The topology of elementary submodels

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Received 30 October 1996; revised 18 February 1997

Abstract

Given a topological space \((X, T)\), we take an elementary submodel \(M\) of a sufficiently large initial fragment of the universe containing \((X, T)\) and naturally define a space \(X_M\). We consider under what circumstances \(X_M\) is a (nice image of a) subspace of \(X\), and what properties of \(X\) are retained by \(X_M\). © 1998 Elsevier Science B.V.

Keywords: Elementary submodel; Compact; Normal; Cardinal invariants; Reflection; Pointwise countable type

AMS classification: Primary 54A10; 03C62; 54D30, Secondary 54A25; 54B99; 54D15; 54D55

This paper is dedicated to the memory of Kiiti Morita, who profoundly influenced the development of topology in Japan. On a personal note, the second author notes that the first time he met Professor Morita was also the first time he ever had Japanese food, in 1969 or '70 at the home of Professor Kunio Murasugi. Since then he has always associated good mathematics with good food!

1. Introduction

Elementary submodels have been playing an increasingly prominent role in set-theoretic topology over the past few years, e.g., [6,7,11,13,25]. Our approach in this note is rather different than in these other papers—we will consider the operation of taking an elementary submodel to be yet another operation, like taking a subspace or an...
image under a nice map, and our goal will be to see what is preserved under this operation. The genesis of this article is twofold: the realization that many of the techniques we have used in reflection arguments, e.g., [23,17], are applicable in general elementary submodel contexts, and a comment by S. Todorcevic that if we were going to study preservation of topological properties by forcing, we should also study preservation by elementary extension. Rather than repeating the introductory material in the other papers, we will assume the reader has been exposed to the basics before. We particularly recommend Section 4 of [13].

To fix notation, we consider the following general situation: \( (X, T) \) is a topological space; \( M \) is an elementary submodel of \( H_\theta \) (the set of all sets of hereditary cardinality \( < \theta, \theta \) a cardinal) containing \( X \) and \( T \). For purposes of intuition, one may sloppily think of \( H_\theta \) as \( V \), the class of all sets, since we always take \( \theta \) sufficiently large so that \( H_\theta \) contains all sets of interest in the context under discussion. Where convenient, we will omit mention of \( H_\theta \) entirely. Let \( X_M \) be \( X \cap M \) with the topology \( T_M \) on \( X_M \) generated by \( T \cap M = \{ U \cap M : U \in T \cap M \} \) (note \( T \cap M \) is a basis since by elementarity it is closed under finite intersections and contains \( \emptyset \) and \( X \cap M \)). The question is, how do \( (X, T) \) and \( M \) constrain \( X_M \)? One might expect to answer "not much", since we are merely weakening the topology of a subspace of \( X \), but we do get some nontrivial results, e.g.:

**Theorem 1.1.** If \( (X, T) \) is \( T_i \), \( i \leq 3 \frac{1}{2} \), so is \( X_M \). This is not true for \( T_4, T_5, \) or \( T_6 \).

**Proof.** The negative case is proved by Examples 7.15, 7.19 and 7.20 in Section 7. We will only show the positive case \( i = 3 \frac{1}{2} \). For that, it is enough to show that for every \( x \in X \cap M \) and for every \( V \in T \cap M \) containing \( x \), there is a continuous function \( f : X_M \to I \), such that \( f(x) = 0 \) and \( f(y) = 1 \) for every \( y \in (X \setminus V) \cap M \).

Fix \( x \in X \cap M \) and \( V \in T \cap M \) such that \( x \in V \). Then

\[
H_\theta \models "x \in X, V \in T \text{ and } x \in V".
\]

Since \( (X, T) \) is \( T_{3 \frac{1}{2}} \),

\[
H_\theta \models "\text{there is a continuous } g : X \to I \text{ such that } g(x) = 0 \text{ and } g(y) = 1 \text{ for every } y \in X \setminus V".
\]

Thus, by elementarity, \( M \) models the same thing, and therefore in \( M \) there is a function \( f : X_M \to I \cap M \), which \( M \) thinks is continuous, such that \( f(x) = 0 \) and \( f(y) = 1 \) for every \( y \in (X \setminus V) \cap M \). But then

\[
H_\theta \models "f : X_M \to I \cap M \text{ is continuous}"
\]

(since \( M \) contains a base for \( I \)), and therefore \( f : X_M \to I \) is as we wanted. \( \Box \)

**Note.** We will loosely quote such inheritance results as, e.g., "an elementary submodel of a \( T_i \) space is \( T_i \), for \( i \leq 3 \frac{1}{2} \)."
The operation of taking an elementary submodel of a topological space will obviously give different results for the many different kinds of spaces; at present though, the classification of elementary submodels seems rather crude in comparison—in this paper we just use “arbitrary”, “countable”, “uncountable”, “countably closed” and “$\omega$-covering”. This may suggest further study. The reason that countable elementary submodels will not play a prominent role is

**Corollary 1.2** (to Theorem 1.1). A countable elementary submodel of a $T_3$ space is metrizable.

**Proof.** It is $T_3$ with a countable basis. □

This result shows that topologically characterizing the elementary submodels of $T_3$ nonmetrizable spaces is doomed to failure; perhaps one can however characterize all those elementary submodels of a certain size, or of certain spaces.

We will be interested in when $X_M$ is a (nice image of a) subspace of $X$. Here are some sample results:

**Theorem 1.3.** If $\langle X, T \rangle$ is first countable, then $X_M$ is a subspace of $\langle X, T \rangle$. This is not true in general.

**Theorem 1.4.** If $\langle X, T \rangle$ is first countable $T_2$ and $M$ is countably closed, then $X_M$ is a closed subspace of $\langle X, T \rangle$. None of the three hypotheses can be omitted.

**Theorem 1.5.** If $\langle X, T \rangle$ is $T_3$ and of pointwise countable type, then $X_M$ is a perfect image of a subspace of $X$. It need not be a perfect image of a closed subspace, even if $X$ is compact. It need not be a perfect image of a subspace of $X$ if $X$ is not necessarily of pointwise countable type, even if $X$ is a normal Fréchet space.

From Theorem 1.5 we get that properties that are hereditary and preserved by perfect maps are preserved by elementary submodels of spaces of pointwise countable type. A nontrivial instance is:

**Corollary 1.6.** Assume $\langle X, T \rangle$ is $T_3$ and of pointwise countable type. Then $X_M$ is $T_3$. The hypothesis of pointwise countable type cannot be omitted.

There are two easy results from classical model theory that we will use several times:

**Theorem 1.7** (Löwenheim–Skolem Theorem). $H_\theta$ ($\theta$ regular uncountable) has elementary submodels of all infinite cardinalities less than $\theta$. It has countably closed elementary submodels of all cardinalities $\lambda < \theta$ which are of form $\lambda = $ $\kappa^{+\theta}$.

**Theorem 1.8.** Given $\{M_\alpha\}_{\alpha < \beta}$ such that each $M_\alpha$ is an elementary submodel of $H_\theta$ and $\alpha < \alpha' < \beta$ implies $M_\alpha$ is an elementary submodel of $M_{\alpha'}$, then $\bigcup_{\alpha < \beta} M_\alpha$ is an elementary submodel of $H_\theta$. 


The plan of the paper is as follows. First we investigate the circumstances under which $X_M$ is a subspace of $X$. Then we consider when it is a nice image of a subspace of $X$. Then we compare the cardinal invariants of $X$ and $X_M$. Then we look at another method of determining a new space, given a space and an elementary submodel. Then we compare the elementary submodel context to the (large cardinal) reflection context. The final section contains many examples illustrating the necessity of our various hypotheses. Since some of them serve several purposes, we decided to collect them all together. The reader is expected to flip back and forth to this section as s/he goes along.

Our investigations in this new area are still in the preliminary stage. It is not yet clear what are the important problems. Nonetheless, having accumulated a large number of results, we decided to publish now in the hope of stimulating other researchers to look at this area.

2. Subspaces

When is $X_M$ a subspace of $X$? First, some definitions.

**Definition 2.1.** A space $X$ is a $k$-space if a subset of $X$ is closed if and only if its intersection with every compact subset of $X$ is closed. A space $X$ is a $k'$-space [2, Definition 3.3] if for any $x \in X$, we have that for any $A \subseteq X$, $x \in A$ if and only if for some compact $K \subseteq X$, $x \in A \cap K$. A space $X$ is Fréchet if whenever $A \subseteq X$ and $x \in \overline{A}$, there is a sequence from $A$ converging to $x$. A $\subseteq X$ is sequentially closed if any sequence from $A$ that converges, converges to a point in $A$. $X$ is sequential if every sequentially closed set is closed.

**Definition 2.2.** Let $X$ be a topological space and $A \subseteq X$. A family $B$ of open sets including $A$ is called an outer base for $A$ if for every open $V \supseteq A$, there is a $B \in B$ such that $B \subseteq V$. The character of $A$ in $X$ is

$$\chi(A, X) = \min \{ \tau : \text{there is an outer base for } A \text{ of cardinality } \tau \}.$$

**Definition 2.3.** A space $X$ has pointwise countable type if, for each $x \in X$, there is a compact $K$ in $X$ such that $x \in K$ and $\chi(K, X) \leq \aleph_0$.

**Definition 2.4.** The tightness of a point $x$ in a space $X$ is the smallest cardinal number $\kappa \geq \aleph_0$ with the property that if $x \in \overline{C}$, then there exists a $C_0 \subseteq C$ such that $|C_0| \leq \kappa$ and $x \in \overline{C_0}$.

Fréchet spaces are $k'$ and sequential; sequential spaces are $k$ and have countable tightness; $k'$-spaces are $k$-spaces. None of these implications reverses; however a space is Fréchet if and only if it is hereditarily $k$ [3, Theorem 1]. Neither of the classes of $k'$-spaces and spaces of pointwise countable type includes the other [2, Section 3]. (Locally) compact $T_2$ spaces are of pointwise countable type.

Our first result is folklore.
Theorem 2.5. If $X$ is first countable, $X_M$ is a subspace of $X$.

Proof. Fix $x \in X \cap M$ and $U \in T$ containing $x$. Since $X$ is first countable and $x \in M$, by elementarity we have that there is a $B_x \in M$ such that $B_x$ is countable and is a base at $x$. But then $B_x \subseteq M$ and therefore there is a $V \in T \cap M$ such that $x \in V$ and $V \subseteq U$. \□

Examples 7.2–7.4, are such that $X_M$ is not a subspace of $X$. Example 7.13 shows that “first countable” cannot be weakened to “Fréchet”, even if $M$ is countably closed.

Theorem 2.6. If $X$ is sequential $T_2$ and $M$ is countably closed, then $X \cap M$ is a closed subset of $X$.

Proof. It suffices to show that $X \cap M$ is sequentially closed. Take $\{x_n\}_{n<\omega} \subseteq M$, $x_n \to x$. We claim that $x \in M$: $\{x_n\}_{n<\omega}$ converges and $\{x_n\}_{n<\omega} \subseteq M$, so $M$ thinks it converges, so there is an $x'$ such that $M$ thinks $x_n \to x'$; but then it really does, hence by $T_2$, $x' = x$. \□

It follows, for example, that

Corollary 2.7. If $X$ is Lindelöf sequential $T_2$ and $M$ is countably closed, then $X_M$ is Lindelöf.

Proof. Let $\mathcal{U}$ be an open cover of $X_M$. Then $\mathcal{U}$ covers $X \cap M$—which is Lindelöf—so has a countable subcover. \□

Corollary 2.8. If $X$ is first countable $T_2$, and $M$ is countably closed, then $X_M$ is a closed subspace of $X$.

