t})\}$ are introduced by similar way.

Also we make the conventions

$$h\big\{T(X_1,\dots,X_s)\big\} = \big(T(X_1,\dots,X_s)\big)^h \quad \text{and} \quad v\big\{T(X_1,\dots,X_s)\big\} = \big(T(X_1,\dots,X_s)\big)^v.$$

Thus $h\{X\} = X^h$ and $v\{X\} = X^v$, for each vector field X on M.

From the preceding quantities, one can define vector fields on *TU* in the following way: If $u = \sum_i u^i (\frac{\partial}{\partial x^i})_x$ is a fixed point in *TM* and $X_1,\dots,X_{s-1}$ are vector fields on U, then we denote by

$$h\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\} \quad \text{(respectively } v\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\}\text{)}$$

the horizontal (respectively vertical) vector field on *TU* defined by

$$h\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\} = \sum_\lambda u^\lambda \left[T\left(X_1,\dots,\frac{\partial}{\partial x^\lambda},\dots,X_{s-1}\right)\right]^h$$

$$\text{(respectively } v\big\{T(X_1,\dots,u,\dots,X_{s-1})\big\} = \sum_\lambda u^\lambda \left[T\left(X_1,\dots,\frac{\partial}{\partial x^\lambda},\dots,X_{s-1}\right)\right]^v\text{)}.$$

Moreover, for vector fields $X_1,\dots,X_{s-1}$ on U, the vector fields $h\{T(X_1,\dots,u,\dots,u,\dots,X_{s-t})\}$ and $v\{T(X_1,\dots,u,\dots,u,\dots,X_{s-t})\}$, on *TU*, are introduced by similar way.

The Riemannian curvature R of g is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{1.11}$$

Now, for $(r,s) \in \mathbb{N}^2$, we write $p_M : TM \to M$ for the natural projection and F for the natural bundle with

$$FM = p_M^*(\underbrace{T^* \otimes \dots \otimes T^*}_{r \text{ times}} \otimes \underbrace{T \otimes \dots \otimes T}_{s \text{ times}})M \to M,$$

$$Ff(X_x, S_x) = \big(Tf.X_x, (T^* \otimes \dots \otimes T^* \otimes T \otimes \dots \otimes T)f.S_x\big)$$

for all manifolds M, local diffeomorphisms f of M, $X_x \in T_xM$ and $S_x \in (T^* \otimes \dots \otimes T^* \otimes T \otimes \dots \otimes T)_x M$. We call the sections of the canonical projection $FM \to M$ *F-tensor fields of type* (r,s). So, if we denote by $\oplus$ the fibered product of fibered manifolds, then F-tensor fields are mappings

$$A : TM \oplus \underbrace{TM \oplus \cdots \oplus TM}_{s \text{ times}} \to \bigsqcup_{x \in M} \otimes^r M_x$$

which are linear in the last s summands such that $\pi_2 \circ A = \pi_1$, where π_1 and π_2 are the natural projections of the source and target fiber bundles of A, respectively. For $r = 0$ and $s = 2$, we obtain the classical notion of *F-metrics*. So, F-metrics are mappings $TM \oplus TM \oplus TM \to \mathbb{R}$ which are linear in the second and the third argument.

Note that we can prove that our definition of F-tensor fields of type $(0, s)$ on M is equivalent to that of *M-tensor fields of type* $(0, s)$ *on TM* introduced in [28] (see also [13]).

If we fix an F-metric δ on M, then there are three distinguished constructions of metrics on the tangent bundle TM, which are given as follows [11]:

(a) If we suppose that δ is symmetric, then the *Sasaki lift* δ^s of δ is defined as follows:

$$\begin{cases} \delta^s_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^s_{(x,u)}(X^h, Y^v) = 0, \\ \delta^s_{(x,u)}(X^v, Y^h) = 0, & \delta^s_{(x,u)}(X^v, Y^v) = \delta(u; X, Y), \end{cases}$$

for all $X, Y \in M_x$. If δ is non degenerate and positive definite, then the same holds for δ^s.

(b) The *horizontal lift* δ^h of δ is a pseudo-Riemannian metric on TM which is given by:

$$\begin{cases} \delta^h_{(x,u)}(X^h, Y^h) = 0, & \delta^h_{(x,u)}(X^h, Y^v) = \delta(u; X, Y), \\ \delta^h_{(x,u)}(X^v, Y^h) = \delta(u; X, Y), & \delta^h_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all $X, Y \in M_x$. If δ is positive definite, then δ^s is of signature (m, m).

(c) The *vertical lift* δ^v of δ is a degenerate metric on TM which is given by:

$$\begin{cases} \delta^v_{(x,u)}(X^h, Y^h) = \delta(u; X, Y), & \delta^v_{(x,u)}(X^h, Y^v) = 0, \\ \delta^v_{(x,u)}(X^v, Y^h) = 0, & \delta^v_{(x,u)}(X^v, Y^v) = 0, \end{cases}$$

for all $X, Y \in M_x$. The rank of δ^v is exactly that of δ.

If $\delta = g$ is a Riemannian metric on M, then the three lifts of δ just constructed coincide with the three classical lifts g^s, g^h and g^v of the metric g, respectively.

Let us define some notions from [6] and some conventions.

For $m \geqslant n$, a non-constant smooth map $\pi : (M^m, g) \to (N^n, h)$, and $x \in M$, put $\nu_x := \ker d\pi_x \subset M_x$ and $\mathcal{H}_x := \nu_x^\perp \subset M_x$. If $C_\pi := \{x \in M \mid d\pi_x = 0\}$ and $\hat{M} = M \setminus C_\pi$, then $\pi : (M, g) \to (N, h)$ is said to be *horizontally* (*weakly*) *conformal* if there exists a function $\lambda : \hat{M} \to \mathbb{R}^{+*}$ such that

$$\lambda^2(x) g(X, Y) = h\big(d\pi(X), d\pi(Y)\big),$$

for all $X, Y \in \mathcal{H}_x$, and $x \in \hat{M}$. The function λ is extended to the whole of M by putting $\lambda \restriction C_\pi \equiv 0$. The extended function $\lambda : M \to \mathbb{R}^+$ is called *the dilation of* π.

It follows from the definitions that $d\pi_x : M_x \to N_{\pi(x)}$ is of rank n on $\hat{M}$ and 0 on C_π, and that $\lambda^2 : M \to \mathbb{R}^+$ is smooth (cf. [6]). We shall denote by $\operatorname{grad}(\lambda^2)$ the gradient of λ^2, which is a smooth section of TM. On $(\hat{M}, g)$, $\nu := \{\nu_x \mid x \in \hat{M}\}$ and $\mathcal{H} := \{\mathcal{H}_x \mid x \in \hat{M}\}$ are smooth distributions or subbundles of $T\hat{M}$, the tangent bundle of $\hat{M}$. They are called the *vertical* and the *horizontal distributions* defined by π. By ν and $\mathcal{H}$ we also denote the projections onto ν_x and $\mathcal{H}_x$ at each point $x \in \hat{M}$. On $\hat{M}$, we have the unique orthogonal decomposition of the gradient of λ^2 into its vertical and horizontal parts given by

$$\mathrm{grad}(\lambda^2) = \mathrm{grad}_{\mathcal{V}}(\lambda^2) + \mathrm{grad}_{\mathcal{H}}(\lambda^2).$$

A non-constant smooth map $\pi : (M, g) \to (N, h)$ is said to be *horizontally homothetic* if it is horizontally conformal and $\mathrm{grad}_{\mathcal{H}}(\lambda^2) \equiv 0$ on $\hat{M}$.

Note that in this case π is necessarily a Riemannian submersion up to a fixed homothety, i.e. $\hat{M} = M$ (cf. [5]).

The horizontal homothety is therefore equivalent to $\lambda^2 : M \to \mathbb{R}^+$ being constant along horizontal curves in (M, g).

If furthermore $\mathrm{grad}_{\mathcal{V}}(\lambda^2) \equiv 0$ on M, then we say that π is *strongly horizontally homothetic*, or that g *is strongly horizontally homothetic to* h. In this case λ^2 is constant on M.

Riemannian submersions are examples of strongly horizontally homothetic maps (with constant dilation $\lambda^2 \equiv 1$). Another example is the following:

Let (M, g) be a Riemannian manifold, TM its tangent bundle and G a Riemannian metric on TM. If we take π as the canonical projection $p_M : (TM, G) \to (M, g)$, then it is easy to check that G is strongly horizontally homothetic to g if and only if there is a constant $c \geqslant 0$ such that $G_{(x,u)}(X^h, Y^h) = c.g_x(X, Y)$, for all vectors $X, Y \in M_x$, $x \in M$, where the lifts are taken at a point $(x, u) \in M_x$. If $c = 1$, then p_M is a Riemannian submersion, and equivalently we shall say that G is *horizontally isometric to* g.

1.2. g-natural metrics

Now, we shall describe all first order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+ T^*$ and $S^2 T^*$ denote the bundle functors of all Riemannian metrics and all symmetric two-forms over m-manifolds, respectively. For the concept of naturality and related notions, see [9] for more details.

Let us call every section $G : TM \to (S^2 T^*)TM$ a (possibly degenerate) *metric*. Then there is a bijective correspondence between the triples of first order natural F-metrics $(\zeta_1, \zeta_2, \zeta_3)$ and first order natural (possibly degenerate) metrics G on the tangent bundles given by (cf. [11]):

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v.$$

Therefore, to find all first order natural operators $S^2_+ T^* \rightsquigarrow (S^2 T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, it suffices to describe all first order natural F-metrics, i.e., first order natural operators $S^2_+ T^* \rightsquigarrow (T, F)$. In this sense, it is shown in [11] (see also [1,9]) that all first order natural F-metrics ζ in dimension $m > 1$ form a family parametrized by two arbitrary smooth functions $\alpha_0, \beta_0 : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, in the following way: For every Riemannian manifold (M, g) and tangent vectors $u, X, Y \in M_x$

$$\zeta_{(M,g)}(u)(X, Y) = \alpha_0\big(g(u, u)\big) g(X, Y) + \beta_0\big(g(u, u)\big) g(u, X) g(u, Y). \tag{1.12}$$

If $m = 1$, then the same assertion holds, but we can always choose $\beta_0 = 0$.

In particular, all first order natural F-metrics are symmetric.

Definition 1.1. Let (M, g) be a Riemannian manifold. We shall call a metric G on TM which comes from g by a first order natural operator $S^2_+ T^* \rightsquigarrow (S^2 T^*)T$ a g-*natural metric*.

Thus, all g-natural metrics on the tangent bundle of a Riemannian manifold (M,g) are completely determined as follows:

Proposition 1.2 [3]. *Let (M,g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, such that for every u, X, $Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2) g_x(X,Y) + (\beta_1 + \beta_3)(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g_x(X,Y) + \beta_1(r^2) g_x(X,u) g_x(Y,u), \end{cases} \tag{1.13}$$

where $r^2 = g_x(u,u)$.

For $m = 1$, the same holds with $\beta_i = 0$, $i = 1,2,3$.

Notations 1.3. In the sequel, we shall use the following notations:

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Riemannian g-natural metrics are characterized as follows:

Proposition 1.4 [3]. *The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold (M,g) to be Riemannian are that the functions of Proposition 1.2, defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{1.14}$$

for all $t \in \mathbb{R}^+$.