Proof. Immediate. \□

Example 7.5 is an example of a first countable $T_1$ space $X$ and a countably closed $M$ such that $X_M$ is not a subspace of $X$. Example 7.1, with $M$ countable, is an example of a first countable $T_2$ space $X$ and a $M$ such that $X_M = X \cap M$ is not a closed subspace of $X$. Example 7.6 is an example of a compact $T_2$ space and a countably closed $M$ such that $X_M$ is not Lindelöf, even though it is a subspace of $X$.

Corollary 2.9. If $X$ is compact, $T_2$, sequential, and $M$ is countably closed, then $X_M$ is a closed subspace of $X$.

Proof. $X \cap M$ is a closed, hence compact, subset of $X$. $T_M$ is a weaker $T_2$ topology on $X \cap M$, so it is equal to the subspace topology on $X \cap M$. \□

Example 7.4 (with $D$ uncountable) is an example of a compact $T_2$ space and a countably closed $M$ such that $X_M$ is not a subspace of $X$. Example 7.7 is an example of a compact $T_2$ sequential space and an $M$ such that $X_M$ is not a subspace of $X$.
Recall an elementary submodel $M$ is $\omega$-covering if for every countable set $A \subseteq M$ there is a countable set $B \in M$ such that $A \subseteq B$. In ZFC one can construct $\omega$-covering elementary submodels of size $\aleph_1$—see, e.g., [11, Section 3].

The next result is included in the proof of Proposition 3.4 in [11]:

**Proposition 2.10.** If $M$ is $\omega$-covering and has size $\omega_1$, and $X$ has countable tightness and is regular, initially $\omega_1$-compact and such that every countable subspace is first countable, then $X_M$ is a subspace of $X$.

Using a similar proof we will improve Corollary 2.9 by showing

**Theorem 2.11.** If $X$ is $T_2$ with countable tightness and of pointwise countable type, and $M$ is countably closed, then $X_M$ is a subspace of $X$.

We will need the following lemma here and in the next section.

**Lemma 2.12.** Suppose $X$ is $T_2$. If $x \in M$, $K$ is compact such that $x \in K$, $K \in M$ and $\chi(K, X) \leq \kappa$, and $V \in T \cap M$ with $x \in V$, then there is a compact $K'$, $K' \in M$, such that $x \in K' \subseteq K \cap V$ and $\chi(K', X) \leq \kappa$.

**Proof.** Since $K$ is regular and $K \in M$, we have that $M \models K$ is regular. We now work in $M$ only. Using that $x$, $K$, $V$, $T \in M$ and that $M \models K$ is regular, we can construct a sequence $\{V_n: n \in \omega\}$ (in $M$) such that $V_0 = V$, and for each $n \in \omega$, $V_n \in T$, and

$$x \in V_{n+1} \cap K \subseteq \overline{V_{n+1} \cap K} \subseteq V_n \cap K,$$

(all this relative to $M$).

Define $K' = \bigcap_{n \in \omega} \overline{V_n \cap K}$. Note that, since $M \models K$ is closed, we have that $M \models K'$ is closed.

Since all the work was done inside $M$, we have that $K' \in M$. By construction, $x \in K'$. Also, $M \models K'$ is closed and $K' \subseteq V \cap K$. Therefore, by elementarity, we have that $K'$ is closed (in $T$) and $K' \subseteq V \cap K$. Thus, $K'$ is compact (in $\langle X, T \rangle$). Finally, by the construction of $K'$, we have that $\chi(K', K) \leq \aleph_0$. Then, since $\chi(F, X) \leq \chi(F, K)$, $\chi(K', X)$, for any compact $F$ and $K$ such that $F \subseteq K$ (see, e.g., [14, Exercise 3.1E]), $\chi(K', X) \leq \kappa$. So $K'$ is as we want. $\square$

**Proof of Theorem 2.11.** We first prove the special case when $X$ is compact. Suppose that $X_M$ is not a subspace. Then there is a $U \in T$ and $x \in U \cap M$ such that $T \cap M \setminus U \neq \emptyset$, for every $T \in T \cap M$ with $x \in T$.

Since $X$ is compact, we can pick $z \in \bigcap \{\overline{T \cap M \setminus U}: T \in T \cap M \text{ and } x \in T\}$. By countable tightness, there a countable set $D \subseteq X \cap M \setminus U$ such that $z \in \overline{D}$. Since $D$ is a countable subset of $M$, and $M$ is countably closed, we have that $D \in M$. Also, $x \notin \overline{D}$, because $D \subseteq X \cap M \setminus U$ and $x \in U$. Therefore, by elementarity, since $x$ and $D$ are in $M$, we have that $M \models x \notin \overline{D}$, and then there is a $V \in T \cap M$ such that $x \in V$ and
$V \cap D = \emptyset$. By regularity, we can pick $V$ such that $\overline{V} \cap \overline{D} = \emptyset$. But this is a contradiction since $z \in \overline{D} \cap \overline{V}$.

Now for the general case. Fix $x \in X \cap M$ and $U \in \mathcal{T}$ such that $x \in U$. Since $X$ has pointwise countable type and $x \in M$, there is a compact $K$, $K \subseteq M$, such that $x \in K$ and $\chi(K, X) \leqslant \aleph_0$.

Now $K$ is compact and therefore closed, which implies that $K$ has countable tightness (as a subspace of $X$). Since $K \subseteq M$, applying the special case for $K$, we then have that $K_M$ is a subspace of $K$. Thus, since $U \cap K$ is open in $\langle K, \mathcal{T} \upharpoonright K \rangle$, there is a $V \in \mathcal{T} \cap M$ such that $x \in V \cap K \cap M \subseteq U \cap K \cap M \subseteq U$.

Take $K'$ as in Lemma 2.12. We then have that $K' \cap M \subseteq U$.

Let $V = \{V_n : n \in \omega\}$ be an outer base of $K'$ such that $V \subseteq M$ and $V_{n+1} \subseteq V_n$, for every $n \in \omega$. Since $x \in K'$, the following claim finishes the proof:

**Claim.** There is an $n \in \omega$ such that $V_n \cap M \subseteq U$.

**Proof.** Suppose not. Then, for every $n \in \omega$, there is an $x_n \in V_n \cap M$ such that $x_n \notin U$. Since $V$ is an outer base for $K'$, there is a $y \in K'$ such that $y$ is an accumulation point of $\{x_n : n \in \omega\}$ (if not, for every $z \in K'$ there is an open set $V_z$ such that $V_z \cap \{x_n : n \in \omega\} = \emptyset$; but then $\bigcup_{z \in K'} V_z \supseteq V_n$, for some $n \in \omega$). Also, $\{x_n : n \in \omega\} \subseteq M$ and $M$ countably closed imply that $\{x_n : n \in \omega\} \subseteq M$. Then, since $H_\theta \models$ there is a $y \in K'$ such that $y$ is an accumulation point of $\{x_n : n \in \omega\}$, and $K'$ and $\{x_n : n \in \omega\}$ are in $M$, by elementarity we have that there is a $y \in K' \cap M$ such that $y$ is an accumulation point of $\{x_n : n \in \omega\}$. But then $y \in K' \cap M \subseteq U$ and $U \cap \{x_n : n \in \omega\} = \emptyset$, a contradiction. □

Example 7.7 shows that the hypothesis of $M$ being countably closed in the previous theorem cannot be dropped.

Corollary 2.9 plus Theorem 2.11 naturally raise the questions of whether the hypothesis of being compact in Corollary 2.9 can be weakened to pointwise countable type, and whether the hypothesis of being sequential can be weakened to having countable tightness. We answer these questions below, the second with a surprising independence result.

**Corollary 2.13.** If $X$ is sequential $T_2$ and of pointwise countable type, and $M$ is countably closed, then $X_M$ is a closed subspace of $X_M$.

**Proof.** By Theorem 2.11 it is a subspace; by Theorem 2.6 it is closed. □

**Corollary 2.14.** PFA implies if $X$ is compact $T_2$ with countable tightness, and $M$ is countably closed, then $X_M$ is a closed subspace of $X$.

**Proof.** By Theorem 2.2 in [4], $X$ is sequential. □

Example 7.8 (Fedorčuk’s compact S-space) is a space constructed from $\diamond$ which is compact $T_2$, has countable tightness, and for which there is a countably closed $M$ such that $X_M$ is not a closed subspace of $X$. 


We next weaken the compactness hypothesis of Corollary 2.14 as follows:

**Corollary 2.15.** PFA implies if $X$ is a $T_2$ $k$-space with countable tightness, and $M$ is countably closed, then $X \cap M$ is a closed subset of $X$.

**Proof.** It suffices to show that PFA implies $T_2$ $k$-spaces with countable tightness are sequential. Suppose $F \subseteq X$ is not closed. We claim $F$ is not sequentially closed. Since $X$ is a $k$-space, there is a compact $K$ such that $F \cap K$ is not closed. Countable tightness is inherited by closed subspaces, so $K$ is sequential. $K$ is closed, so $F \cap K$ is not closed in $K$, so it is not sequentially closed in $K$. Let $\{x_n\}_{n \in \omega}$ be a sequence in $F \cap K$ converging in $K$ to $x \in K \setminus F$. Then $\{x_n\}_{n \in \omega}$ converges to $x$ in $X$. By $T_2$ then, $F$ is not sequentially closed. \( \square \)

Example 7.9 shows that "$k$" cannot be removed from the hypothesis.

**Corollary 2.16.** PFA implies if $X$ has countable tightness, is $T_2$, and is of pointwise countable type, then if $M$ is countably closed, $XM$ is a closed subspace of $X$.

Example 7.13 below shows that Corollary 2.15 cannot be improved to get $XM$ a subspace of $X$, even if $X$ is Fréchet. However, just having "closed subset" is useful:

**Theorem 2.17.** If $X$ is a sequential $T_2$ space (or if $X$ is a $T_2$ $k$-space with countable tightness and PFA is assumed), then if $M$ is countably closed and $P$ is a property of $X$ preserved by continuous images and inherited by closed subspaces, $XM$ has property $P$.

In general we want to conclude properties of $XM$ from those of $X$ rather than vice versa, but the following result has an interesting corollary.

**Theorem 2.18.** $M$ countably closed and $t(X_M) \leq \aleph_0$ imply $X_M$ is a subspace of $X$.

**Proof.** Suppose $H$ is closed in $X$. Claim $H \cap M$ is closed in $X_M$. It suffices to show that, for every countable $F \subseteq H \cap M$, $\overline{F}^M \subseteq H$. Suppose $x \in \overline{F}^M$. By countable closure, $F \in M$. Then $M \models x \in \overline{F}$, so by elementarity, $x \in \overline{F}$, but then $x \in H$. \( \square \)

**Corollary 2.19.** $M$ countably closed, $X$ $T_2$ and sequential, $X_M$ sequential, imply $X_M$ is a closed subspace of $X$.

By Example 7.13 below, countably closed elementary submodels of sequential spaces—even of Fréchet spaces—need not be sequential.

**Corollary 2.20.** If $X$ is hereditarily separable and $M$ is countably closed, then $X_M$ is a subspace of $X$.

**Proof.** $X$ hereditarily separable implies $X_M$ is hereditarily separable and hence has countable tightness. \( \square \)
In fact if $X$ is regular, we can do better than this.

**Theorem 2.21.** If $X$ is locally separable and regular and $M$ is countably closed, then $X_M$ is a subspace of $X$.

We first need the following known result:

**Lemma 2.22.** If $M$ is countably closed, then any member of $M$ of size $\leq 2^{\aleph_0}$ is included in $M$.

**Proof.** First note that $M$ contains and includes $P(\omega)$. The former is by elementarity; the latter because $\omega \subseteq M$ and every subset of $\omega$ is a countable subset of $M$. Now if $S \in M$, $|S| \leq 2^{\aleph_0}$, then by elementarity there is a surjection $f \in M$ mapping $P(\omega)$ to $S$. But since $P(\omega) \subseteq M$, by elementarity we have that for every $t \subseteq \omega$, $f(t) \in M$. Therefore $S \subseteq M$. □

Now the same proof as for Theorem 2.5 shows

**Theorem 2.23.** If $M$ is countably closed and $\chi(X) \leq 2^{\aleph_0}$, then $X_M$ is a subspace of $X$.

Theorem 2.21 is then an immediate corollary since separable regular spaces have weight $\leq 2^{\aleph_0}$.

**Corollary 2.24.** If $X$ is locally separable, $T_3$, and sequential, and $M$ is countably closed, then $X_M$ is a closed subspace of $X$. If $X$ is separable $T_3$, and sequential, and $M$ is countably closed, then $X_M = X$.

**Proof.** The second part follows from the following lemma which we shall use several times. □

**Lemma 2.25.** If $X$ is separable, $X \cap M$ is separable and dense in $X$.

**Proof.** By elementarity, $X \cap M$ contains and hence includes a countable set $M$ thinks is dense in $X$. But then it is. □

**Theorem 2.26.**

(i) Suppose there is a $Y \in [X]^{<\kappa}$ such that the weight (least cardinal of a basis) $w(Y) \geq \kappa$, and $M$ is an elementary submodel of $H_\theta$ such that $Y \subseteq M$, $\kappa \in M$ and $|M| < \kappa$. Then $X_M$ is not a subspace of $(X, T)$.

(ii) Suppose $Y \subseteq X$ witnesses that $t(x, X) \geq \kappa$, i.e., there is an $x \in \widetilde{Y}$ such that $x \notin \widetilde{Y}'$ for any $Y' \subseteq Y$, $|Y'| < \kappa$. Suppose $M$ is an elementary submodel of $H_\theta$ such that $x, Y \in M$, and $|M| < \kappa$. Then $X_M$ is not a subspace of $(X, T)$.

**Proof.** We first prove (i). Suppose $Y \in [X]^{<\kappa}$ such that $w(Y) \geq \kappa$, and $M$ is the elementary submodel. Then $Y \cap M = Y$. But then $w(Y, T) \geq \kappa$ and $w(Y, T_M) \leq \alpha < \kappa$ (since $|M| = \alpha < \kappa$). Therefore $X_M$ cannot be a subspace of $(X, T)$. 
In particular, if \( w(X) > |X| \), take an elementary submodel \( M \supseteq X \), such that \(|M| = |X|\). Then clearly \( X_M \) is not a subspace of \( X \). (Note that on the other hand, if for every \( Y \in [X]^{\leq \omega_1}, w(Y) = \omega \), then \( w(X) = \omega \) \([20]\) and therefore \( X_M \) is always a subspace of \( X \).

To show (ii), let \( Y, x \) and \( M \) be as in the statement and suppose \( X_M \) is a subspace of \( (X, T) \). Since \( x \in \overline{Y} \), by elementarity, we have that \( M \models "x \in \overline{Y}" \). Thus, \( x \in \overline{Y \cap M^T} \).

But since we are assuming that \( X_M \) is a subspace, this implies that \( x \in Y \cap M \), a contradiction, since \(|Y \cap M| < \kappa\). \( \square \)

3. Images

If \( X_M \) is not a subspace of \( X \), perhaps it is a nice image of a subspace of \( X \). For example, doing some small modifications we can get the elementary submodel version of a result proved in a supercompact reflection context in \([17, \text{Theorem 5.1.1}]\).

**Definition 3.1.** For a topological space \( X \), \( h(X) \) is the least cardinal \( \lambda \) with the property that for every point \( x \in X \) there is a compact set \( K \subseteq X \) such that \( x \in K \) and \( \chi(K, X) \leq \lambda \).

**Theorem 3.2.** Let \( (X, T) \) be a regular space with \( h(X) \leq \kappa \) and let \( M \) be a elementary submodel of \( H_\kappa \) such that \( (X, T) \in M \) and such that \( \kappa \subseteq M \). Then there is a \( Y \subseteq X \) and \( \pi : (Y, T|Y) \rightarrow X_M \) such that \( \pi \) is perfect.

**Proof.** We first introduce some notation. Let

\[
\mathcal{K} = \{ K \subseteq X : \text{K is compact and } \chi(K, X) \leq \kappa \},
\]

\[
\mathcal{K}_x = \{ K \in \mathcal{K} \cap M : x \in K \}, \quad \text{for } x \in X_M,
\]

\[
\mathcal{V}_x = \{ V \in T \cap M : x \in V \}, \quad \text{for } x \in X_M.
\]

For each \( x \in X_M \) define \( K_x = \bigcap \mathcal{V}_x \). Note that, since \( X \) is Hausdorff, a simple elementary submodel argument shows that if \( x, y \in M \) and \( x \neq y \), then \( K_x \cap K_y = \emptyset \).

Define

\[
Y = \bigcup\{ K_x : x \in X_M \}, \quad \text{and}
\]

\[
\pi : (Y, T|Y) \rightarrow (X_M, T_M), \quad \text{by } \pi(y) = x \text{ if and only if } y \in K_x.
\]

We shall show that \( \pi \) is perfect. First we show that \( \pi \) is continuous.

**Claim 1.** \( \pi \) is continuous.

**Proof.** Fix \( F \in M \) such that \( X \setminus F \in T_M \). We want to show that \( \pi^{-1}(F) \) is closed in \( (Y, T|Y) \). Let \( y \in Y \setminus \pi^{-1}(F) \). Since \( y \in Y \), there must be \( x \in X_M \) such that \( y \in K_x \).

Clearly \( x \notin F \). Using that \( X \) is regular and that \( x \) and \( F \) are in \( M \), by elementarity we can conclude that \( M \models \text{ there are open disjoint sets } U \text{ and } W \text{ such that } x \in U \text{ and } F \subseteq W \).
Thus, we can fix $U, W \in \mathcal{T}_M$ disjoint such that $x \in U$ and $F \subseteq W$.

Now, $x \in U$ and $U \in \mathcal{T}_M$ implies that $\pi^{-1}(x) = K_x \subseteq U$, by the definition of $K_x$. Similarly, $F \subseteq W$ and $W \in \mathcal{T}_M$ implies that $\pi^{-1}(F) \subseteq W$. We then have that $U \cap \pi^{-1}(F) = \emptyset$ and that $y \in U$, which proves that $\pi^{-1}(F)$ is closed. □

The next claim shows that $\pi^{-1}(x)$ is compact, for every $x \in X_M$.

Claim 2. $K_x = \bigcap K_x$, for each $x \in X_M$.

Proof. We first show that $K_x \subseteq \bigcap K_x$. Suppose $y \notin \bigcap K_x$. Then there is a $K \in K_x$ such that $y \notin K$. Since $\chi(K, X) \leq \kappa$, we have that

$$H_\theta \models \text{there is } \mathcal{V}_K \text{ such that } |\mathcal{V}_K| \leq \kappa \text{ and } \mathcal{V}_K \text{ is an outer base for } K.$$ 

But $K \in M$, so by elementarity of $M$, there is a $\mathcal{V}_K \in M$ such that (in $H_\theta$) $\mathcal{V}_K$ has cardinality $\kappa$ and is an outer base for $K$. Since $y \notin K$, there must be $V \in \mathcal{V}_K$ such that $y \notin V$. Also $\mathcal{V}_K \in M$, $|\mathcal{V}_K| \leq \kappa$, and $\kappa \in M$ imply that $\mathcal{V}_K \subseteq M$. Thus, there is a $V \in \mathcal{T}_M$ such that $x \in V \subseteq K_x$ and $y \notin V$, which implies that $y \notin K_x$.

To show that $K_x \supseteq \bigcap K_x$, fix $y \in \bigcap K_x$ and $V \in \mathcal{T}_M$ such that $x \in V$. We have to show that $y \notin V$. This clearly follows from Lemma 2.12. □

Claim 3. $\pi$ is closed.

Proof. Suppose $A$ is closed in $Y$ and $x \in X_M \setminus \pi(A)$. Let $F$ closed in $X$ such that $F \cap Y = A$. We first show that there is a $K \in K_x$ such that $K \cap A = \emptyset$. Suppose not. Then $K \cap A \neq \emptyset$, and therefore $K \cap F \neq \emptyset$, for every $K \in K_x$. Consider the family $\mathcal{F} = \{K \cap F: K \in K_x\}$. Clearly $\mathcal{F}$ is centered and $K \cap F$ is compact for each $K \in K_x$. But then there must be $z \in \bigcap \{K \cap F: K \in K_x\}$, which implies that $z \in \bigcap K_x \cap F = K_x \cap F = K_x \cap A$. But this means that $x \in \pi(A)$, a contradiction.

So fix $K \in M$ such that $K$ is compact, $\chi(K, X) \leq \kappa$, $x \in K$, and $K \cap A = \emptyset$. Since $\chi(K, X) \leq \kappa$ and $K \in M$, by elementarity, as before, we have that there is a $\mathcal{V}_K \in M$ such that $\mathcal{V}_K$ has cardinality $\kappa$ and $\mathcal{V}_K$ is an outer base for $K$. Since $K \cap A = \emptyset$, there is a $V \in \mathcal{V}_K$ such that $V \cap A = \emptyset$. But $\mathcal{V}_K \in M$ and has cardinality $\kappa$, therefore $\mathcal{V}_K \subseteq M$, since $\kappa \subseteq M$. We then have that there is a $V \in \mathcal{T}_M$ such that $V \cap A = \emptyset$ and $x \in K \subseteq V$. We will be done if we show that $V \cap \pi(A) = \emptyset$. Suppose there is a $y \in V \cap \pi(A)$. Then $y = \pi(a)$ for some $a \in A$, which means that there is an $a \in A$ such that $a \in K_y$. But $y \in V$ and $V \in \mathcal{T}_M$. We then have that $a \in K_y \subseteq V$, for some $a \in A$, a contradiction. □

The first part of Corollary 1.6 follows since $T_3$ is preserved by closed maps. Here is another corollary:

**Corollary 3.3.** Assuming the consistency of a supercompact cardinal, there is a model of set theory in which elementary submodels of $T_3$ spaces of pointwise countable type are hereditarily collectionwise normal.
This follows from Balogh's Theorem 1.1 in [5] that adjoining supercompact many Cohen or random reals yields a model in which normal spaces of pointwise countable type are collectionwise normal. (For a short proof using the methods of the proof of Theorem 3.2, see [17, Theorem 5.3.2].) Just observe that
(a) pointwise countable type is inherited by open sets,
(b) that a space is hereditarily collectionwise normal if open subsets are collectionwise normal, and
(c) hereditary collectionwise normality is preserved by closed maps.
Here is another application of Theorem 3.2.

**Definition 3.4.** A space \((X, \mathcal{T})\) is **cometrizable** if there is a weaker metric separable topology \(S\) on \(X\) such that for every \(U \in \mathcal{T}\) and \(x \in U\), there is a \(V \in \mathcal{T}\) satisfying \(x \in V \subseteq \overline{V}^S \subseteq U\).

**Theorem 3.5.** MA implies that if \(X\) is a cometrizable space of pointwise countable type, then every elementary submodel of \(X\) of size \(< 2^{\aleph_0}\) is normal.