For $m = 1$ the system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.

Important Conventions.

(1) In the sequel, when we consider an arbitrary Riemannian g-natural metric G on TM, we implicitly suppose that it is defined by the functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3$, given in Proposition 1.2 and satisfying (1.14).

(2) Unless otherwise stated, all real functions α_i, β_i, ϕ_i, α and ϕ and their derivatives are evaluated at $r^2 := g_x(u,u)$.

In [3], we have calculated the Levi-Civita connection $\bar{\nabla}$ of an arbitrary g-natural metric on TM. Our result can be presented as follows:

Proposition 1.5. *Let (M,g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a Riemannian g-natural metric on TM. Then the Levi-Civita connection $\bar\nabla$ of (TM,G) is characterized by*

(i) $(\bar\nabla_{X^h}Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u;X_x,Y_x)\} + v\{B(u;X_x,Y_x)\}$,
(ii) $(\bar\nabla_{X^h}Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u;X_x,Y_x)\} + v\{D(u;X_x,Y_x)\}$,
(iii) $(\bar\nabla_{X^v}Y^h)_{(x,u)} = h\{C(u;Y_x,X_x)\} + v\{D(u;Y_x,X_x)\}$,
(iv) $(\bar\nabla_{X^v}Y^v)_{(x,u)} = h\{E(u;X_x,Y_x)\} + v\{F(u;X_x,Y_x)\}$,

for all vector fields X, Y on M and $(x,u)\in TM$, where A, B, C, D, E and F are the F-tensor fields of type $(1,2)$ on M defined, for all $u, X, Y \in M_x$, $x\in M$, by:

$$
\begin{aligned}
A(u;X,Y) = {}& -\frac{\alpha_1\alpha_2}{2\alpha}\big[R(X,u)Y + R(Y,u)X\big] + \frac{\alpha_2(\beta_1+\beta_3)}{2\alpha}\big[g_x(Y,u)X + g_x(X,u)Y\big] \\
& + \frac{1}{\alpha\phi}\Big\{\alpha_2\big[\alpha_1(\phi_1(\beta_1+\beta_3) - \phi_2\beta_2) + \alpha_2(\beta_1\alpha_2 \\
& - \beta_2\alpha_1)\big]g_x(R(X,u)Y,u) + \phi_2\alpha(\alpha_1+\alpha_3)' g_x(X,Y) \\
& + \big[\alpha\phi_2(\beta_1+\beta_3)' + (\beta_1+\beta_3)\big[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) \\
& + (\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1)\big]\big]g_x(X,u)g_x(Y,u)\Big\}u, \\
B(u;X,Y) = {}& \frac{\alpha_2^2}{\alpha}R(X,u)Y - \frac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha}R(X,Y)u \\
& - \frac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha}\big[g_x(Y,u)X + g_x(X,u)Y\big] \\
& + \frac{1}{\alpha\phi}\Big\{\alpha_2\big[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)) + (\alpha_1+\alpha_3)(\beta_2\alpha_1 \\
& - \beta_1\alpha_2)\big]g_x(R(X,u)Y,u) - \alpha(\phi_1+\phi_3)(\alpha_1+\alpha_3)' g_x(X,Y) \\
& + \big[-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)' + (\beta_1+\beta_3)(\alpha_1+\alpha_3)\big[(\phi_1+\phi_3)\beta_1 - \phi_2\beta_2\big] \\
& + \alpha_2\big[\alpha_2(\beta_1+\beta_3) - (\alpha_1+\alpha_3)\beta_2\big]\big]g_x(X,u)g_x(Y,u)\Big\}u, \\
C(u;X,Y) = {}& -\frac{\alpha_1^2}{2\alpha}R(Y,u)X - \frac{\alpha_1(\beta_1+\beta_3)}{2\alpha}g_x(X,u)Y \\
& + \frac{1}{\alpha}\left[\alpha_1(\alpha_1+\alpha_3)' - \alpha_2\left(\alpha_2' - \frac{\beta_2}{2}\right)\right]g_x(Y,u)X \\
& + \frac{1}{\alpha\phi}\Bigg\{\frac{\alpha_1}{2}\big[\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) + \alpha_1(\phi_1(\beta_1+\beta_3) \\
& - \phi_2\beta_2)\big]g_x(R(X,u)Y,u) + \alpha\left[\frac{\phi_1}{2}(\beta_1+\beta_3) + \phi_2\left(\alpha_2' - \frac{\beta_2}{2}\right)\right]g_x(X,Y) \\
& + \Big[\alpha\phi_1(\beta_1+\beta_3)' + \big[\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_1(\phi_2\beta_2-(\beta_1+\beta_3)\phi_1)\big]\bigg[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}\bigg]\\
&+\big[\alpha_2(\beta_1(\phi_1+\phi_3)-\beta_2\phi_2)+\alpha_1(\beta_2(\alpha_1+\alpha_3))\\
&-\alpha_2(\beta_1+\beta_3)\big]\left(\alpha_2'-\frac{\beta_2}{2}\right)\bigg]g_x(X,u)g_x(Y,u)\bigg\}u,
\end{aligned}
$$

$$
\begin{aligned}
D(u;X,Y)={}&\frac{1}{\alpha}\bigg\{\frac{\alpha_1\alpha_2}{2}R(Y,u)X-\frac{\alpha_2(\beta_1+\beta_3)}{2}g_x(X,u)Y\\
&+\bigg[-\alpha_2(\alpha_1+\alpha_3)'+(\alpha_1+\alpha_3)\left(\alpha_2'-\frac{\beta_2}{2}\right)\bigg]g_x(Y,u)X\bigg\}\\
&+\frac{1}{\alpha\phi}\bigg\{\frac{\alpha_1}{2}\big[(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)+\alpha_2(\phi_2\beta_2-\phi_1(\beta_1+\beta_3))\big]g_x(R(X,u)Y,u)\\
&-\alpha\bigg[\frac{\phi_2}{2}(\beta_1+\beta_3)+(\phi_1+\phi_3)\left(\alpha_2'-\frac{\beta_2}{2}\right)\bigg]g_x(X,Y)\\
&+\bigg[\alpha\phi_2(\beta_1+\beta_3)'+\big[(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)\\
&+\alpha_2(\phi_1(\beta_1+\beta_3)-\phi_2\beta_2)\big]\bigg[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}\bigg]\\
&+\big[(\alpha_1+\alpha_3)(\beta_2\phi_2-\beta_1(\phi_1+\phi_3))+\alpha_2(\beta_2(\alpha_1+\alpha_3))\\
&-\alpha_2(\beta_1+\beta_3)\big]\left(\alpha_2'-\frac{\beta_2}{2}\right)\bigg]g_x(X,u)g_x(Y,u)\bigg\}u,
\end{aligned}
$$

$$
\begin{aligned}
E(u;X,Y)={}&\frac{1}{\alpha}\bigg[\alpha_1\left(\alpha_2'+\frac{\beta_2}{2}\right)-\alpha_2\alpha_1'\bigg]\big[g_x(Y,u)X+g_x(X,u)Y\big]\\
&+\frac{1}{\alpha\phi}\Big\{\alpha\big[\phi_1\beta_2-\phi_2(\beta_1-\alpha_1')\big]g_x(X,Y)\\
&+\big[\alpha(2\phi_1\beta_2'-\phi_2\beta_1')+2\alpha_1'\big[\alpha_1(\alpha_2(\beta_1+\beta_3)\\
&-\beta_2(\alpha_1+\alpha_3))+\alpha_2(\beta_1(\phi_1+\phi_3)-\beta_2\phi_2)\big]\\
&+(2\alpha_2'+\beta_2)\big[\alpha_1(\phi_2\beta_2-\phi_1(\beta_1+\beta_3))+\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)\big]\big]g_x(X,u)g_x(Y,u)\Big\}u,
\end{aligned}
$$

$$
\begin{aligned}
F(u;X,Y)={}&\frac{1}{\alpha}\bigg[-\alpha_2\left(\alpha_2'+\frac{\beta_2}{2}\right)+(\alpha_1+\alpha_3)\alpha_1'\bigg]\big[g_x(Y,u)X+g_x(X,u)Y\big]\\
&+\frac{1}{\alpha\phi}\Big\{\alpha\big[(\phi_1+\phi_3)(\beta_1-\alpha_1')-\phi_2\beta_2\big]g_x(X,Y)\\
&+\big[\alpha((\phi_1+\phi_3)\beta_1'-2\phi_2\beta_2')+2\alpha_1'\big[\alpha_2(\beta_2(\alpha_1+\alpha_3)\\
&-\alpha_2(\beta_1+\beta_3))+(\alpha_1+\alpha_3)(\beta_2\phi_2-\beta_1(\phi_1+\phi_3))\big]\\
&+(2\alpha_2'+\beta_2)\big[\alpha_2(\phi_1(\beta_1+\beta_3)-\phi_2\beta_2)\\
&+(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)\big]\big]g_x(X,u)g_x(Y,u)\Big\}u.
\end{aligned}
$$

For $m=1$ the same holds with $\beta_i=0$, $i=1,2,3$.

2. Some notations and properties of F-tensor fields

Fix $(x,u) \in TM$ and a system of normal coordinates $S := (U; x^1, \ldots, x^m)$ of (M,g) centered at x. Then we can define on U the vector field $\mathrm{U} := \sum_i u^i \frac{\partial}{\partial x^i}$, where $(u^1, \ldots, u^m)$ are the coordinates of (x,u) with respect to the basis $((\frac{\partial}{\partial x^i})_x; i = 1, \ldots, m)$ of M_x.

Let P be an F-tensor field of type (p,q) on M. Then, on U, we can define a (p,q)-tensor field P_u^S (or P_u if there is no risk of confusion), associated to u and S, by

$$P_u(X_1, \ldots, X_q) := P(U_z; X_1, \ldots, X_q), \tag{2.1}$$

for all $(X_1, \ldots, X_q) \in M_z$, $z \in U$.

Informally, we can say that we have "*tensorized*" P at u with respect to S.

On the other hand, if we fix $x \in M$ and q vectors $X_1, \ldots, X_q$ in M_x, then we can define a C^∞-mapping $P_{(X_1,\ldots,X_q)} : M_x \to \otimes^p M_x$, associated to $(X_1, \ldots, X_q)$, by

$$P_{(X_1,\ldots,X_q)}(u) := P(u; X_1, \ldots, X_q), \tag{2.2}$$

for all $u \in M_x$.

Let $s > t$ be two non-negative integers, T be a $(1,s)$-tensor field on M and P^T be an F-tensor field, of type $(1,t)$, of the form

$$P^T(u; X_1, \ldots, X_t) = T(X_1, \ldots, u, \ldots, u, \ldots, X_t), \tag{2.3}$$

for all $(u, X_1, \ldots, X_t) \in TM \oplus \cdots \oplus TM$, i.e., u appears $s-t$ times at positions $i_1, \ldots, i_{s-t}$ in the expression of T. Then

- P_u^T is a $(1,t)$-tensor field on a neighborhood U of x in M, for all $u \in M_x$;
- $P^T_{(X_1,\ldots,X_t)}$ is a C^∞-mapping $M_x \to M_x$, for all $X_1, \ldots, X_t$ in M_x.