**Proof.** It suffices to show \(\pi^{-1}(X_M)\) is normal. As a subspace of a cometrizable space, \(\pi^{-1}(X_M)\) is cometrizable; it is the union of \(< 2^{\aleph_0}\) compact sets, so by [1, Theorem 1], it is normal. \(\Box\)

4. Cardinal invariants

Next we compare the cardinal invariants of \(X_M\) to those of \(X\). (We use "hL" and "hd" for hereditary Lindelöf number and hereditary density, respectively.)

**Theorem 4.1.**

(i) Let \(f \in \{c, hL, hd, \chi, \Psi, s, w\}\). Then \(f(X_M) \leq f(X)\). The inequality may be strict.

(ii) Let \(f \in \{L, t\}\). There are \(X\) and \(M\) such that \(f(X) < f(X_M)\) and \(X'\) and \(M'\) such that \(f(X'_{M'}) < f(X')\).

(iii) Let \(f \in \{d, \pi\}\). If \(f(X) \leq \aleph_0\), so is \(f(X_M)\).

**Proof.** (i) is straightforward; taking a countable \(M\) for a space with \(f\) uncountable witnesses strict inequality. Similarly for the second inequality in (ii). To establish the first inequality for \(L\), first note that by Theorem 4.2 below, the elementary submodes of Examples 7.6, 7.19 and 7.20 are countably compact but not compact, hence not Lindelöf. To establish it for \(t\), see Example 7.13.

To prove (iii), if, e.g., \(X\) has a countable \(\pi\)-base, then by elementarity, \(M \models "X\ has a countable \(\pi\)-base". But then \(X_M\) has a countable \(\pi\)-base. Similarly for density. \(\Box\)

The following result is not surprising.
Theorem 4.2. Countably closed elementary submodels of (countably) compact spaces are countably compact.

Proof. Take a countable open cover \( \mathcal{U} = \{ U_i \cap M : i \in \omega \} \) of \( X \cap M \), where \( U_i \in M \). By countable closure, \( \mathcal{U}' = \{ U_i : i \in \omega \} \in M \). Then \( M \models \" \mathcal{U}' \text{ is an open cover} \" \), so \( \mathcal{U}' \) is an open cover of \( X \) by elementarity. Then it has a finite subcover \( \{ U_{ij} : j \leq n \} \). Then \( M \models \{ U_{ij} : j \leq n \} \) is a cover, so \( \{ U_{ij} \cap M : j \leq n \} \) is a cover of \( X \cap M \), as required. \( \Box \)

Also not surprisingly, there is a compact \( T_2 \) space \( X \) and a countably closed elementary submodel \( M \) such that \( X_M \) is not compact—see Examples 7.6, 7.19 or 7.20 below. By Corollary 2.9, no such example can be sequential. We also have the following result.

Theorem 4.3. \( MA \) implies that if \( M \) is countably closed and \( X \) is a (countably) compact \( T_2 \) separable space with \( hL(X) < 2^{\aleph_0} \), then \( X_M = X \).

Proof. \( X_M \) is countably compact, \( T_3 \), separable, and has Lindelöf number \( < 2^{\aleph_0} \), so Hechler's theorem [18] applies. Thus, \( X_M \) is compact, hence closed, so equals \( X \). \( \Box \)

On the other hand, \( 2^{\aleph_0} < 2^{\aleph_1} \) yields an example (Example 7.19 below) of a separable compact \( T_2 \) space with weight and hence \( hL \leq \aleph_1 < 2^{\aleph_0} \), and a countably closed \( M \) such that \( X_M \) is not compact. (Simply take a countably closed \( M \) of size \( 2^{\aleph_0} \).)

We also have an independence result when \( \text{"} hL < 2^{\aleph_0} \text{"} \) is replaced by \( \text{"} T_5 \text{"} \).

Theorem 4.4. \( PFA \) implies if \( X \) is \( T_5 \), separable, and countably compact, then if \( M \) is countably closed, \( X_M = X \).

Proof. By [21] \( X \) is compact and has countable tightness. By Corollary 2.14 \( X_M \) is a closed subspace of \( X \) and hence is \( X \). \( \Box \)

On the other hand, a countably closed elementary submodel \( M \) of size \( 2^{\aleph_0} \) of Fedorčuk's compact \( T_5 \) S-space (Example 7.8) [15] yields a proper subspace but it will not be closed in this case.

5. The space \( X(M) \)

The ideas in Section 3 can be embedded in a more general context; the following definition occurred to us when we tried to understand [7], which is couched in the language of uniform spaces.

Let \( \langle X, T \rangle \) be a topological space and \( \mathcal{F} \) be a (not necessary open) cover of \( X \). Let \( M \) be an elementary submodel of \( H_\theta \) such that \( X, T, \mathcal{F} \in M \).

Definition 5.1. For every \( x, y \in X \), define \( x \sim y \) if and only if \( x \in V \iff y \in V \), for every \( V \in \mathcal{F} \cap M \).
Clearly \( \sim \) is an equivalence relation. We can then define the quotient space \( X_\mathcal{F}(M) = X/\sim \).

Also let

\[
\varphi = \varphi_\mathcal{F}^M : X \to X(M)
\]

be the quotient map. We will denote by \([x]\) the equivalence class of \(x\). We will suppress the "\(\mathcal{F}\)" in \(X(M)\) when it is clear from context.

**Example 5.2.** If \(X\) is metrizable and \(\mathcal{F}\) is a countable base of \(X\), then \(X = X(M)\).

**Proof.** Since \(\mathcal{F}\) is countable, \(\mathcal{F} \subseteq M\); since it is a base, \(\bigcap\{V \in \mathcal{F}: x \in V\} = \{x\}\). □

**Example 5.3.** If \(X = 2^\kappa\) (or \(X = I^\kappa\)) and \(\mathcal{F}\) is the usual base for \(X\), then \(X(M)\) is homeomorphic to \(2^{\kappa \cap M}\) (\(I^{\kappa \cap M}\)).

**Proof.** For any \(f\) and \(g\) in \(X\), we have that \(f \sim g\) if and only if \(x \in V \iff y \in V\), for every \(V \in \mathcal{F} \cap M\). Since we are picking \(\mathcal{F}\) to be the usual base for \(X\), \(f \sim g\) if and only if \(f \in [p] \iff g \in [p]\), for every finite \(p\) with \(\text{dom}(p) \subseteq \kappa \cap M\), which is equivalent to \(f \upharpoonright (\kappa \cap M) = g \upharpoonright (\kappa \cap M)\). □

**Theorem 5.4.** If \(X\) is \(T_{3\frac{1}{2}}\), \(\mathcal{F}\) is a base of \(X\) formed by functionally open sets, and \(C(X) \in M\), then \(x \sim y\) if and only if \(f(x) = f(y)\) for every \(f \in C(X) \cap M\).

**Proof.** Suppose there is a \(f \in C(X) \cap M\) such that \(f(x) \neq f(y)\). Then, there are disjoint open intervals \(I_x\) and \(I_y\) with rational end-points, containing \(f(x)\) and \(f(y)\), respectively. Since \(I_x\) and \(I_y\) have rational end-points, \(I_x, I_y \in M\) and therefore \(f^{-1}(I_x), f^{-1}(I_y) \in \mathcal{F} \cap M\). Also \(f^{-1}(I_x) \cap f^{-1}(I_y) = \emptyset\) and \(x \in f^{-1}(I_x), y \in f^{-1}(I_y)\). Thus, \(x \sim y\).

Suppose \(x \sim y\). Then there is a \(V \in \mathcal{F} \cap M\) witnessing it, and we can suppose \(x \in V\) and \(y \notin V\). By our choice of \(\mathcal{F}\) and since \(V \in M\), \(V = f^{-1}(U)\) for some \(f \in C(X) \cap M\) and \(U\) open in \(I\). But then \(x \in V\) implies that \(f(x) \in U\), and \(y \notin V\) implies that \(f(y) \notin U\). Thus, \(f(x) \neq f(y)\). □

In the notation of Theorem 3.2, recall that \(K_x = \bigcap \mathcal{V}_x = \bigcap \mathcal{K}_x\). We then have

**Theorem 5.5.** If \(X\) is \(T_{3\frac{1}{2}}\) with pointwise countable type, \(\mathcal{F}\) is a base of \(X\) formed by functionally open sets, and \(C(X) \in M\), then \(y \sim x\) if and only if \(y \in K_x\), for \(x \in X_M\). Thus \(X_M\) is homeomorphic to a subspace of \(X(M)\).

**Proof.** Since the set of the functionally open sets is a base for a \(T_{3\frac{1}{2}}\) space, we have that \([x] \subseteq K_x\). Suppose that \(x \in X_M\) and \(y \sim x\). By the previous theorem, there is an \(f \in C(X) \cap M\) such that \(f(x) \neq f(y)\). Thus, there is an open interval \(I_x \in M\) such that \(f(x) \in I_x\) and \(f(y) \notin I_x\). But then, \(x \in f^{-1}(I_x), f^{-1}(I_x)\) is an open set in \(M\) and \(y \notin f^{-1}(I_x)\), which implies that \(u \notin K_x\).
To see that $X_M$ is homeomorphic to a subspace of $X(M)$, define

$$Y = \{ [x] \in X(M): x \in M \}$$

and define $\pi: Y \to X_M$ by $\pi([x]) = x$. Then $\pi$ is one-to-one and onto. Also, since $[x] = K_x$ if $x \in M$, by the proof of Theorem 3.2, we have that $\pi$ is continuous and closed. 

We have a similar result for regular spaces:

**Theorem 5.6.** Suppose $X$ is a $T_3$ space with pointwise countable type. If $x \in X \cap M$ and $\mathcal{F} = \{K \subseteq X: K$ is compact with countable character$, then $y \sim x$ if and only if $y \in K_x$. Thus $X_M$ is homeomorphic to a subspace of $X(M)$. 

**Proof.** Denote by $[x]$ the equivalence class of $x$. Clearly, by the definition of the equivalence relation, $[x] \subseteq K_x$. Suppose $y \sim x$. If there is a $K \in \mathcal{F} \cap M$ such that $x \in K$ and $y \notin K$, then $y \notin K_x$ and we are done. We can suppose then that there is a $K \in \mathcal{F} \cap M$ such that $y \in K$ but $x \notin K$. But then, since $K, x \in M$ and $K$ is compact, there is a $V \in \mathcal{V}_x \cap M$ such that $K \cap V = \emptyset$. Thus, there is a $V \in \mathcal{V}_x \cap M$ such that $y \notin V$, which implies that $y \notin K_x$.

The proof that $X_M$ is homeomorphic to a subspace of $X(M)$ is the same as in the previous theorem. 

Suppose now that $X$ is only regular and let $\mathcal{F}$ be the set of all regular closed sets. We then have that

**Lemma 5.7.** For each $x \in X \cap M$, $[x] = \bigcap\{F \in \mathcal{F} \cap M: x \in F\}$ (which we will denote by $\bigcap \mathcal{F}_x \cap M$).

**Proof.** Clearly $y \sim x$ implies that $y \in \bigcap \mathcal{F} \cap M$. Suppose $y \in \bigcap \mathcal{F} \cap M$ and fix $F \in \mathcal{F} \cap M$. If $x \in F$, then $y \in F$ by assumption. Suppose $x \notin F$. Then by regularity, and since $x$ and $F$ are in $M$, there is an $F' \in \mathcal{F} \cap M$ such that $x \in F' \subseteq X \setminus F$. But $x \in F'$ implies $y \in F'$. Therefore $y \notin F$. 