Furthermore, we have

Lemma 2.1. (1) *The covariant derivative of P_u^T, with respect to the Levi-Civita connection of (M,g), is given by*:

$$\big(\nabla_X P_u^T\big)(X_1, \ldots, X_t) = (\nabla_X T)(X_1, \ldots, u, \ldots, u, \ldots, X_t), \tag{2.4}$$

for all vectors X, $X_1, \ldots, X_t$ in M_x, where u appears at positions $i_1, \ldots, i_{s-t}$ in the right-hand side of the preceding formula.

(2) *The differential of $P^T_{(X_1,\ldots,X_t)}$, at $u \in M_x$, is given by*:

$$d\big(P^T_{(X_1,\ldots,X_t)}\big)_u(X) = T(X_1, \ldots, X, \ldots, u, \ldots, X_t) + \cdots + T(X_1, \ldots, u, \ldots, X, \ldots, X_t), \tag{2.5}$$

for all $X \in M_x$.

Proof. (1) If we extend $X_1, \ldots, X_t$ to vector fields on U denoted by the same letters, then we can write

$$\begin{aligned}
&(\nabla_X P_u^T)(X_1, \ldots, X_t) \\
&\quad = \nabla_X\big[P_u^T(X_1, \ldots, X_t)\big] - P_u^T(\nabla_X X_1, \ldots, X_t) - \cdots - P_u^T(X_1, \ldots, \nabla_X X_t) \\
&\quad = \nabla_X\big[T(X_1, \ldots, U, \ldots, U, \ldots, X_t)\big] - T(\nabla_X X_1, \ldots, u, \ldots, u, \ldots, X_t)
\end{aligned}$$

$$- \cdots - T(X_1, \ldots, u, \ldots, u, \ldots, \nabla_X X_t)$$
$$= (\nabla_X T)(X_1, \ldots, u, \ldots, u, \ldots, X_t) + T(X_1, \ldots, \nabla_X U, \ldots, u, \ldots, X_t)$$
$$+ \cdots + T(X_1, \ldots, u, \ldots, \nabla_X U, \ldots, X_t).$$

But $\nabla_X U = \sum_i u^i \nabla_X \frac{\partial}{\partial x^i}$, since u^i is constant on U.

On the other hand, the coordinate system $(U; x^1, \ldots, x^m)$ is normal and hence

$$\Gamma_{ij}^k(x) = 0, \quad i, j, k = 1, \ldots, m. \tag{2.6}$$

We deduce that $\nabla_X \frac{\partial}{\partial x^i} = \sum_{i,j,k} X^j \Gamma_{ij}^k(x) \frac{\partial}{\partial x^k}(x) = 0$, where (X^i) are the components of X with respect to the basis $((\frac{\partial}{\partial x^i})_x, i = 1, \ldots, m)$ of M_x. Hence

$$\nabla_X U = 0. \tag{2.7}$$

(2) $P^T_{(X_1,\ldots,X_t)}$ is the composite of an $(s-t)$-linear mapping $M_x \times \cdots \times M_x \to M_x$ and the diagonal mapping $M_x \to M_x \times \cdots \times M_x$, $u \mapsto (u, \ldots, u)$. A classical calculation gives obviously the required identity. □

We have also the following:

Lemma 2.2. *Let T be a $(1, s)$-tensor field on M. Then*

(1) $$\bar{\bar{\nabla}}_{X^h} h\{T(X_1, \ldots, u, \ldots, u, \ldots, X_t)\}$$
$$= h\{(\nabla_X P_u^T)((X_1)_x, \ldots, (X_t)_x) + A(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\}$$
$$+ v\{B(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\},$$

(2) $$\bar{\bar{\nabla}}_{X^v} h\{T(X_1, \ldots, u, \ldots, u, \ldots, X_t)\}$$
$$= h\{d(P^T_{((X_1)_x,\ldots,(X_t)_x)})_u(X) + C(u; T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t), X)\}$$
$$+ v\{D(u; T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t), X)\},$$

(3) $$\bar{\bar{\nabla}}_{X^h} v\{T(X_1, \ldots, u, \ldots, u, \ldots, X_t)\}$$
$$= h\{C(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\}$$
$$+ v\{(\nabla_X P_u^T)((X_1)_x, \ldots, (X_t)_x) + D(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\},$$

(4) $$\bar{\bar{\nabla}}_{X^v} v\{T(X_1, \ldots, u, \ldots, u, \ldots, X_t)\}$$
$$= h\{E(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\}$$
$$+ v\{d(P^T_{((X_1)_x,\ldots,(X_t)_x)})_u(X)) + F(u; X, T_x(X_1, \ldots, u, \ldots, u, \ldots, X_t))\},$$

for all vector fields $X_1, \ldots, X_t$ on M and $X \in M_x$, where u appears at positions $i_1, \ldots, i_{s-t}$ in any expression of T. Here, X^h and X^v are taken at (x, u).

Proof. We shall prove only (1), the proof of the other identities being similar.

$$\begin{aligned}
&\bar{\nabla}_{X^h} h\bigl\{T(X_1,\dots,u,\dots,u,\dots,X_t)\bigr\}\\
&\quad=\sum_{\lambda_1,\dots,\lambda_{s-t}} \bar{\nabla}_{X^h}\left\{u^{\lambda_1}\cdots u^{\lambda_{s-t}}\left[T\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}},\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}},\dots,X_t\right)\right]^h\right\}\\
&\quad=\sum_{\lambda_1,\dots,\lambda_{s-t}} \left\{u^{\lambda_1}\cdots u^{\lambda_{s-t}}\bar{\nabla}_{X^h}\left[T\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}},\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}},\dots,X_t\right)\right]^h\right.\\
&\qquad\left.-\sum_k X^j\Gamma^{\lambda_k}_{j\mu_k}u^{\lambda_1}\cdots u^{\mu_k}\cdots u^{\lambda_{s-t}}\left[T\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}}(x),\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}}(x),\dots,X_t\right)\right]^h\right\},
\end{aligned}$$

where we have used (1.10) in the second identity. But, by virtue of (2.6), we deduce

$$\begin{aligned}
&\bar{\nabla}_{X^h} h\bigl\{T(X_1,\dots,u,\dots,u,\dots,X_t)\bigr\}\\
&\quad=\sum_{\lambda_1,\dots,\lambda_{s-t}} u^{\lambda_1}\cdots u^{\lambda_{s-t}}\bar{\nabla}_{X^h}\left[T\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}},\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}},\dots,X_t\right)\right]^h\\
&\quad=\sum_{\lambda_1,\dots,\lambda_{s-t}} u^{\lambda_1}\cdots u^{\lambda_{s-t}}\left\{\left[(\nabla_X T)\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}},\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}},\dots,X_t\right)\right]^h\right.\\
&\qquad+h\left\{A\left(u;X,T_x\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}}(x),\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}}(x),\dots,X_t\right)\right)\right\}\\
&\qquad\left.+v\left\{B\left(u;X,T_x\left(X_1,\dots,\frac{\partial}{\partial x^{\lambda_1}}(x),\dots,\frac{\partial}{\partial x^{\lambda_{s-t}}}(x),\dots,X_t\right)\right)\right\}\right\}\\
&\quad=\bigl[(\nabla_X T)\bigl(X_1,\dots,u,\dots,u,\dots,X_t\bigr)\bigr]^h+h\bigl\{A\bigl(u;X,T_x(X_1,\dots,u,\dots,u,\dots,X_t)\bigr)\bigr\}\\
&\qquad+v\bigl\{B(u;X,T_x(X_1,\dots,u,\dots,u,\dots,X_t))\bigr\}\\
&\quad=\bigl[(\nabla_X P^T_u)(X_1,\dots,X_t)\bigr]^h+h\bigl\{A(u;X,T_x(X_1,\dots,u,\dots,u,\dots,X_t))\bigr\}\\
&\qquad+v\bigl\{B(u;X,T_x(X_1,\dots,u,\dots,u,\dots,X_t))\bigr\},
\end{aligned}$$

where we have used in the last equality formula (2.4). □

Now, let P be the F-tensor field of type $(1,t)$ of the form

$$P(u;X_1,\dots,X_t)=\sum_i f_i^P(r^2)T_i(X_1,\dots,u,\dots,u,\dots,X_t), \tag{2.8}$$

where $f_i:\mathbb{R}^+\to\mathbb{R}$ are real-valued functions on $\mathbb{R}^+$, and any T_i is a $(1,s_i)$-tensor field on M, $s_i>t$, with the s_i's not necessarily equal. Then, we have

Lemma 2.3. *Let P be an F-tensor field, of type $(1,t)$, on M given by (2.8). Then*

(1) $(\nabla_X P_u)(X_1,\dots,X_t)=\sum_i f_i^P(r^2)(\nabla_X T_i)(X_1,\dots,u,\dots,u,\dots,X_t)$,

$$\text{(2)}\quad d(P_{(X_1,\dots,X_t)})_x(X) = 2\sum_i (f_i^P)'(r^2) g(X,u) T_i(X_1,\dots,u,\dots,u,\dots,X_t) + \sum_i f_i^P(r^2)\big\{T_i(X_1,\dots,X,\dots,u,\dots,X_t) + \cdots + T_i(X_1,\dots,u,\dots,X,\dots,X_t)\big\},$$

for all $u, X, X_1, \dots, X_t \in M_x$.

Proof. (1) It is clear $P_u(X_1,\dots,X_t) = \sum_i f_i(r^2) P_u^{T_i}(X_1,\dots,X_t)$. We deduce that

$$\begin{aligned}(\nabla_X P_u)(X_1,\dots,X_t) &= 2\sum_i (f_i^P)'(r^2) g(\nabla_X \mathrm{U}, u) P_u^{T_i}(X_1,\dots,X_t) + \sum_i f_i^P(r^2)(\nabla_X P_u^{T_i})(X_1,\dots,X_t)\\ &= \sum_i f_i^P(r^2)(\nabla_X P_u^{T_i})(X_1,\dots,X_t),\end{aligned}$$

by virtue of (2.7). Using (2.4), we obtain the desired identity.