We also have

**Theorem 5.8.** If $X$ is regular, and $\mathcal{F}$ is the set of all regular closed sets, then $X_M$ is a continuous image of the subspace $\{[x]: x \in X \cap M\}$ of $X(M)$.

**Proof.** We will show that $[x] \subseteq \bigcap\{V \in \mathcal{T} \cap M: x \in V\}$, for every $x \in X \cap M$. Then we will have that the function that takes $[x]$ to $x$, for every $x \in X \cap M$ is continuous (the proof is the same as the proof of continuity of $\pi$ in Theorem 3.2).

Fix $x \in X \cap M$ and $y \sim x$. If $V \in \mathcal{T} \cap M$ and $x \in V$, since $x \in M$, by regularity, there is an $F \in \mathcal{F} \cap M$ such that $x \in X \setminus F \subseteq V$. But then $x \notin F$, which implies that $y \notin F$. Therefore, $y \in V$. 

The following result is in [7, Proposition 3.9]:

**Theorem 5.9.** Let $A$ be a Boolean algebra and $X = S(A)$ (the Stone space of $A$). If $\mathcal{F}$ is the usual base for the Stone space, then $X(M)$ is homeomorphic to $S(A \cap M)$.

**Proof.** For each $a \in A$, let

$$O(a) = \{ u \in S(A) : a \in u \}.$$  

We will show that, for every $u, v \in S(A)$, $u \sim v$ if and only if $u \cap M = v \cap M$. Suppose $u \sim v$. This is equivalent to $u \in O(a)$ if and only if $v \in O(a)$, for every $O(a) \in M \cap \mathcal{F}$. But $O(a) \in M$ if and only if $a \in M$. Therefore, we can conclude that $u \sim v$ if and only if $u \cap M = v \cap M$.

It is now easy to define an homeomorphism between $X(M)$ and $S(A \cap M)$. Define

$$\pi : X(M) \to S(A \cap M),$$

by $\pi([u]) = u \cap M$. \hfill $\Box$

**Remark 5.10.** Bandlow defined the space $X(M)$ in [7] using uniformities. The definition of $X(M)$ for $X \in T_3$, $X$ considered to be a subspace of $[0, 1]^{C(X)}$, and using the equivalence relation $x \sim y$ if and only if $f(x) = f(y)$ for every $f \in C(X) \cap M$, was given by Dow in [12].

### 6. Reflection

We will briefly compare the elementary submodel context to the reflection context. In the elementary submodel context, the three important objects are $\langle X, T \rangle$, $\langle X \cap M, T \upharpoonright M \rangle$ (i.e., the subspace topology), and $X_M$. $X_M$ is elementarily equivalent to $\langle X, T \rangle$ and $T_M$ is a weakening of $T \upharpoonright M$ on $X \cap M$. In the reflection context, we have an elementary embedding $j$, which is sufficiently closed, and the three objects are $\langle j(X), j(T) \rangle$, $\langle j''X, j(T) \upharpoonright j''X \rangle$, and $\langle j''X, \{ j''U : U \in T \} \rangle$. Last is just homeomorphic to $\langle X, T \rangle$. We have $\langle X, T \rangle$ is elementarily equivalent to $\langle j(X), j(T) \rangle$ and $\langle j''X, \{ j''U : U \subseteq T \} \rangle$ is a weakening of $j(T) \upharpoonright j''X$ on $j''X$. In fact, we are really just in the elementary submodel context again. However, in most applications we do not have an elementary embedding $j$ in $V$, but rather a generic elementary embedding in some forcing extension $V[G]$. The difference this makes is that $\langle j''X, \{ j''U : U \in T \} \rangle$ is no longer homeomorphic to $\langle X, T \rangle$—which is no longer a topological space—but rather to $\langle X, T(G) \rangle$, where $T(G)$ is the topology generated by $T$ in $V[G]$. This makes life more difficult, since one has to worry about whether properties of $\langle X, T \rangle$ are preserved by the forcing. On the other hand, having the elementary embedding $j$ to use is more powerful than merely knowing that $X_M$ is an elementary submodel of $\langle X, T \rangle$.

In the elementary submodel context, the chief concern is whether properties of $X$ are retained by $X_M$, i.e., although by elementarity $X_M$ thinks it has all the properties $X$
does, it may be mistaken. In the generic elementary embedding context, the additional concern is whether properties of $X$ are preserved by forcing.

The analogue of $X(M)$ in the reflection context is the quotient of $j(X)$ obtained by identifying two points if they are in all the same $j(U)$’s, $U$ open in $X$.

7. Examples

To help the reader become used to the operation of obtaining $X_M$ from $X$, we first discuss some familiar examples. References to $X(M)$ may be ignored until the reader gets to Section 5.

Example 7.1. $X = [0, 1]$.

$X_M = [0, 1] \cap M$ is a subspace of $X$ since $X$ is first countable. By elementarity, we always have that $\mathbb{Q} \in X_M$. We have then that $X_M$ is a dense subspace of $X$. In fact, $X_M$ contains, e.g., all algebraic numbers in $[0, 1]$. If $|M| < \mathfrak{c}$, then $X_M \neq X$. Note that, by elementarity, every convergent sequence in $M$ converges to a point in $X_M$, but we may have sequences included in $X_M$ that converge to points outside $M$.

Also $X(M) = X$ if one takes $\mathcal{F}$ to be a countable base for $X$.

Not every elementary submodel of size less than $\kappa$ intersects $\kappa$ in an ordinal, but one can get such a model as the union of a countable chain of elementary submodels, if $\kappa$ has uncountable cofinality.

Example 7.2. $X = \kappa + 1$ with the usual topology and $M$ is such that $M \cap \kappa = \alpha < \kappa$, and $\kappa \in M$.

Then $X_M = \alpha \cup \{\kappa\}$ with the following topology: since $\alpha \subseteq M$, the topology $T_M$ in $\alpha$ is the same as $T$; the basic open sets at $\kappa$ are $(\beta, \kappa) \cap M = (\beta, \alpha) \cup \kappa$, for $\beta < \alpha$. Therefore $X_M$ is not a subspace of $X$. But note that $X_M$ is homeomorphic to $\alpha + 1$.

Since $\kappa + 1$ is compact $T_5$, $X_M$’s are $T_5$ for nontrivial reasons, since they may not be subspaces.

Let $\mathcal{F} = \{V \in T \cap M : V$ is an interval$\}$. Then for every $\beta \in \alpha$, $[\beta] = \{\beta\}$. Note that $V \in M$ if and only if the endpoints of $V$ are also in $M$. Therefore, $[\kappa] = [\alpha, \kappa]$. Thus, $X(M) = \alpha \cup [\kappa]$, which is homeomorphic to $X_M$.

Example 7.3. $X = 2^\kappa$ with the product topology, and $M$ is an elementary submodel such that $\kappa \in M$.

Let $V_p = \{f \in X : p \subseteq f\}$.

First suppose $\kappa \subseteq M$. Since finite subsets of elements of $X$ are in $M$, $p \in M$, for every $p \subseteq f$, if $f \in X \cap M$. Also $p \in M$ implies $V_p \in M$. Therefore, $X_M$ is a subspace of $X$.

If $\kappa \not\subseteq M$ then this is not true. Take $\alpha \in \kappa \setminus M$. Pick for example $V_p$, where $p = \{(\alpha, 0)\}$. Then $V_p$ is not open in $T \cap M$. 

We showed before that $X(M) = 2^{\kappa \cap M}$. Since for every $f, g \in X \cap M$, $f \neq g$ implies that there is $\alpha \in \kappa \cap M$ such that $f(\alpha) \neq g(\alpha)$, it is easy to see that $X_M$ is homeomorphic to $2^{\kappa \cap M} \cap M$.

**Example 7.4.** $X = \beta D$, where $D$ is a discrete space of size $\kappa$.

Recall that $X = \{u: u$ is an ultrafilter on $D\}$. The topology is given by $O(a) = \{u \in X: a \in u\}$, for every $a \subseteq D$. Then $X_M = \beta D \cap M = \{u \in M: u$ is an ultrafilter on $D\}$ and $T_M$ is the topology generated by $\{O(a) \cap M: a \in P(D) \cap M\}$.

Note that $X_M$ is a subspace of $X$ if and only if $M$ is closed under subsets of $D$. Therefore, for suitable $M$, e.g., $M$ countable, this can be an example of a compact space such that $X_M$ is not a subspace of $X$.

We showed before that $X(M)$ is homeomorphic to $S(P(D) \cap M)$, and therefore it is a compactification of $D$ (if $D \subseteq M$), but not necessarily the Stone–Čech compactification.

**Example 7.5.** A first countable $T_\sigma$ space $X$ and a countably closed elementary submodel $M$ such that $X_M$ is not a closed subspace of $X$.

**Proof.** Let $X$ be $\mathbf{c}^+$ with the following topology: $\omega$ is isolated; for each $\alpha \in \mathbf{c}^+ \setminus \omega$ and each finite $F \subseteq \omega$, $\{\alpha\} \cup (\omega \setminus F)$ is open. Take a countably closed elementary submodel of size $\mathbf{c}$. Then $X_M$ is not a closed subspace of $X$. $\Box$

**Example 7.6.** A compact space $X$ and a countably closed elementary submodel $M$ such that $X_M$ is not a perfect image of a closed subspace of $X$.

**Proof.** Let $X = 2^\kappa$ with the usual topology, where $\kappa \geq 2^{\aleph_0}$. Let $M$ be an elementary submodel of $H_\theta$ such that $M^\omega \subseteq M$, $|M| = \kappa$, $X \in M$, $\kappa \subseteq M$, and also such that $M$ includes a dense subset of $X$. As in Example 7.3, $X_M$ is a subspace of $X$. But $|X \cap M| < |X|$, and therefore $X \cap M \subseteq X$. Since $M$ includes a dense subset of $X$, $X \cap M$ is dense in $X$, and therefore cannot be closed. Therefore it is not compact and so is not even a continuous image of a closed subspace of $X$. By Theorem 4.2, $X_M$ is countably compact. $\Box$

We are grateful to A. Dow for providing the following example as well as Example 7.18 for us (private communication).

**Example 7.7.** A compact sequential space $X$ and an elementary submodel $M$ such that $X_M$ is not a subspace of $X$.

**Proof.** The space is the one-point compactification of a $\Psi$-space. Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of $\omega$. Let $\Psi(\mathcal{A})$ be $\mathcal{A} \cup \omega$ with the following topology: $\omega$ is discrete; the basic open sets at $A \in \mathcal{A}$ are $\{A\} \cup A \setminus n$, for each $n \in \omega$.

Let $X = \Psi(\mathcal{A}) \cup \{\infty\}$ be the one-point compactification of $\Psi(\mathcal{A})$. Then the neighbourhoods at $\{\infty\}$ are of the form $\{\infty\} \cup \Psi(\mathcal{A}) \setminus K$, where $K$ is a compact set in $\Psi(\mathcal{A})$. 

Let $M$ be an elementary submodel such that $A \in M$ and $|M| < |A|$. Fix $A \in A \setminus M$. Then, $\{A\} \cup A$ is compact in $\mathcal{P}(A)$ and therefore

$$V = \{\infty\} \cup \mathcal{P}(A) \setminus (\{A\} \cup A) = \{\infty\} \cup (A \setminus \{A\}) \cup \omega \setminus A$$

is open in $X$.