(2) is obtained, in the same manner, using (2.5) instead of (2.4). □

If we denote by $h\{P(u; X_1,\dots,X_t)\}$ (respectively $v\{P(u; X_1,\dots,X_t)\}$) the quantity

$$h\big\{P(u; X_1,\dots,X_t)\big\} = \sum_i f_i^P(r^2) h\big\{T_i(X_1,\dots,u,\dots,u,\dots,X_t)\big\} \tag{2.9}$$

$$\text{(respectively } v\big\{P(u; X_1,\dots,X_t)\big\} = \sum_i f_i^P(r^2) v\big\{T_i(X_1,\dots,u,\dots,u,\dots,X_t)\big\}\text{)}, \tag{2.10}$$

then we can assert

Lemma 2.4.

$$\begin{aligned}\text{(1)}\quad &\bar{\bar\nabla}_{X^h} h\big\{P(u; X_1,\dots,X_t)\big\}\\ &= h\big\{(\nabla_X P_u)((X_1)_x,\dots,(X_t)_x) + A(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\}\\ &\quad + v\big\{B(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\},\\ \text{(2)}\quad &\bar{\bar\nabla}_{X^v} h\big\{P(u; X_1,\dots,X_t)\big\}\\ &= h\big\{d(P_{((X_1)_x,\dots,(X_t)_x)})_u(X) + C(u; P(u; (X_1)_x,\dots,(X_t)_x), X)\big\}\\ &\quad + v\big\{D(u; P(u; (X_1)_x,\dots,(X_t)_x), X)\big\},\\ \text{(3)}\quad &\bar{\bar\nabla}_{X^h} v\big\{P(u; X_1,\dots,X_t)\big\}\\ &= h\big\{C(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\}\\ &\quad + v\big\{(\nabla_X P_u)((X_1)_x,\dots,(X_t)_x) + D(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\},\\ \text{(4)}\quad &\bar{\bar\nabla}_{X^v} v\big\{P(u; X_1,\dots,X_t)\big\}\\ &= h\big\{E(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\}\\ &\quad + v\big\{d(P_{((X_1)_x,\dots,(X_t)_x)})_u(X)) + F(u; X, P(u; (X_1)_x,\dots,(X_t)_x))\big\},\end{aligned}$$

for all vector fields $X_1,\dots,X_t$ *on* M *and* $X \in M_x$. *Here* X^h *and* X^v *are taken at* (x,u).

Proof. We shall give the proof of (1), the proof of the other identities being similar. We have by (2.9)

$$\begin{aligned}\bar\nabla_{X^h} h\big\{P(u;X_1,\dots,X_t)\big\} &= \sum_i X^h\big(f_i(r^2)\big)h\big\{T_i((X_1)_x,\dots,u,\dots,u,\dots,(X_t)_x)\big\}\\ &\quad+\sum_i f_i(r^2)\bar\nabla_{X^h}h\big\{T_i(X_1,\dots,u,\dots u,\dots,X_t)\big\}\\ &= \sum_i f_i(r^2)\bar\nabla_{X^h}h\big\{T_i(X_1,\dots,u,\dots u,\dots,X_t)\big\},\end{aligned}$$

by virtue of (1.4). Hence, due to Lemma 2.2 and Proposition 1.5, we obtain

$$\begin{aligned}\bar\nabla_{X^h} h\big\{P(u;X_1,\dots,X_t)\big\} = \sum_i f_i(r^2)\Big\{&h\big\{(\nabla_X P_u^{T_i})\big((X_1)_x,\dots,(X_t)_x\big)\\ &+A\big(u;X,T_i((X_1)_x,\dots,u,\dots,u,\dots,(X_t)_x)\big)\big\}\\ &+v\big\{B\big(u;X,T_i((X_1)_x,\dots,u,\dots,u,\dots,(X_t)_x)\big)\big\}\Big\}.\end{aligned}$$

Using the linearity of A and B and (1) of Lemma 2.3, we obtain the required formula. □

Finally, the following lemma will be used in the proof of Theorem 0.2:

Lemma 2.5. *Let P be an F-tensor field, of type $(1,t+s)$, of the form*

$$P(u;X_1,\dots,X_{t+s}) = T\big(u;X_1,\dots,X_t,S(u;X_{t+1},\dots,X_{t+s})\big), \tag{2.11}$$

where T and S are F-tensor fields of types $(1,t+1)$ and $(1,s)$, respectively. Then we have

$$\begin{aligned}(\nabla_X P_u)(X_1,\dots,X_{t+s}) &= (\nabla_X T_u)\big(X_1,\dots,X_t,S_u(X_{t+1},\dots,X_{t+s})\big)\\ &\quad+T_u\big(X_1,\dots,X_t,(\nabla_X S_u)(X_{t+1},\dots,X_{t+s})\big),\end{aligned} \tag{2.12}$$

for all $X_1,\dots,X_{t+s}\in M_x$, $x\in M$.

Proof. Notice that, for each $u\in M_x$, $x\in M$, the $(1,t+s)$-tensor field P_u is a contraction of the $(2,t+s+1)$-tensor field $T_u\otimes S_u$, say $P_u=\mathbf{C}(T_u\otimes S_u)$. It follows that (cf. [8], I, p. 123)

$$\nabla_X P_u = \nabla_X\big[\mathbf{C}(T_u\otimes S_u)\big] = \mathbf{C}(\nabla_X T_u\otimes S_u)+\mathbf{C}(T_u\otimes\nabla_X S_u),$$

which gives, clearly, the result. □

3. Riemannian curvatures of g-natural metrics

Proposition 3.1. *Let (M,g) be a Riemannian manifold and G be a Riemannian g-natural metric on TM. Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. Then, with the notations of Section 2, the Riemannian curvature tensor $\bar R$ of (TM,G) is completely determined by*

(i) $\bar R(X^h,Y^h)Z^h = \big[R(X,Y)Z\big]^h + h\big\{(\nabla_X A_u)(Y,Z)-(\nabla_Y A_u)(X,Z)$

$$\begin{aligned}
&+ A(u; X, A(u; Y, Z)) - A(u; Y, A(u; X, Z)) + C(u; X, B(u; Y, Z)) \\
&- C(u; Y, B(u; X, Z)) + C(u; Z, R(X, Y)u)\big\} + v\big\{(\nabla_X B_u)(Y, Z) \\
&- (\nabla_Y A_u)(X, Z) + B(u; X, A(u; Y, Z)) - B(u; Y, A(u; X, Z)) \\
&+ D(u; X, B(u; Y, Z)) - D(u; Y, B(u; X, Z)) + D(u; Z, R(X, Y)u)\big\},
\end{aligned}$$

(ii) $$\begin{aligned}
\bar{R}(X^h, Y^h)Z^v = {}&\big[R(X, Y)Z\big]^v + h\big\{(\nabla_X C_u)(Y, Z) - (\nabla_Y C_u)(X, Z) \\
&+ A(u; X, C(u; Y, Z)) - A(u; Y, C(u; X, Z)) + C(u; X, D(u; Y, Z)) \\
&- C(u; Y, D(u; X, Z)) + E(u; R(X, Y)u, Z)\big\} + v\big\{(\nabla_X D_u)(Y, Z) \\
&- (\nabla_Y D_u)(X, Z) + B(u; X, C(u; Y, Z)) - B(u; Y, C(u; X, Z)) \\
&+ D(u; X, D(u; Y, Z)) - D(u; Y, D(u; X, Z)) + F(u; R(X, Y)u, Z)\big\},
\end{aligned}$$

(iii) $$\begin{aligned}
\bar{R}(X^h, Y^v)Z^h = {}&h\big\{(\nabla_X C_u)(Z, Y) + A(u; X, C(u; Z, Y)) + C(u; X, D(u; Z, Y)) \\
&- C(u; A(u; X, Z), Y) - E(u; Y, B(u; X, Z)) - d(A_{(X,Z)})_u(Y)\big\} \\
&+ v\big\{(\nabla_X D_u)(Z, Y) + B(u; X, C(u; Z, Y)) + D(u; X, D(u; Z, Y)) \\
&- D(u; A(u; X, Z), Y) - F(u; Y, B(u; X, Z)) - d(B_{(X,Z)})_u(Y)\big\},
\end{aligned}$$

(iv) $$\begin{aligned}
\bar{R}(X^h, Y^v)Z^v = {}&h\big\{(\nabla_X E_u)(Y, Z) + A(u; X, E(u; Y, Z)) + C(u; X, F(u; Y, Z)) \\
&- C(u; C(u; X, Z), Y) - E(u; Y, D(u; X, Z)) - d(C_{(X,Z)})_u(Y)\big\} \\
&+ v\big\{(\nabla_X F_u)(Y, Z) + B(u; X, E(u; Y, Z)) + D(u; X, F(u; Y, Z)) \\
&- D(u; C(u; X, Z), Y) - F(u; Y, D(u; X, Z)) - d(D_{(X,Z)})_u(Y)\big\},
\end{aligned}$$

(v) $$\begin{aligned}
\bar{R}(X^v, Y^v)Z^h = {}&h\big\{d(C_{(Z,Y)})_u(X) - d(C_{(Z,X)})_u(Y) + C(u; C(u; Z, Y), X) \\
&- C(u; C(u; Z, X), Y) + E(u; X, D(u; Z, Y)) - E(u; Y, D(u; Z, X))\big\} \\
&+ v\big\{d(D_{(Z,Y)})_u(X) - d(D_{(Z,X)})_u(Y) + D(u; C(u; Z, Y), X) \\
&- D(u; C(u; Z, X), Y) + F(u; X, D(u; Z, Y)) - F(u; Y, D(u; Z, X))\big\},
\end{aligned}$$

(vi) $$\begin{aligned}
\bar{R}(X^v, Y^v)Z^v = {}&h\big\{d(E_{(Y,Z)})_u(X) - d(E_{(X,Z)})_u(Y) + C(u; E(u; Y, Z), X) \\
&- C(u; E(u; X, Z), Y) + E(u; X, F(u; Y, Z)) - E(u; Y, F(u; X, Z))\big\} \\
&+ v\big\{d(F_{(Y,Z)})_u(X) - d(F_{(X,Z)})_u(Y) + D(u; E(u; Y, Z), X) \\
&- D(u; E(u; X, Z), Y) + F(u; X, F(u; Y, Z)) - F(u; Y, F(u; X, Z))\big\},
\end{aligned}$$

for all $x \in M$ and $X, Y, Z \in M_x$, where the lifts are taken at $u \in M_x$.

Proof. We shall prove the first formula, the proof of the others being the same. Remark that any of A, B, C, D, E and F, of Proposition 1.5, is an F-tensor field, of type $(1,2)$, of the form (2.8). Using the identity (1.11), Proposition 1.5 and Lemma 2.4, we can write

$$\begin{aligned}
\bar{R}(X^h, Y^h)Z^h &= \bar{\nabla}_{X^h}\bar{\nabla}_{Y^h}Z^h - \bar{\nabla}_{Y^h}\bar{\nabla}_{X^h}Z^h - \bar{\nabla}_{[X^h, Y^h]}Z^h \\
&= \bar{\nabla}_{X^h}(\nabla_Y Z)^h + \bar{\nabla}_{X^h}h\big\{A(u; Y, Z)\big\} + \bar{\nabla}_{X^h}v\big\{B(u; Y, Z)\big\} \\
&\quad - \bar{\nabla}_{Y^h}(\nabla_X Z)^h - \bar{\nabla}_{Y^h}h\big\{A(u; X, Z)\big\} + \bar{\nabla}_{Y^h}v\big\{B(u; X, Z)\big\} \\
&\quad - \bar{\nabla}_{[X,Y]^h}Z^h + \bar{\nabla}_{v\{R(X,Y)u\}}Z^h
\end{aligned}$$

$$
\begin{aligned}
&= \big[R(X,Y)Z\big]^h + h\big\{(\nabla_X A_u)(Y,Z) - (\nabla_Y A_u)(X,Z) \\
&\quad + A(u;X,A(u;Y,Z)) - A(u;Y,A(u;X,Z)) + C(u;X,B(u;Y,Z)) \\
&\quad - C(u;Y,B(u;X,Z)) + C(u;Z,R(X,Y)u)\big\} + v\big\{(\nabla_X B_u)(Y,Z) \\
&\quad - (\nabla_Y A_u)(X,Z) + B(u;X,A(u;Y,Z)) - B(u;Y,A(u;X,Z)) \\
&\quad + D(u;X,B(u;Y,Z)) - D(u;Y,B(u;X,Z)) + D(u;Z,R(X,Y)u)\big\}. \qquad \square
\end{aligned}
$$

4. Proofs of the main theorems

Proof of Theorem 0.1. Remark, at first, that the F-tensor fields A, B, C, D, E and F of Proposition 1.5 and also the quantities $\nabla_X P_u$, for F-tensors P of the form (2.8), are identically zero for $u=0$. Suppose, now, that (TM,G) is flat, i.e., its Riemannian curvature tensor $\bar{R}$ vanishes identically. Then, restricting formula (i) of Proposition 3.1 to the zero section of TM, we deduce, by virtue of the preceding remark, that

$$0 = \bar{R}_{(x,0)}(X^h,Y^h)Z^h = \big[R(X,Y)Z\big]^h_{(x,0)}, \tag{4.1}$$

for all $x \in M$ and $X, Y, Z \in \mathfrak{X}(M)$. It follows that R vanishes identically on M. Hence, (M,g) is flat, which shows the second part of Theorem 0.1.

This implies, in particular, that all F-tensor fields A, B, C, D, E and F of Proposition 1.5 reduce to the following form

$$
\begin{aligned}
P(u;X,Y) &= f_3^P g(Y,u)X + f_4^P g(X,u)Y + f_5^P g(X,Y)u \\
&\quad + f_6^P g(X,u)g(Y,u)u.
\end{aligned} \tag{4.2}
$$

Now, in order to make the calculations easier, we shall use the preceding functions. Since the identities of Proposition 3.1 involve quantities of the form $\nabla_X P_u$ and $(dP_{(Y,Z)})_u(X)$, we shall give explicitly these quantities for an F-tensor field of the form (4.2):

Lemma 4.1. *Let P be an F-tensor field, of type $(1,2)$, of the form (4.2). Then with the notations of Section 2, we have for all $u, X, Y, Z \in M_x$, $x \in U$,*

(1) $\nabla_X P_u = 0$,

(2)
$$
\begin{aligned}
d(P_{(X,Y)})_u(Z) &= \big[f_3^P g(Y,Z) + 2(f_3^P)' g(Y,u)g(Z,u)\big]X \\
&\quad + \big[f_4^P g(X,Z) + 2(f_4^P)' g(X,u)g(Z,u)\big]Y + f_5^P g(X,Y)Z \\
&\quad + \big\{f_6^P\big[g(X,Z)g(Y,u) + g(Y,Z)g(X,u)\big] \\
&\quad + 2(f_5^P)' g(X,Y)g(Z,u) + 2(f_6^P)' g(X,u)g(Y,u)g(Z,u)\big\}u.
\end{aligned}
$$

Proof. Fix $u \in M_x$, $x \in M$. Notice that P_u is of the form $\sum_{i=3}^5 f_i^P P_u^{T_i} + f_6^P P_u^{T_6}$, where

- $T_i = (g \otimes I) \circ \sigma_i$, $i = 3, 4, 5$,
- $T_6 = (g \otimes g \otimes I) \circ \sigma_6$,

and σ_i are the mappings defined by

- $\sigma_3 : X_1 \otimes X_2 \otimes X_3 \mapsto X_1 \otimes X_3 \otimes X_2$,
- σ_4 is the identity mapping,
- $\sigma_5 : X_1 \otimes X_2 \otimes X_3 \mapsto X_2 \otimes X_3 \otimes X_1$, and
- $\sigma_6 : X_1 \otimes X_2 \otimes X_3 \mapsto X_1 \otimes X_2 \otimes X_1 \otimes X_3 \otimes X_1$,

I being the identity $(1,1)$-tensor field on U. It follows that $\nabla T_i = 0$ on U, $i = 3, \ldots, 6$, and consequently, by virtue of (1) of Lemma 2.4, we obtain $\nabla_X P_u^{T_i} = 0$, for all $X \in M_x$. Using (1) of Lemma 2.3, we deduce that $\nabla_X P_u = 0$. The proof of the second property is similar, but using Lemmas 2.4 and 2.3. □

We shall now prove the first part of Theorem 0.1, i.e., that G is strongly horizontally homothetic to g. For this, it is sufficient to show, according to the first formula of (1.13), that $\alpha_1 + \alpha_3$ is constant and $\beta_1 + \beta_3$ vanishes identically on $\mathbb{R}^+$.

We shall start with the first property. In fact, if we put $Y = Z$ and we suppose that $\{u, X, Y\}$ is an orthogonal system in M_x, then for any F-tensor field P of the form (4.2), we have:

$$\begin{aligned} &P(u; X, Y) = 0, \quad P(u; X, u) = r^2 f_3^P.X, \\ &P(u; u, X) = r^2 f_4^P.X, \quad P(u; X, X) = \|X\|^2 f_5^P.u, \\ &P(u; u, u) = r^2\big[f_3^P + f_4^P + f_5^P + r^2 f_6^P\big]u. \end{aligned} \tag{4.3}$$

Substituting from (4.3) and from the first identity of Lemma 4.1 into (i) of Proposition 3.1, we obtain

$$\bar{R}_{(x,u)}(X^h, Y^h)Y^h = r^2\|Y\|^2\big[\big(f_3^C f_5^B + f_3^A f_5^A\big).X_u^h + \big(f_3^B f_5^A + f_3^D f_5^B\big).X_u^v\big].$$

Consequently, we have on $\mathbb{R}^{+*}$

$$\begin{cases} f_3^C f_5^B + f_3^A f_5^A = 0, \\ f_3^B f_5^A + f_3^D f_5^B = 0. \end{cases}$$

Substituting from Proposition 1.5, we obtain on $\mathbb{R}^{+*}$

$$\begin{cases} \alpha_2 \Phi + \alpha_1 \Psi = 0, \\ -(\alpha_1 + \alpha_3)\Phi - \alpha_2 \Psi = 0, \end{cases}$$

where $\Phi = (\alpha_1 + \alpha_3)'[\phi_2 \frac{\beta_1+\beta_3}{2} - (\phi_1 + \phi_3)(\alpha_2' - \frac{\beta_2}{2})]$ and $\Psi = (\phi_1 + \phi_3)[(\alpha_1 + \alpha_3)']^2$.

From the linear equations above we get $\Phi = \Psi = 0$ (by virtue of $\alpha \neq 0$ everywhere). But $\Psi = 0$ implies that $(\alpha_1 + \alpha_3)' = 0$ on $\mathbb{R}^{+*}$, since $\phi_1 + \phi_3 \neq 0$ everywhere. By continuity, $(\alpha_1 + \alpha_3)' = 0$ on $\mathbb{R}^+$.

We prove now that $\beta_1 + \beta_3$ vanishes identically on $\mathbb{R}^+$. In fact, if we put $X = u$ and $Y = Z$ orthogonal to u, then substituting from analogous formulas of (4.3) and from Lemma 4.1 into (i) of Proposition 3.1, we obtain

$$\bar{R}_{(x,u)}(u^h, Y^h)Y^h = -r^2\|Y\|^2 f_4^B\big[f_5^E.u^h + f_5^D.u^h\big].$$

Here, we have used the fact that $f_5^A = f_5^B = 0$ on $\mathbb{R}^+$, since $(\alpha_1 + \alpha_3)' = 0$ on $\mathbb{R}^+$. It follows that on $\mathbb{R}^{+*}$, we have

$$\begin{cases} f_4^B f_5^E = 0, \\ f_4^B f_5^F = 0. \end{cases}$$

Substituting from Proposition 1.5 and using the facts that $\alpha_1+\alpha_3\neq 0$ and $\alpha\neq 0$ everywhere on $\mathbb{R}^{+*}$, we obtain on $\mathbb{R}^{+*}$

$$\begin{cases}(\beta_1+\beta_3)[\phi_1\frac{\beta_1+\beta_3}{2}+\phi_2(\alpha_2'-\frac{\beta_2}{2})]=0,\\ (\beta_1+\beta_3)[\phi_2\frac{\beta_1+\beta_3}{2}+(\phi_1+\phi_3)(\alpha_2'-\frac{\beta_2}{2})]=0.\end{cases}$$

We claim that $\beta_1+\beta_3=0$ everywhere on $\mathbb{R}^{+*}$. Indeed, suppose that there is some $t_0\in\mathbb{R}^{+*}$ such that $(\beta_1+\beta_3)(t_0)\neq 0$. Then the previous system reduces at t_0 to the system

$$\begin{cases}\phi_1(t_0)\frac{\beta_1+\beta_3}{2}(t_0)+\phi_2(t_0)(\alpha_2'-\frac{\beta_2}{2})(t_0)=0,\\ \phi_2(t_0)\frac{\beta_1+\beta_3}{2}(t_0)+(\phi_1+\phi_3)(t_0)(\alpha_2'-\frac{\beta_2}{2})(t_0)=0,\end{cases}$$

and hence, by virtue of $\phi(t_0)\neq 0$,

$$\frac{\beta_1+\beta_3}{2}(t_0)=\left(\alpha_2'-\frac{\beta_2}{2}\right)(t_0)=0,$$

which contradicts our assumption.

Thus $\beta_1+\beta_3=0$ on $\mathbb{R}^{+*}$, and by continuity on $\mathbb{R}^{+}$. □

Proof of Theorem 0.2. Remark, at first, that any F-tensor field A, B, C, D, E and F of Proposition 1.5 is of the form

$$\begin{aligned}P(u;X,Y)={}&f_1^P(r^2).R(X,u)Y+f_2^P(r^2).R(Y,u)X\\&+f_3^P(r^2).g(Y,u)X+f_4^P(r^2).g(X,u)Y+f_5^P(r^2).g(X,Y)u\\&+\big[f_7^P(r^2).g(R(X,u)Y,u)+f_6^P(r^2).g(X,u)g(Y,u)\big]u.\end{aligned}\tag{4.4}$$

We begin by calculating $(\bar\nabla_{W^h}\bar R)(X^h,Y^h)Z^h$, for all $X,Y,Z\in M_x$. If we extend X,Y,Z to vector fields on M, which we denote also by the same letters, then we can write

$$\begin{aligned}(\bar\nabla_{W^h}\bar R)(X^h,Y^h)Z^h={}&\bar\nabla_{W^h}\big[\bar R(X^h,Y^h)Z^h\big]-\bar R(\bar\nabla_{W^h}X^h,Y^h)Z^h\\&-\bar R(X^h,\bar\nabla_{W^h}Y^h)Z^h-\bar R(X^h,Y^h)\bar\nabla_{W^h}Z^h.\end{aligned}$$