But $V$ is not open in $X_M$. Note that if $W \in T \cap M$, to get $W$ we can just remove points of $A \cap M$, i.e., $W$ must contain a set of the form $A \setminus \{B_1, \ldots, B_n\}$, where $B_1, \ldots, B_n \in A \cap M$ (by the definition of the topologies). Therefore, $A \notin W$. But $A \notin V$. Thus, $W \not\subseteq V$, for every $W \in T \cap M$. □

Note that, assuming $2^{\aleph_0} > \aleph_1$ and taking $|A| = 2^{\aleph_0}$, we can have $M$ be $\omega$-covering. This shows that in Corollary 2.9, "countably closed" cannot be weakened to "$\omega$-covering" in ZFC. If CH is assumed, it is not clear how to get an $\omega$-covering elementary submodel that is not countably closed.

**Example 7.8.** ♦ implies there is a compact $T_2$ space with countable tightness and a countably closed $M$ such that $X_M$ is not a closed subspace of $X$.

**Proof.** The space is Fedorčuk's compact S-space [15, Theorem 41. Since it is hereditarily separable, it has countable tightness. Fedorčuk remarks that any infinite closed subspace has cardinality $2^c$, so by taking $M$ of size $c$, we have the desired result. □

The following example shows that "$k$" cannot be removed from Corollary 2.15.

**Example 7.9.** A countably tight $T_2$ space $X$ and a countably closed $M$ such that $X \cap M$ is not a closed subset of $X$.

**Proof.** The example is Example 1.22 in [19]. Let $X = \beta\omega$. For every $A \subseteq X$ define a new closure operation by

$$\overline{A}^c = \bigcup \{\overline{B}: B \subseteq A \text{ and } |B| \leq \aleph_0\}.$$ 

Let $X^c$ be the space $X$ with the topology generated by the closure operation defined above. Then $t(X^c) = \omega$. Also $\overline{N}^c = X$ and therefore $N$ is dense in $X^c$. Let $M$ be a countably closed elementary submodel of size $c$. Then $|X \cap M| < |X|$ and therefore $X \cap M \subseteq X$. But $N \subseteq X \cap M$ and $\overline{N}^c = X$. Therefore $M \cap X$ is not a closed subset of $X^c$. □

**Example 7.10.** A completely regular—in fact normal—space $X$ such that $(X_M, T_M)$ is not a perfect (or even closed) image of a subspace of $(X, T)$.

**Proof.** Suppose $\kappa$ is a regular cardinal. Let $X = \kappa + 1$ and $T$ be the following topology in $X$: $\{\alpha\}$ is open for every $\alpha < \kappa$ and $\{[\beta, \kappa): \beta < \kappa\}$ is a base at $\kappa$. Let $M$ be an elementary submodel of $H_\kappa$ such that $\kappa \in M$ and $M \cap \kappa = \alpha < \kappa$. 
It is easy to see that \( \langle X_M, T_M \rangle \) is isomorphic to \( \langle \alpha + 1, T_\alpha \rangle \), where \( T_\alpha \) is the topology defined on \( \alpha + 1 \) in which \( \{ \beta \} \) is open for each \( \beta < \alpha \) and \( \{ (\beta, \alpha) : \beta < \alpha \} \) is a base at \( \alpha \). We will show that \( \langle \alpha + 1, T_\alpha \rangle \) is not a closed image of a subspace of \( \langle X, T \rangle \).

Suppose there is a \( B \subseteq X \) such that there is a \( \pi : B \to X_M \) continuous and closed. First note that, since we want \( \pi \) to be closed, we cannot have \( \pi^{-1}(\{ \alpha \}) \) open. If it were open, we would have \( F = B \setminus \pi^{-1}(\{ \alpha \}) \) closed, which would imply \( \pi(F) = (\alpha + 1) \setminus \{ \alpha \} = \alpha \) closed, a contradiction.

So we must have \( \kappa \in B \) and \( \pi(\kappa) = \alpha \). Moreover, since \( \pi^{-1}(\{ \alpha \}) \) cannot be open (in \( B \)), we must have an increasing sequence \( \{ \beta_\lambda : \lambda < \kappa \} \), cofinal in \( \kappa \), such that \( \beta_\lambda \in B \) and \( \pi(\beta_\lambda) < \alpha \), for each \( \lambda < \kappa \). Since \( \kappa \) is regular and \( \alpha < \kappa \), there must be \( \beta < \alpha \) such that \( \pi^{-1}(\{ \beta \}) \) is cofinal in \( \kappa \). But then \( \pi^{-1}(\{ \beta + 1, \alpha \}) \) cannot be open in \( B \), contradicting the continuity of \( \pi \). \( \square \)

The following examples show there is no nice analogue of Theorem 3.2 for \( k' \)-spaces or even for Fréchet spaces. They destroy the reasonable conjecture that for a countably closed elementary submodel \( M \) of a Fréchet space \( X \), \( X_M \) is a subspace of \( X \). One would have expected there to be an analogue of Theorem 3.2 because reflection arguments can be carried out in the context of \( k' \)-spaces [10, Theorem 4], [16, Theorem 1.4].

The first example is one used by Arhangel’skiï [2, Example 3.3] to show that \( k' \)-spaces need not be of pointwise countable type. The second example is a minor modification of the first, used to deal with countable closure.

**Example 7.11.** A Fréchet space \( \langle X, T \rangle \) without pointwise countable type such that \( X_M \) is not a closed image of a subspace of \( X \), where \( M \) is an elementary submodel of \( H_\theta \), \( \langle X, T \rangle \in M \) and \( M \) is countable.

**Proof.** Let \( X = (\omega \times \omega) \cup \{ s \} \). The topology on \( X \) is given by: \( \{(i, j)\} \) is open for every \( i, j \in \omega \), and \( \{ V_f : f \in \omega^\omega \} \) is a neighbourhood base at \( s \), where

\[
V_f = \{ s \} \cup \{(i, j) : i \in \omega, j \geq f(i) \}.
\]

Clearly, this defines a topology on \( X \). Also, it is easy to see that \( X \) is normal and \( \chi(s, X) > \omega \).

We will first give Arhangel’skiï’s proof that \( X \) is a (Fréchet and hence) \( k' \)-space without pointwise countable type.

First, suppose that we can find a compact set \( K \), with countable character in \( X \), such that \( s \in K \). Then, since \( X \) is countable, \( K \) is also countable. Also, \( \omega(K) \leq |K| = \aleph_0 \), since \( K \) is compact. Therefore, we must have \( \chi(s, K) = \aleph_0 \). But then, since \( \chi(s, X) \leq \chi(s, K) \cdot \chi(K, X) \), for any compact \( K \) (see, e.g., [14, Exercise 3.1E]), \( \chi(s, X) = \aleph_0 \), a contradiction.

To see that \( X \) is a Fréchet space, first note that \( s \) is a limit point for some set \( A \subseteq X \) if and only if for some \( i_0 \in \omega \), the set \( \{ j : (i_0, j) \in A \} \) is infinite. But then, letting \( s_j = (i_0, j) \), \( \{ s_j \}_{j \in \omega} \to s \), which shows that \( X \) is a Fréchet space.

Suppose now that \( M \) is a countable elementary submodel of \( H_\theta \) containing \( X \) and \( T \). Since \( X \) is countable, \( X \subseteq M \). Thus, \( X_M \) is a first countable space. We first show there
is no perfect map, and then use that to show there is no closed map. Suppose there is a
\( Y \subseteq X \) and a perfect mapping \( \pi : (Y, T) \to X_M \), we will work for a contradiction.

We will first show that we must have \( s \in Y \) and \( \pi(s) = s \). Suppose \( s \notin Y \) or \( s \notin Y \)
and \( \pi(s) \neq s \). Then \( \pi^{-1}(s) \) contains only isolated points and therefore is open. But then
\( \pi^{-1}(X \setminus \{ s \}) = Y \setminus \pi^{-1}(s) \) is closed in \( Y \), and since we are assuming \( \pi \) to be closed,
this implies that \( \pi(\pi^{-1}(X \setminus \{ s \})) = X \setminus \{ s \} \) is closed, a contradiction. This argument
shows that \( \pi^{-1}(s) \) cannot be open in \( Y \). In particular, we must have \( s \in Y \) and \( \pi(s) = s \).
Also this implies that \( s \) cannot be an isolated point in \( Y \).

We will need a lemma, which can be found for example in [14, Corollary 3.7.28].
First recall that \( nw(X) \) is the smallest cardinal number of the form \( |N| \), where \( N \) is a
network for \( X \) (i.e., \( N \) is a family of subsets of \( X \) such that for every \( x \in X \) and for
any neighbourhood \( U \) of \( x \), there is an \( N \in N \) such that \( x \in N \subset U \)).

**Lemma 7.12.** If \( nw(X) \leq \kappa \) and there is a perfect mapping \( f : X \to Y \) such that
\( \chi(Y) \leq \kappa \), then \( \chi(X) \leq \kappa \).

Since \( X \) is countable, \( nw(X) = \aleph_0 \). So, by Lemma 7.12, \( Y \) must be first countable.
Then \( F = \{ i \in \omega : \{ j \in \omega : (i, j) \in Y \} \text{ is infinite} \} \) is finite (otherwise, we could show
that \( Y \) is not first countable the same way we can show \( X \) is not first countable).

By the definition of \( F \), it is easy to see that \( K = ((\omega \setminus F) \times \omega) \cap Y \) is closed, and
therefore \( \pi(K) \) is closed. Note that we can suppose \( s \notin \pi(K) \) (since \( \pi^{-1}(s) \) is compact
and therefore there could only be finitely many points in \( K \setminus \pi^{-1}(s) \)). We then have
that there is an \( f \in \omega^\omega \cap M \) such that \( V_f \cap \pi(K) = \emptyset \). Therefore \( j < f(i) \) for every
\( (i, j) \in K \).

But \( \pi \) is onto, so we must have points in \( Y \) that are mapped into \( V_f \). Since \( V_f \cap \pi(K) = \emptyset \),
these points must come from \( (F \times \omega) \cap Y \). Using that \( F \) is finite, we can conclude
that there is an \( i \in F \) such that \( \{ j : \pi(i, j) = (n, f(n) + 1) \} \), for some \( n \in \omega \) \) is infinite.

Define \( g(n) = f(n) + 2 \) for every \( n \in \omega \) (note that \( g \in M \)). We then have that \( \pi^{-1}(V_g) \)
is not open: for every \( h \in \omega^\omega \) there is a \( j > h(i) \) such that \( \pi(i, j) = (n, f(n) + 1) \) for
some \( n \in \omega \), and therefore \( \pi(i, j) \notin \pi^{-1}(V_g) \). But this, together with the
fact that \( s \in Y \) and \( \pi(s) = s \), contradicts the continuity of \( \pi \).

We will show now that \( X_M \) is not even a closed image of a subspace of \( X \). Suppose
there is a \( Y \subseteq X \) and \( \pi : Y \to X_M \) closed. For each \( x \in X_M \), choose \( y_x \in Y \) such that
\( \pi(y_x) = x \), and such that if \( x = s \), then \( y_s = s \) (recall that we showed before that we
must have \( s \in Y \) and \( \pi(s) = s \)). Define \( Y' = \{ y_x : x \in X_M \} \). Since \( y_s = s \), \( s \in Y' \)
and therefore \( Y' \) is closed in \( Y \). Then, \( \pi \upharpoonright Y' : Y' \to X_M \) is a closed mapping. Also,
\( \pi^{-1}(x) \) is a singleton for every \( x \in X_M \) and thus is compact. We conclude then that
\( \pi : Y' \to X_M \) is a perfect mapping, contradicting what we proved before.