Using (i) of Proposition 1.5 and (i) of Proposition 3.1, we deduce that

$$\begin{aligned}&(\bar\nabla_{W^h}\bar R)(X^h,Y^h)Z^h\\&\quad=\big[\nabla_W(R(X,Y)Z)\big]^h+\bar\nabla_{W^h}h\big\{(\nabla_XA_u)(Y,Z)-(\nabla_YA_u)(X,Z)\\&\qquad+A(u;X,A(u;Y,Z))-A(u;Y,A(u;X,Z))+C(u;X,B(u;Y,Z))\\&\qquad-C(u;Y,B(u;X,Z))+C(u;Z,R(X,Y)u)\big\}+\bar\nabla_{W^h}v\big\{(\nabla_XB_u)(Y,Z)\\&\qquad-(\nabla_YA_u)(X,Z)+B(u;X,A(u;Y,Z))-B(u;Y,A(u;X,Z))\\&\qquad+D(u;X,B(u;Y,Z))-D(u;Y,B(u;X,Z))+D(u;Z,R(X,Y)u)\big\}\\&\qquad-\bar R((\nabla_WX)^h,Y^h)Z^h-\bar R(X^h,(\nabla_WY)^h)Z^h-\bar R(X^h,Y^h)(\nabla_WZ)^h\\&\qquad-\bar R\big(h\{A(u;W,X)\},Y^h\big)Z^h-\bar R\big(X^h,h\{A(u;W,Y)\}\big)Z^h\\&\qquad-\bar R(X^h,Y^h)h\{A(u;W,Z)\}-\bar R\big(v\{B(u;W,X)\},Y^h\big)Z^h\\&\qquad-\bar R\big(X^h,v\{B(u;W,Y)\}\big)Z^h-\bar R(X^h,Y^h)v\{B(u;W,Z)\}.\end{aligned}$$

If we restrict ourselves to the zero section of TM, then we can write, for each F-tensor field P, of the form (4.4)

$$P_0 = 0. \tag{4.5}$$

We have, also, by (1) of Lemma 2.4 and (4.5),

$$\begin{aligned}
&\big[\bar\nabla_{W^h} h\big\{(\nabla_X P_u)(Y,Z)\big\}\big]_{(x,0)} \\
&\quad = \big[(\nabla_W \nabla_X P_0)(Y,Z)\big]^h_{(x,0)} + h\big\{A(0;W,(\nabla_X P_0)(Y,Z))\big\} \\
&\qquad + v\big\{B(0;W,(\nabla_X P_0)(Y,Z))\big\} \\
&\quad = 0.
\end{aligned} \tag{4.6}$$

If P' is another F-tensor field of the form (4.4), then we obtain, using (1) of Lemma 2.4, (2.12) and (4.5)

$$\begin{aligned}
&\big[\bar\nabla_{W^h} h\big\{P(u;X,P'(u;Y,Z))\big\}\big]_{(x,0)} \\
&\quad = h\big\{(\nabla_W P_0)(X,P'_0(Y,Z)) + P_0(X,(\nabla_W P'_0)(Y,Z)) \\
&\qquad + A(0;W,P(0;X,P'(0;Y,Z))\big\} + v\big\{B(0;W,P(0;X,P'(0;Y,Z))\big\} \\
&\quad = 0.
\end{aligned} \tag{4.7}$$

Similarly, we have

$$\big[\bar\nabla_{W^h} v\big\{(\nabla_X P_u)(Y,Z)\big\}\big]_{(x,0)} = 0 \quad \text{and} \tag{4.8}$$

$$\big[\bar\nabla_{W^h} v\big\{P(u;X,P'(u;Y,Z))\big\}\big]_{(x,0)} = 0. \tag{4.9}$$

By virtue of (i) of Proposition 3.1 and (4.5), we have

$$\bar R_{(x,0)}\big((\nabla_W X)^h, Y^h\big)Z^h = \big[R(\nabla_W X, Y)Z\big]^h_{(x,0)}, \tag{4.10}$$

$$\bar R_{(x,0)}\big(X^h, (\nabla_W Y)^h\big)Z^h = \big[R(X, \nabla_W Y)Z\big]^h_{(x,0)}, \tag{4.11}$$

$$\bar R_{(x,0)}(X^h, Y^h)(\nabla_W Z)^h = \big[R(X,Y)\nabla_W Z\big]^h_{(x,0)}. \tag{4.12}$$

By a substitution from (4.5)–(4.12), we conclude that

$$\begin{aligned}
\big[(\bar\nabla_{W^h}\bar R)(X^h,Y^h)Z^h\big]_{(x,0)} &= \big[\nabla_W(R(X,Y)Z)\big]^h_{(x,0)} - \big[R(\nabla_W X,Y)Z\big]^h_{(x,0)} \\
&\quad - \big[R(X,\nabla_W Y)Z\big]^h_{(x,0)} - \big[R(X,Y)\nabla_W Z\big]^h_{(x,0)}.
\end{aligned}$$

It follows that

$$\big[(\bar\nabla_{W^h}\bar R)(X^h,Y^h)Z^h\big]_{(x,0)} = \big[(\nabla_W R)(X,Y)Z\big]^h_{(x,0)}, \tag{4.13}$$

for all $X, Y, Z, W \in M_x$, $x \in M$. Hence, if we suppose that (TM, G) is locally symmetric, i.e., $\bar\nabla \bar R = 0$ identically, then we have, in particular, by virtue of (4.13), $\nabla R = 0$ identically. This completes the proof. □

Proof of Theorem 0.3. Let G be any Riemannian g-natural metric on TM.

We start by proving the heredity of the constant sectional curvature. Suppose that (TM, G) is a space of constant sectional curvature K. Then we have, in particular,

$$\bar{R}_{(x,u)}(X^h, Y^h)Z^h = K\big(G_{(x,u)}(Y^h, Z^h)X^h - G_{(x,u)}(X^h, Z^h)Y^h\big), \tag{4.14}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $(x, u) \in TM$. If we take $u = 0$ in (4.14) and we use the first identity of (1.13), then we get

$$\bar{R}_{(x,0)}(X^h, Y^h)Z^h = K(\alpha_1 + \alpha_3)(0)\big(g_x(Y, Z)X^h_{(x,0)} - g_x(X, Z)Y^h_{(x,0)}\big). \tag{4.15}$$

Substituting from (4.1) into (4.15), we deduce that

$$\big[R(X, Y)Z\big]^h_{(x,0)} = \big[K(\alpha_1 + \alpha_3)(0)(g_x(Y, Z)X - g_x(X, Z)Y)\big]^h_{(x,0)}. \tag{4.16}$$

Since the map $X \to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,0)}$, formula (4.16) implies that

$$R_x(X, Y)Z = K(\alpha_1 + \alpha_3)(0)\big(g_x(Y, Z)X - g_x(X, Z)Y\big), \tag{4.17}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $x \in M$, which shows that (M, g) is a space of constant sectional curvature $K(\alpha_1 + \alpha_3)(0)$.

We shall now prove the second part of Theorem 0.3, i.e., the heredity of the constant scalar curvature. Let us evaluate the scalar curvature $\bar{S}$ of (TM, G) at an arbitrary point $(x, 0)$ in the zero section of TM, $x \in M$.

Notice that the F-tensor fields A, B, C, D, E and F, of Proposition 1.5, are of the form (4.4). For an arbitrary F-tensor field of the form (4.4), we have by virtue of (1) of Lemma 2.3 and (1) of Lemma 4.1

$$(\nabla_X P_u)(Y, Z) = f_1^P(\nabla_X R)(Y, u)Z + f_2^P(\nabla_X R)(Z, u)Y + f_7^P g\big((\nabla_X R)(Y, u)Z, u\big),$$

for all X, Y, Z and $u \in M_x$. It follows that

$$\nabla_X P_0 = 0. \tag{4.18}$$

Also, we have, by virtue of (2) of Lemma 2.3 and (2) of Lemma 4.1

$$\begin{aligned} d(P_{(X,Y)})_u(Z) = 2g(Z, u)\big\{&(f_1^P)'R(X, u)Y + (f_2^P)'R(Y, u)X + (f_3^P)'g(Y, u)X \\ &+ (f_4^P)'g(X, u)Y + \big[(f_5^P)'g(X, Y) + (f_6^P)'.g(X, u)g(Y, u) \\ &+ (f_7^P)'g(R(X, u)Y, u)\big]u\big\} + f_1^P R(X, Z)Y + f_2^P R(Y, Z)X \\ &+ f_3^P g(Y, Z)X + f_4^P g(X, Z)Y + \big[f_5^P g(X, Y) \\ &+ f_6^P g(X, u)g(Y, u) + f_7^P g(R(X, u)Y, u)\big]X \\ &\big\{f_6^P\big[g(X, Z)g(Y, u) + g(Y, Z)g(X, u)\big] \\ &+ f_7^P\big[g(R(X, Z)Y, u) + g(R(X, u)Y, Z)\big]\big\}u, \end{aligned}$$

for all X, Y, Z and $u \in M_x$. It follows that

$$\begin{aligned} d(P_{(Y,Z)})_0(X) = f_1^P(0)R(Y, X)Z + f_2^P(0)R(Z, X)Y \\ + f_3^P(0)g(X, Z)Y + f_4^P(0)g(X, Y)Z + f_5^P(0)g(Y, Z)X. \end{aligned} \tag{4.19}$$

Using (4.5), (4.18) and (4.19), we deduce from Propositions 1.5 and 3.1 that

$$\bar{R}_{(x,0)}(X^h, Y^h)Z^h = \big[R(X,Y)Z\big]^h, \tag{4.20}$$

$$\begin{aligned}\bar{R}_{(x,0)}(X^h, Y^v)Z^h = {}& \frac{\alpha_2(0)}{2\alpha(0)} h\big\{\alpha_1(0)\big[R(X,Y)Z + R(Z,Y)X\big] \\ & - (\beta_1+\beta_3)(0)\big[g(Y,Z)X + g(X,Y)Z\big] - 2(\alpha_1+\alpha_3)'(0)g(X,Z)Y\big\} \\ & + \frac{1}{2\alpha(0)} v\big\{-2(\alpha_2(0))^2 R(X,Y)Z + \alpha_1(0)(\alpha_1+\alpha_3)(0)R(X,Z)Y \\ & + (\alpha_1+\alpha_3)(0)(\beta_1+\beta_3)(0)\big[g(Y,Z)X + g(X,Y)Z\big] \\ & + 2(\alpha_1+\alpha_3)(0)(\alpha_1+\alpha_3)'(0)g(X,Z)Y\big\},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\bar{R}_{(x,0)}(X^v, Y^v)Z^v = {}& \frac{1}{\alpha(0)}\left\{h\left\{\left[-\alpha_1\left(\alpha_2' - \frac{\beta_2}{2}\right) + \alpha_2(2\alpha_1' - \beta_1)\right](0)\big[g(Y,Z)X - g(X,Z)Y\big]\right\}\right. \\ & \left. + v\left\{\left[\alpha_2\left(\alpha_2' - \frac{\beta_2}{2}\right) - (\alpha_1+\alpha_3)(2\alpha_1' - \beta_1)\right](0)\big[g(Y,Z)X - g(X,Z)Y\big]\right\}\right\}.\end{aligned} \tag{4.22}$$

Here, the lifts are taken at $(x,0)$.

Let $\{E_1,\dots,E_m\}$ be an orthonormal basis for M_x. Then putting

$$F_i = \frac{1}{\sqrt{(\alpha_1+\alpha_3)(0)}}.E_i^h \quad \text{and}$$

$$F_{m+i} = \frac{\alpha_2(0)}{\sqrt{\alpha(0)(\alpha_1+\alpha_3)(0)}}.E_i^h - \frac{\sqrt{(\alpha_1+\alpha_3)(0)}}{\sqrt{\alpha(0)}}.E_i^v; \quad i = 1,\dots,m, \tag{4.23}$$

we get an orthonormal basis $\{F_1,\dots,E_{2m}\}$ for the tangent space $(TM)_{(x,0)}$. Here, the lifts are also taken at $(x,0)$. Note that each F_A, $A = 1,\dots,2m$, is well-defined due to formulas (1.14).

The scalar curvature $\bar{S}$ is, by definition, given by

$$\begin{aligned}\bar{S}_{(x,0)} = {}& \sum_{\substack{A,B\\ A\neq B}} G\big(\bar{R}(F_A,F_B)F_B, F_A\big) \\ = {}& \sum_{i\neq j} G\big(\bar{R}(F_i,F_j)F_j, F_i\big) + \sum_{i\neq j} G\big(\bar{R}(F_{m+i},F_{m+j})F_{m+j}, F_{m+i}\big) \\ & + 2\sum_{i,j} G\big(\bar{R}(F_i,F_{m+j})F_{m+j}, F_i\big) \\ = {}& \sum_{i\neq j}\left\{\frac{1}{((\alpha_1+\alpha_3)(0))^2}\left(1 + \frac{(\alpha_2(0))^2}{\alpha(0)}\right)^2 G\big(\bar{R}(E_i^h,E_j^h)E_j^h, E_i^h\big)\right. \\ & + \frac{2}{\alpha(0)}\left(1 + \frac{(\alpha_2(0))^2}{\alpha(0)}\right)\left[\frac{-2\alpha_2(0)}{(\alpha_1+\alpha_3)(0)} G\big(\bar{R}(E_i^h,E_j^v)E_j^h, E_i^h\big)\right. \\ & \left. + G\big(\bar{R}(E_i^h,E_j^v)E_j^v, E_i^h\big)\right] + \frac{2(\alpha_2(0))^2}{(\alpha(0))^2}\big[G\big(\bar{R}(E_i^h,E_j^h)E_j^v, E_i^v\big)\end{aligned}$$

$$+ G\big(\bar R(E_i^h, E_j^v)E_j^h, E_i^v\big)\big] - 2\frac{(\alpha_1+\alpha_3)(0)}{(\alpha(0))^2}\big[G\big(\bar R(E_i^v, E_j^v)E_j^v, E_i^h\big)$$
$$+ G\big(\bar R(E_i^v, E_j^v)E_j^h, E_i^v\big)\big] + \frac{((\alpha_1+\alpha_3)(0))^2}{(\alpha(0))^2}G\big(\bar R(E_i^v, E_j^v)E_j^v, E_i^v\big)\Big\}$$
$$+ \frac{2}{\alpha(0)}\sum_i G\big(\bar R(E_i^h, E_i^v)E_i^v, E_i^h\big).$$

Using the three formulas (4.20)–(4.22) of the Riemannian curvature tensor $\bar R_{(x,0)}$ and the identities of (1.13), we find

$$\begin{aligned}
G_{(x,0)}\big(\bar R(E_i^h, E_j^h)E_j^h, E_i^h\big) &= (\alpha_1+\alpha_3)(0)g\big(R(E_i, E_j)E_j, E_i\big),\\
G_{(x,0)}\big(\bar R(E_i^h, E_j^v)E_j^h, E_i^h\big) &= \alpha_2(0)g\big(R(E_i, E_j)E_j, E_i\big),\\
G_{(x,0)}\big(\bar R(E_i^h, E_j^v)E_j^v, E_i^h\big) &= -(\alpha_1+\alpha_3)'(0) - (\beta_1+\beta_3)(0)(\delta_{ij})^2,\\
G_{(x,0)}\big(\bar R(E_i^h, E_j^h)E_j^v, E_i^v\big) &= G\big(\bar R(E_i^h, E_j^v)E_j^h, E_i^v\big) - G\big(\bar R(E_j^h, E_j^v)E_i^h, E_i^v\big)\\
&= \alpha_1(0)g\big(R(E_i, E_j)E_j, E_i\big),\\
G_{(x,0)}\big(\bar R(E_i^h, E_j^v)E_j^h, E_i^v\big) &= \frac{\alpha_1(0)}{2}g\big(R(E_i, E_j)E_j, E_i\big) + \frac{(\beta_1+\beta_3)(0)}{2}\\
&\quad + \left[\frac{(\beta_1+\beta_3)(0)}{2} + (\alpha_1+\alpha_3)'(0)\right](\delta_{ij})^2,\\
G_{(x,0)}\big(\bar R(E_i^v, E_j^v)E_j^v, E_i^h\big) &= \left[\frac{\beta_2(0)}{2} - \alpha_2'(0)\right]\big[1-(\delta_{ij})^2\big],\\
G_{(x,0)}\big(\bar R(E_i^v, E_j^v)E_i^h, E_j^v\big) &= \left[\frac{\beta_2(0)}{2} - \alpha_2'(0)\right]\big[1-(\delta_{ij})^2\big],\\
G_{(x,0)}\big(\bar R(E_i^v, E_j^v)E_i^v, E_j^v\big) &= \big[\beta_1(0) - 2\alpha_1'(0)\big]\big[1-(\delta_{ij})^2\big].
\end{aligned}$$

Hence, by virtue of

$$1 + \frac{(\alpha_2(0))^2}{\alpha(0)} = \frac{\alpha_1(0)(\alpha_1+\alpha_3)(0)}{\alpha(0)} \quad \text{and} \quad S = \sum_{i\neq j} g\big(R(E_i, E_j)E_j, E_i\big),$$

we deduce by simple calculation that

$$\begin{aligned}
\bar S_{(x,0)} = \frac{\alpha_1(0)}{\alpha(0)}.S_x + \frac{m}{(\alpha(0))^2}\big\{&-2\big[(m-1)\alpha_1(0)(\alpha_1+\alpha_3)(0)\\
&+\alpha(0)\big](\alpha_1+\alpha_3)'(0) + \big[(m-1)(\alpha_2(0))^2 - 2\alpha(0)\big](\beta_1+\beta_3)(0)\\
&+(m-1)\big[((\alpha_1+\alpha_3)(0))^2(\beta_1(0) - 2\alpha_1'(0))\\
&-2\alpha_2(0)(\alpha_1+\alpha_3)(0)(\beta_2(0) - 2\alpha_2'(0))\big]\big\},
\end{aligned} \tag{4.24}$$

where S denotes the scalar curvature of (M, g).

Now, if $\bar S$ is a constant $\bar S_0$ on TM, then in particular the function $x \mapsto \bar S_{(x,0)}$ is constant on M equal to $\bar S_0$. It follows then from (4.24) that S is constant. More precisely, we have

$$S = \frac{\alpha(0)}{\alpha_1(0)}\bar S_0 - \frac{m}{\alpha(0)\alpha_1(0)}\big\{-2\big[(m-1)\alpha_1(0)(\alpha_1+\alpha_3)(0) + \alpha(0)\big](\alpha_1+\alpha_3)'(0)$$

$$+\big[(m-1)(\alpha_2(0))^2-2\alpha(0)\big](\beta_1+\beta_3)(0)+(m-1)\big[((\alpha_1+\alpha_3)(0))^2(\beta_1(0)-2\alpha_1'(0))-2\alpha_2(0)(\alpha_1+\alpha_3)(0)(\beta_2(0)-2\alpha_2'(0))\big]\big\}. \qquad \square$$

Proof of Theorem 0.4. Fix $x\in M$. As in the proof of Theorem 0.3, we consider the orthonormal basis $\{F_1,\dots,F_{2m}\}$ of $(TM)_{(x,0)}$, given by (4.23), where $\{E_1,\dots,E_m\}$ is an orthonormal basis of M_x. Then the Ricci tensor field $\overline{\mathrm{Ric}}$ of (TM,G) is, by definition, given by

$$\overline{\mathrm{Ric}}_{(x,0)}(V,W)=\sum_{i=1}^{m}\big[G_{(x,0)}(\bar R(V,F_i)F_i,W)+G_{(x,0)}(\bar R(V,F_{m+i})F_{m+i},W)\big], \tag{4.25}$$

for all $V,W\in(TM)_{(x,0)}$. If we put $V=X^h$ and $W=Y^h$, for $X,Y\in M_x$, then (4.25) becomes

$$\begin{aligned}\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h)=\sum_{i=1}^{m}\bigg\{&\frac{\alpha_1(0)}{\alpha(0)}G_{(x,0)}\big(\bar R(X^h,E_i^h)E_i^h,Y^h\big)\\&-\frac{\alpha_2(0)}{\alpha(0)}\big[G_{(x,0)}(\bar R(X^h,E_i^h)E_i^v,Y^h)+G_{(x,0)}(\bar R(X^h,E_i^v)E_i^h,Y^h)\big]\\&+\frac{(\alpha_1+\alpha_3)(0)}{\alpha(0)}G_{(x,0)}(\bar R(X^h,E_i^v)E_i^v,Y^h)\bigg\}.\end{aligned} \tag{4.26}$$

Using (4.20) and the first identity of (1.13), we obtain

$$G_{(x,0)}\big(\bar R(X^h,E_i^h)E_i^h,Y^h\big)=(\alpha_1+\alpha_3)(0)g\big(R(X,E_i)E_i,Y\big). \tag{4.27}$$

Similarly, using (4.20) and the second identity of (1.13), we get

$$G_{(x,0)}\big(\bar R(X^h,E_i^h)E_i^v,Y^h\big)=\alpha_2(0)g\big(R(X,E_i)E_i,Y\big), \tag{4.28}$$

$$G_{(x,0)}\big(\bar R(X^h,E_i^v)E_i^h,Y^h\big)=\alpha_2(0)g\big(R(X,E_i)E_i,Y\big). \tag{4.29}$$

Finally, using (4.21) and the third identity of (1.13), we find

$$\begin{aligned}&G_{(x,0)}\big(\bar R(X^h,E_i^v)E_i^v,Y^h\big)=-G_{(x,0)}\big(\bar R(X^h,E_i^v)Y^h,E_i^v\big)\\&\quad=-\frac{(\alpha_2(0))^2}{2\alpha(0)}\big\{2\alpha_1(0)g(R(X,E_i)Y,E_i)-2(\beta_1+\beta_3)(0)g(X,E_i)g(Y,E_i)\\&\qquad-2(\alpha_1+\alpha_3)'(0)g(X,Y)\big\}-\frac{\alpha_1(0)}{2\alpha(0)}\big\{-2(\alpha_2(0))^2g(R(X,E_i)Y,E_i)\\&\qquad+2(\alpha_1+\alpha_3)(0)\big[(\beta_1+\beta_3)(0)g(X,E_i)g(Y,E_i)+(\alpha_1+\alpha_3)'(0)g(X,Y)\big]\big\}.\end{aligned}$$

It follows that

$$\begin{aligned}G_{(x,0)}\big(\bar R(X^h,E_i^v)E_i^v,Y^h\big)=&-(\beta_1+\beta_3)(0)g(X,E_i)g(Y,E_i)\\&-(\alpha_1+\alpha_3)'(0)g(X,Y).\end{aligned} \tag{4.30}$$