**Example 7.13.** A Fréchet space \( X \) of cardinality \( c \) such that \( X_M \) is not a subspace if
\( |M| = c \) and \( X \subseteq M \). Moreover, if \( M \) is countably closed, then \( X_M \) is not a \( k' \)-space
(and therefore it is not Fréchet or \( k' \)), and furthermore, \( X_M \) is not a quotient of any
subspace of \( X \). Also, \( t(X_M) > t(X) \).
Proof. Take $X = \varepsilon \times \omega \cup \{s\}$. The topology is as in the above example: let $\varepsilon \times \omega$ be discrete and the neighbourhoods of $s$ are given by $V_f = \{s\} \cup \{ (\alpha, m) : m \geq f(\alpha) \}$, for $f \in \omega^\varepsilon$. Then, as before $X$ is a Fréchet space.

First note that $\chi(s, X) > c$: let $\{f_\alpha : \alpha < c\} \subseteq \omega^\varepsilon$ and define $f \in \omega^\varepsilon$ by $f(\alpha) = f_\alpha(\alpha) + 1$; then $V_{f_\alpha} \not\subseteq V_f$, for every $\alpha < c$.

Suppose $M$ is an elementary submodel such that $|M| = c$ and $X \subseteq M$. Then $\chi(s, XM) = c$ and $X \cap M = X$. Therefore $XM$ is not a subspace of $X$.

Suppose now that $M$ is countably closed; we will show that $XM$ is not a $k$-space. Let $\mathcal{F} = \{f_\alpha : \alpha < c\}$ be an enumeration of $\omega^\varepsilon \cap M$. Define $A = \{ (\alpha, f_\alpha(\alpha)) : \alpha < c \}$. Then, $s \in \overline{A}^{TM}$, since $\{ V_{f_\alpha} : \alpha < c \}$ is a base at $s$ in $TM$ and clearly $V_{f_\alpha} \cap A \not= \emptyset$, for every $\alpha < c$.

Therefore $A$ is not closed in $XM$. Then, to show that $XM$ is not a $k$-space, we just have to show that $A \cap K$ is finite (and therefore compact), for every compact $K$ in $XM$.

Fix $K$ a compact subspace of $XM$. First note that if $s \not\in K$, then $K$ is finite since $X \setminus \{s\}$ is discrete. Thus, we can suppose $s \in K$. Suppose $A \cap K$ is infinite and let $B \subseteq A \cap K$ be a countable subset of it. Since $B$ is a countable subset of $M$ and $M$ is countably closed, we have that $B \in M$. Write $B = \{ (\alpha_n, f_\alpha(\alpha_n)) : n \in \omega \}$. Define $f \in \omega^\varepsilon$ by $f(\alpha_n) = f_\alpha(\alpha_n) + 1$, for every $n \in \omega$, and $f(\beta) = 0$ otherwise. Then $f \in M$ since $B \in M$. But also $A \cap K \setminus V_f$ is finite. Therefore $K \setminus V_f$ is infinite, which contradicts the fact that $K$ is compact in $XM$, since the points in $K \setminus V_f$ are all isolated points.

To see that $XM$ is not a quotient of any subspace of $X$, simply recall that Fréchet spaces are hereditarily $k$ and that a quotient of a $k$-space is a $k$-space.

Since $X$ is Fréchet, $t(X) = \aleph_0$. Again consider $A$ as above. If $t(X_M)$ were countable, there would be a countable $B \subseteq A$ such that $s \in \overline{B}^{TM}$. Supposing $M$ is countably closed and including $X$, we have $B \in M$. But then by elementarity, $s \in \overline{B}$. But that is not true, so $t(X_M)$ must be uncountable. \qed

Remark 7.14. The two examples above can also be described as quotient spaces, e.g., for the second example, let $Y$ be the disjoint union of $\varepsilon$ many copies of $\omega + 1$ and identify all the points $\{ \omega \}$. We then have a quotient map $\pi : Y \to X$. Note that if $M$ is an elementary submodel of size $c$ including such that $Y, X, \pi \in M$, then we still have that $X_M$ is not a subspace of $X$, but $Y_M = Y$ (since it is first countable). We then have that $\pi : Y_M = Y \to X_M$ is not a quotient map. Therefore, the previous example also shows that the property of being a quotient map is not preserved by elementary submodels.

We do not know whether $T_3$ can be weakened to $T_2$ in the hypothesis of Theorem 3.2. One might hope that by weakening $\pi$ to say a closed map, one might be able to weaken pointwise countable type to say $k'$. Example 7.11 destroys that hope, and Example 7.13 destroys the plausible conjecture that if $X$ is Fréchet and $M$ is countably closed, then $X_M$ is a subspace of $X$, as well as the conjecture that $X$ a $T_3$ $k$-space implies $X_M$ is a quotient of a subspace of $X$. 

The following example completes the verification of Corollary 1.6.

**Example 7.15.** A $T_5$ space $(X, T)$ and an elementary submodel $M$ such that $X_M$ is not normal.

**Proof.** We use a variation of an example in [22] (or see [9]). Let $\kappa$ be a regular uncountable cardinal, $L$ and $K$ be two disjoint sets of cardinality $\kappa$, and $\mathcal{A}$ be an independent family on $K$ of size $2^\kappa$.

By a well-ordering argument, we can construct a function

$$f : \mathcal{P}(L) \rightarrow A \cup \{K \setminus A : A \in \mathcal{A}\}$$

such that if $A \subseteq L$, then $f(L \setminus A) = K \setminus f(A)$. Then we have that for every $A \subseteq L$, $f(A) \cap f(L \setminus A) = \emptyset$, and if $A_1, \ldots, A_n \subseteq L$, with $A_i \neq L \setminus A_j$ for every $i, j \leq n$, then $\bigcap_{i \leq n} f(A_i) \neq \emptyset$.

Let $X = K \cup L$. The topology on $X$ is given by the subbasis

$$S = \{A \cup f(A) : A \subseteq L\} \cup \{\{p\} : p \in K\} \cup \{X \setminus \{p\} : p \in K\}.$$

We have then that the points of $K$ are isolated. Also, $L$ is a closed discrete set, since for every $x \in L$, $\{x\} \cup f(\{x\})$ is open and $(\{x\} \cup f(\{x\})) \cap L = \{x\}$.

To see that $X$ is $T_5$ it is enough to show we can separate $F$ and $L \setminus F$, for every $F \subseteq L$ (since all points not in $F$ are isolated). But for that we just have to take the open sets $F \cup f(F)$ and $(L \setminus F) \cup f(L \setminus F)$.

Now let $M$ be an elementary submodel of $H_\theta$ such that $(X, T), L, K, f$ are in $M$, $K \cup L \subseteq M$ and $|M| = \kappa$. We want to show that $X_M$ is not normal.

First, note that since $L \subseteq M$ and $f \in M$, $\{x\} \cup f(\{x\}) \in M$, for every $x \in L$, and therefore $L$ still is closed discrete in $X_M$.

Enumerate $[L]^\kappa \cap M$ as $\{L_\alpha\}_{\alpha < \kappa}$ such that each element of $[L]^\kappa \cap M$ appears cofinally often. Pick $f_\alpha \neq g_\alpha$, $\{f_\alpha, g_\alpha\} \subseteq L_\alpha \setminus \bigcup_{\beta < \alpha} \{f_\beta, g_\beta\}$. Let $F = \{f_\alpha : \alpha < \kappa\}$. Then for every $\alpha < \kappa$, $|L_\alpha \cap F| = |L_\alpha \cap (L \setminus F)| = \kappa$. Thus, if $A \in \mathcal{P}(L) \cap M$ is such that $|A| = \kappa$ and $|L \setminus A| = \kappa$, then $|L \setminus A \cap F| = |(L \setminus A) \cap (L \setminus F)| = \kappa$, since $L \setminus A \in M$. Clearly, if $A \in \mathcal{P}(L) \cap M$ is such that $|A| < \kappa$, then $|(L \setminus A) \cap F| = \kappa$, since $|L \setminus A| = \kappa$.

Suppose there were disjoint open sets $U$ and $V$ in $T \cap M$ such that $F \subseteq U$ and $L \setminus F \subseteq V$. Then, by the definition of the topology, without loss of generality we can write

$$U = \bigcup_{s \in S} \bigcap_{i \leq n_s} (F_i^s \cup f(F_i^s)) \quad \text{and} \quad V = \bigcup_{t \in T} \bigcap_{j \leq m_t} (K_j^t \cup f(K_j^t)),$$

where $F_i^s$ and $K_j^t$ are all in $M$.

Since $L$ is discrete in $X$ and hence in $X_M$, without loss of generality we may suppose

$$U = \bigcup_{s \in F} \bigcap_{i \leq n_s} (F_i^s \cup f(F_i^s)) \quad \text{and} \quad V = \bigcup_{t \in L \setminus F} \bigcap_{j \leq m_t} (K_j^t \cup f(K_j^t)),$$

where $\bigcap_{i \leq n_s} F_i^s = \{s\}$ and $\bigcap_{j \leq m_t} K_j^t = \{t\}$.
For each \( s \in F \), let \( \mathcal{F}_s = \{ F_i^s : i \leq n_s \} \), and for each \( t \in L \setminus F \), let \( \mathcal{K}_t = \{ K_j^t : j \leq m_t \} \).

Take \( T_0 \subseteq L \setminus F \) and \( m \in \omega \) such that \( |T_0| = \kappa \) and \( m_t = m \), for every \( t \in T_0 \). The next claim implies that \( U \cap V \neq \emptyset \), which finishes the proof.

Claim. There is an \( s \in F \) and a \( t \in T_0 \) such that for every \( i \leq n_s \) and for every \( j \leq m \), \( F_i^s \neq L \setminus K_j^t \).

Proof. Suppose not. Then
\[
(\forall s \in S)(\forall t \in T_0)(\exists i \leq n_s)(\exists j \leq m)(F_i^s = L \setminus K_j^t). 
\]

(*)

We work for a contradiction. Fix \( s_0 \in S \). Let \( I_0 = \{ i \leq n_{s_0} : |L \setminus F_{i_0}^{s_0}| < \kappa \} \). Then \( |\bigcup_{t \in I_0}(L \setminus F_t^{s_0})| < \kappa \). Let
\[
K_0 = T_0 \setminus \left( \bigcup_{t \in I_0}(L \setminus F_t^{s_0}) \right). 
\]

Then \( |K_0| = \kappa \).

By (*), there is a \( T_1 \subseteq K_0 \), an \( i_0 \leq n_{s_0} \) and a \( j \leq m \) such that \( |T_1| = \kappa \) and for every \( t \in T_1 \), \( K_j^t = L \setminus F_{i_0}^{s_0} \). Note \( i_0 \notin I_0 \), for if it were, take \( t \in K_0 \). Then
\[
t \in K_0 \cap K_j^t = K_0 \cap (L \setminus F_{i_0}^{s_0}). 
\]

But \( t \notin \bigcup_{t \in I_0}(L \setminus F_t^{s_0}) \). Therefore \( i_0 \notin I_0 \), which implies that \( |L \setminus F_{i_0}^{s_0}| = \kappa \).

By our construction of \( F \), we now have that \( |(L \setminus F_{i_0}^{s_0}) \cap F| = \kappa \). Thus, there is an \( s_1 \in F \) such that \( s_1 \notin F_{i_0}^{s_0} \). Then \( F_{i_0}^{s_0} \notin \mathcal{F}_{s_1} \).

Thus we will recursively find \( s_k \) and \( F_{i_k}^{s_k} \in \mathcal{F}_{s_k} \) such that \( s_k \notin \bigcup_{i < k} F_{i_i}^{s_i} \), thus assuring that the \( F_{i_k}^{s_k} \)'s are distinct. The argument is essentially the same as the case we have just done.