Substituting from (4.27)–(4.30) into (4.26), we deduce that

$$\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h)=\sum_{i=1}^{m}\bigg\{\frac{(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)(0)}{\alpha(0)}g\big(R(X,E_i)E_i,Y\big)$$

$$- \frac{(\alpha_1+\alpha_3)(0)}{\alpha(0)}\big[(\beta_1+\beta_3)(0)g(X,E_i)g(Y,E_i) + (\alpha_1+\alpha_3)'(0)g(X,Y)\big]\bigg\}. \tag{4.31}$$

But since $\{E_1,\dots,E_m\}$ is an orthonormal basis of M_x, then we have

$$\sum_{i=1}^{m} g(X,E_i)g(Y,E_i) = g(X,Y).$$

It follows that

$$\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h) = \frac{1}{\alpha(0)}\bigg\{(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)(0)\sum_{i=1}^{m} g(R(X,E_i)E_i,Y) - (\alpha_1+\alpha_3)(0)\big[(\beta_1+\beta_3)+m(\alpha_1+\alpha_3)'\big](0)g(X,Y)\bigg\}. \tag{4.32}$$

If Ric denotes the Ricci tensor field of (M,g), then (4.32) transforms to

$$\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h) = \frac{1}{\alpha(0)}\big\{(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)(0)\mathrm{Ric}_x(X,Y) - (\alpha_1+\alpha_3)(0)\big[(\beta_1+\beta_3)+m(\alpha_1+\alpha_3)'\big](0)g(X,Y)\big\}. \tag{4.33}$$

Now, if (TM,G) is an Einstein manifold, i.e., $\overline{\mathrm{Ric}} = \lambda G$, for a constant $\lambda\in\mathbb{R}$, then we have in particular

$$\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h) = \lambda G_{(x,0)}(X^h,Y^h) = \lambda(\alpha_1+\alpha_3)(0)g(X,Y), \tag{4.34}$$

for all $x\in M$ and $X,Y\in M_x$.

If we have $(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)(0)\neq 0$, then substituting from (4.34) into (4.33), we deduce that

$$\mathrm{Ric}_x(X,Y) = \frac{(\alpha_1+\alpha_3)[\lambda\alpha+(\beta_1+\beta_3)+m(\alpha_1+\alpha_3)']}{(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)}(0)g_x(X,Y), \tag{4.35}$$

for all $x\in M$ and $X,Y\in M_x$. It follows that (M,g) is an Einstein manifold.

If $(\alpha_1(\alpha_1+\alpha_3)-2\alpha_2^2)(0)=0$, then $\alpha(0)=(\alpha_2(0))^2$ and, in particular, $\alpha_2(0)\neq 0$. By similar way as for the computation of $\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^h)$, we can calculate $\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^v)$ to find

$$\overline{\mathrm{Ric}}_{(x,0)}(X^h,Y^v) = \frac{1}{2\alpha(0)}\big\{-\alpha_1(0)\alpha_2(0)\mathrm{Ric}_x(X,Y) + \big[-\alpha_2(0)\big[(m+1)(\beta_1+\beta_3)(0)+2(\alpha_1+\alpha_3)'(0)\big] + (m-1)(\alpha_1+\alpha_3)(0)(\beta_2-2\alpha_2')\big]g(X,Y)\big\}. \tag{4.36}$$

Using the fact that (TM,G) is an Einstein manifold, (4.36) gives, by virtue of $\alpha_1(0)\neq 0$ and $\alpha_2(0)\neq 0$,

$$\mathrm{Ric}_x(X,Y) = \frac{2}{\alpha_1(0)\alpha_2(0)}\big\{-\lambda\alpha(0)\alpha_2(0) + \big[-\alpha_2(0)\big[(m+1)(\beta_1+\beta_3)(0) + 2(\alpha_1+\alpha_3)'(0)\big] + (m-1)(\alpha_1+\alpha_3)(0)(\beta_2-2\alpha_2')\big]\big\}g_x(X,Y),$$

for all $x\in M$ and $X,Y\in M_x$. It follows, also in this case, that (M,g) is an Einstein manifold. □

Proof of Theorem 0.5. It follows immediately from Theorem A.2 in Appendix A below. □

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Appendix A

In [17], Oproiu defined a family of Riemannian metrics on TM, which depends on 2 arbitrary functions of one variable, in the following way:

For any two smooth functions $v, w : \mathbb{R}^+ \to \mathbb{R}$, such that $v(t) > 0$ and $v(t) + 2tw(t) > 0$, for all $t \in \mathbb{R}^+$, consider the metric $G_{v,w}$ on TM given locally by:

$$G_{v,w} = \zeta(u)_{ij} dx^i\, dx^j + \xi(u)_{ij} \nabla u^i \nabla u^j, \tag{A.1}$$

where

(a) ζ (respectively ξ) is the F-metric on M defined by:

$$\zeta(u; X, Y) = v(\tau) g(X, Y) + w(\tau) g(X, u) g(Y, u)$$
$$\text{(respectively } \xi(u; X, Y) = \frac{1}{v(\tau)} g(X, Y) - \frac{w(\tau)}{v(\tau)(v(\tau) + 2tw(\tau))} g(X, u) g(Y, u)),$$

τ being the energy density, i.e. $\tau = \frac{1}{2} g(u, u)$.

(b) $\nabla u^i = du^i + \Gamma^i_{jk} u^j dx^k$ is the absolute differential of u^i with respect to the Levi-Civita connection ∇ of g.

It is easy to check that $G_{v,w}$ is a Riemannian g-natural metric, where the defining functions α_i, β_i, $i = 1, 2, 3$, satisfy the following equalities:

$$\begin{cases} (\alpha_1 + \alpha_3)(t) = v(t/2), \quad (\beta_1 + \beta_3)(t) = w(t/2), \\ \alpha_1(t) = \frac{1}{v(t/2)}, \quad \beta_1(t) = -\frac{w(t/2)}{v(t/2)(v(t/2) + tw(t/2))}, \\ \alpha_2(t) = \beta_2(t) = 0, \end{cases} \tag{A.2}$$

for all $t \in \mathbb{R}^+$. Now, the following result was proved in [17] (see also [18]):

Theorem A.1. *Let (M, g) be a space of negative constant sectional curvature K and let $v, w : \mathbb{R}^+ \to \mathbb{R}$ be the functions given by*:

$$v(t) = A + \sqrt{A^2 - 2Kt} \quad \text{and} \quad w(t) = -\frac{2K}{A} + \frac{K}{A + \sqrt{A^2 - 2Kt}} \quad \text{with } A > 0. \tag{A.3}$$

Then $(TM, G_{v,w})$ is a Kaehler Einstein manifold.

Note that TM, endowed with any $G_{v,w}$ of Theorem A.1, is a locally symmetric space (cf. [19]).

In the following theorem, we shall show additional properties of the metrics $G_{v,w}$:

Theorem A.2. *Let (M, g) be an m-dimensional space of negative constant sectional curvature K, where $m \geqslant 3$. Then, for any functions v, w given by (A.3), the Riemannian g-natural metric $G_{v,w}$ provides TM with a structure of a space of positive constant scalar curvature.*

Further, for every choice of the constants $K < 0$ and $\bar{S}_0 > 0$, there exists $A > 0$ such that the positive constant scalar curvature of $(TM, G_{v,w})$ is exactly $\bar{S}_0 > 0$.

Proof. Suppose that (M, g) is a space of constant scalar curvature $K < 0$. By virtue of Theorem A.1, $(TM, G_{v,w})$ is an Einstein manifold, for any v, w given by (A.3), and hence a space of constant scalar curvature $\bar{S}_0$. We claim that $\bar{S}_0 > 0$. In fact, since $\alpha_2 = \beta_2 = 0$, $\alpha(0) = \alpha_1(0)(\alpha_1 + \alpha_3)(0)$ and the scalar curvature of (M, g) is equal to $S = m(m-1)K$, (4.24) reduces to

$$\bar{S}_0 = \frac{m(m-1)}{(\alpha_1+\alpha_3)(0)}K + \frac{m}{(\alpha(0))^2}\big\{-2m\alpha(0)(\alpha_1+\alpha_3)'(0) \\ + (m-1)((\alpha_1+\alpha_3)(0))^2(\beta_1(0) - 2\alpha_1'(0))\big\}. \tag{A.4}$$

On the other hand, a simple calculation, using (A.2) and (A.3), yields

$$\alpha_1(0) = \frac{1}{2A}, \quad (\alpha_1+\alpha_3)(0) = 2A, \quad \alpha(0) = 1,$$
$$(\alpha_1+\alpha_3)'(0) = \frac{v'(0)}{2} = -\frac{K}{2A}, \quad \beta_1(0) = -\frac{3K}{8A^3},$$
$$\alpha_1'(0) = -\frac{v'(0)}{2(v(0))^2} = \frac{K}{8A^3}.$$

Substituting into (A.4), we find that

$$\bar{S}_0 = -\frac{m(m-2)K}{A},$$

which is positive for $m \geqslant 3$, since $K < 0$.

For an arbitrary constant $\bar{S}_0 > 0$, if we consider (M, g) a space of arbitrary constant sectional curvature $K < 0$ and v, w given by (A.3), with $A = -\frac{m(m-2)K}{\bar{S}_0}$, then $(TM, G_{v,w})$ is, clearly, a space of constant scalar curvature $\bar{S}_0$. □

Remarks A.3. (1) The family of Riemannian metrics on TM considered by Oproiu is, exactly, the family of Riemannian g-natural metrics on TM characterized by:

- horizontal and vertical distributions are orthogonal,
- $\alpha = \phi = 1$.

Indeed, from (A.2), we have $\alpha = \alpha_1(\alpha_1 + \alpha_3) = 1$ and $\phi = \phi_1(\phi_1 + \phi_3) = (\alpha_1 + t\beta_1)((\alpha_1 + \alpha_3) + t(\beta_1 + \beta_3)) = 1$. Conversely, if $\alpha_2 = \beta_2 = 0$ and α_1, $\alpha_1 + \alpha_3$, β_1 and $\beta_1 + \beta_3$ are given in such a way that $\alpha = \phi = 1$, i.e., $\alpha_1 = \frac{1}{\alpha_1+\alpha_3}$ and $\phi_1 = \frac{1}{\phi_1+\phi_3}$, then we define v and w by: $v(t) = (\alpha_1 + \alpha_3)(2t)$ and $w(t) = (\beta_1 + \beta_3)(2t)$. It is easy to see, by virtue of (A.2), that $G_{v,w}$ is no other than the metric defined by the given α_i, β_i, $i = 1, 2, 3$, via Proposition 1.2.

(2) Let (M, g) be a space of negative constant sectional curvature K. Another g-natural metric on TM, apart the Sasaki metric, which has vanishing scalar curvature is given by (cf. (4.24)):

$$\alpha_1 = \alpha_1 + \alpha_3 = 1, \quad \alpha_2 = \beta_2 = 0, \quad \beta_1 = \frac{-K}{1 - Kt}, \quad \beta_1 + \beta_3 = -K. \tag{A.5}$$

In this case, (TM, G) is locally symmetric (Theorem 8 of [18]).

(3) If, in the conditions of the preceding remark, we choose $K > 0$, then we have also a structure of locally symmetric Riemannian manifold, but in the tube around the zero section in TM, defined by $\|u\|^2 < \frac{1}{2K}$—not on whole TM—(Theorem 8 of [18]).

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