Given \( s_1, \ldots, s_k \) distinct, \( F_{i_k}^{s_k} \in \mathcal{F}_{s_k} \), \( l \leq k \), and \( T_k \subseteq T_{k-1} \) of size \( \kappa \) such that \( t \in T_k \) implies \( \{ L \setminus F_{i_0}^{s_0}, \ldots, L \setminus F_{i_{k-1}}^{s_{k-1}} \} \subseteq \mathcal{K}_t \), let \( I_k = \{ i \leq n_{s_k} : |L \setminus (\bigcup_{l < k} F_{i_l}^{s_l} \cup F_{i_k}^{s_k})| < \kappa \} \).

Then \( |\bigcup_{t \in I_k}(L \setminus (\bigcup_{l < k} F_{i_l}^{s_l} \cup F_{i_k}^{s_k}))| < \kappa \).

Let \( K_k = T_k \setminus \bigcup_{t \in I_k}(L \setminus (\bigcup_{l < k} F_{i_l}^{s_l} \cup F_{i_k}^{s_k})) \). Then \( |K_k| = \kappa \). By (*), there is a \( T_{k+1} \subseteq K_k \), an \( i_k \leq n_{s_k} \) and a \( j \leq m \) such that \( |T_{k+1}| = \kappa \) and for every \( t \in T_{k+1} \), \( K_j^t = L \setminus F_{i_k}^{s_k} \).

Subclaim. \( i_k \notin I_k \).

Proof. Suppose it were. Take \( t \in K_k \). Then \( t \in K_j^t \cap K_k = (L \setminus F_{i_k}^{s_k}) \cap K_k \). Also, we have \( t \notin \bigcup_{l \leq i_k}(L \setminus (\bigcup_{l < k} F_{i_l}^{s_l} \cup F_{i_k}^{s_k})) \), so \( t \in \bigcup_{l < k} F_{i_l}^{s_l} \). But \( t \in K_k \subseteq T_k \), so \( \{ L \setminus F_{i_0}^{s_0}, \ldots, L \setminus F_{i_{k-1}}^{s_{k-1}} \} \subseteq \mathcal{K}_t \). Therefore, \( t \in \bigcap_{l < k} L \setminus F_{i_l}^{s_l} \), a contradiction. \( \Box \)

Since \( i_k \notin I_k \), \( |\bigcup_{l < k} F_{i_l}^{s_l}| = \kappa \). By our construction of \( F \), there is an \( s_{k+1} \in F \) such that \( s_{k+1} \notin \bigcup_{l < k} F_{i_l}^{s_l} \). Then \( \{ F_{i_0}^{s_0}, \ldots, F_{i_k}^{s_k} \} \cap \mathcal{F}_{s_{k+1}} = \emptyset \). \( \Box \)

Having obtained the required \( F_{i_k}^{s_k} \), \( T_{k+1} \) and \( s_{k+1} \), we are done. But if we carry out the recursion up to \( k = m + 1 \), we get a \( t \in T_0 \) with \( |\mathcal{K}_t| = m + 1 \), contradiction. \( \Box \)
Remark 7.16. Using the same technique that makes Bing’s Example G into a perfectly normal Example H [8], we can make this last example perfectly normal. It also can be shown to be not collectionwise Hausdorff. We have then that perfect normality is also not preserved by elementary submodels.

A simple example shows that elementary submodels rather than just weaker topologies on subspaces play a role in Corollary 1.6.

Example 7.17. A $T_\delta$ first countable space that has a weaker topology that is not normal.

Proof. Let $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$. Let the topology $T_1$ be the topology on $X$ inherited from the product of $\omega_1 + 1$ with the discrete topology and $\omega + 1$ with the usual topology, and let $T_2$ be the usual Tychonoff plank topology. Then $(X, T_1)$ is $T_\delta$ and first countable, $T_2 \subseteq T_1$ and $(X, T_2)$ is not normal. □

Example 7.18 (Dow). (CH) $(\beta \omega)_M$ is not normal if $M$ is countably closed and $|M| < 2^\omega$.

Proof. Let $p$ be a P-point such that $p \notin M$. Then, using CH, there are regular closed disjoint sets $F$ and $K$ in $(\beta \omega \setminus \omega) \setminus p$ such that $p \in \overline{F \cap K}$ [24, Theorem 1]. Since $M$ is countably closed, by Lemma 2.22 $(\beta \omega \setminus \omega) \cap M$ is dense in $\beta \omega \setminus \omega$, hence we have $p \in \overline{F \cap M \cap K \cap M}$. By countable closure of $M$ it is also true (by Theorem 2.23) that $(\beta \omega)_M$ is a subspace of $\beta \omega$. Therefore $F \cap M$ and $K \cap M$ are disjoint closed sets in $(\beta \omega)_M$ (since $p \notin M$). Suppose that there are disjoint open sets $U$ and $V$ in $(\beta \omega)_M$ which separate $F \cap M$ and $K \cap M$. But then $p \in \overline{U \cap \omega \cap V \cap \omega}$, which is a contradiction since $U \cap \omega$ and $V \cap \omega$ are disjoint subsets of $\omega$ and $p$ is an ultrafilter. □

Example 7.19. Let $X$ be $2^{\omega_1}$.

(a) If $\omega_1 \subseteq M$, then $X_M$ is a subspace of $X$.

(b) Take a countably closed $M$ of size $2^{\aleph_0}$. $X_M$ is countably compact; if $2^{\aleph_0} < 2^{\aleq_1}$, $X_M$ is not compact. If $2^{\aleph_0} = 2^{\aleq_1}$, then $X_M = X$.

(c) There is an $\omega$-covering $M$ such that $X_M$ is not normal.

Example 7.20. Let $X$ be $2^\omega$.

(a) If $\omega \subseteq M$, e.g., is $M$ is countably closed, $X_M$ is a subspace of $X$.

(b) Take a countably closed $M$ of size $2^{\aleph_0}$. Then $X_M$ is countably compact but not compact.

(c) There is a countably closed $M$ such that $X_M$ is not normal.

Proofs. All may safely be left to the reader except for (c) in both examples and Example 7.19(b). We originally had assumed MA++CH for Example 7.19(b); the improvement is due to Alan Dow. It suffices to show $X \subseteq M$, but it is since $\mathcal{P}(\omega) \subseteq M$ and by hypothesis there is a function from $\mathcal{P}(\omega)$ onto $X$. We will give the proof of Example 7.20(c); it
will then be straightforward for the reader to do the simpler Example 7.19(c). It is noteworthy that Example 7.20(c) gives a ZFC example of a compact $T_2$ $X$ and a countably closed $M$ such that $X_M$ is not normal.

Fix a bijection $h : c \to \omega_1 \times c$. By induction, construct an elementary chain $\{M_\alpha\}_{\alpha < \omega_1}$ of countably closed elementary submodels of $H_\theta$ such that $h \in M_\alpha$, $c \subseteq M_\alpha$, $2^c \in M_\alpha$ and $|M_\alpha| = c$, for each $\alpha < \omega_1$. In addition, given $M_\beta$, $\beta < \alpha$, let $\varphi_\alpha$ be an enumeration $\{f_{\mu,\nu} : \mu < \alpha, \nu < c\}$ of $2^c \cap \bigcup_{\beta < \alpha} M_\beta$ such that for all $\beta < \alpha$, $\varphi_\alpha \upharpoonright M_\beta = \varphi_\beta$. Then take $M_\alpha$ containing $\varphi_\alpha$. Let $M = \bigcup_{\alpha < \omega_1} M_\alpha$. Then $M$ is countably closed and $M \cap 2^c = \{f_{\mu,\nu} : \mu < \omega_1, \nu < c\}$.

Define $f \in 2^c$ by $f(\delta) = f_{\mu,\nu}(\delta)$, where $h(\delta) = (\mu, \nu)$. For $i = 0, 1$ define

$$F_i = \{g \in 2^c : g(\beta) = i \text{ whenever } g(\beta) \neq f(\beta)\}.$$  

We claim $F_i \cap M$ are disjoint nonempty closed subsets of $X_M$ which violate normality.

To see the $F_i$ are nonempty, consider the constant $i$ function. To see that they are closed, if $g \notin F_i$, there is a $\beta$ such that $g(\beta) \neq f(\beta)$ and $g(\beta) = 1 - i$. Consider $B_\beta = \pi_\beta^{-1}(1 - i)$. Then $g \in B_\beta$ and $B_\beta$ is disjoint from $F_i$. To show that $F_0 \cap F_1 \cap M = \emptyset$, it suffices to observe first that $F_0 \cap F_1 = \{f\}$, and second, that $f \notin M$. To see the former, clearly $f \in F_0 \cap F_1$, while if $g \in F_0 \cap F_1$, $g$ must agree with $f$ everywhere. If $f$ were in $M$, the function $e$ defined by $e(\delta) = 1 - f(\delta)$ would also be in $M$, say $e = f_{\mu_0,\nu_0}$. Let $\delta$ be such that $h(\delta) = (\mu_0, \nu_0)$. Then $e(\delta) = f_{\mu_0,\nu_0}(\delta) = f(\delta)$, contradiction.

Now suppose that $U, V$ are open subsets of $2^c$ such that $U \cap M, V \cap M$ are disjoint and include $F_0 \cap M$ and $F_1 \cap M$ respectively. We may suppose that $U \cap F_1 = V \cap F_0 = \emptyset$. For every $\beta < \omega_1$, define $g_\beta \in 2^c$ by $g_\beta(\delta) = f_{\mu,\nu}(\delta)$ if $h(\delta) = (\mu, \nu) \in \beta \times c$, and $g_\beta(\delta) = 0$ otherwise. Then $g_\beta \in M$ since $h$ is and $\varphi_\beta$ is. Also, $g_\beta \in F_0$ since $g(\delta)$ either $= f(\delta)$ or $0$. Hence there is a basic open $B_{p_\beta}$, $p_\beta$ a finite partial function from $c$ into 2, such that $g_\beta \in B_{p_\beta} \subseteq U$. Hence $p_\beta \subseteq g_\beta$. Note that since $B_{p_\beta} \subseteq U$ and $U \cap F_1 = \emptyset$, $p_\beta \notin f$. Claim no countable collection of $B_{p_\beta}$’s can cover $\{g_\beta : \beta < \omega_1\}$. For given a countable $S \subseteq \omega_1$, take

$$\eta > \max\{\alpha : \text{for some } \beta \in S \text{ and for some } \delta \in \text{dom}(p_\beta),$$

$$\text{there is a } \gamma \in c \text{ such that } h(\delta) = (\alpha, \gamma)\}.$$  

Then $g_\eta \notin \bigcup_{\beta \in S} B_{p_\beta}$, for if say $g_\eta \in B_{p_\beta}$, $\beta \in S$, then $p_\beta \subseteq g_\eta$. Take $\delta \in \text{dom}(p_\beta)$ such that $p_\beta(\delta) \neq f(\delta)$. Let $h(\delta) = (\alpha, \gamma)$. Then $\eta > \alpha$ so $g_\eta(\delta) = f(\delta)$, contradiction. Thus we can find an uncountable $S \subseteq \omega_1$ such that all the $B_{p_\beta}$, $\beta \in S$ are distinct.

Let $d_\beta = \text{dom}(p_\beta)$. There is an uncountable $T \subseteq S$ such that $\{d_\beta : \beta \in T\}$ forms a $\Delta$-system with root $d$, say $d, h \subseteq \eta \times c$, some $\eta < \omega_1$. Define $g(\delta) = f(\delta)$ if $h(\delta) \in \eta \times c$, $g(\delta) = 1$ otherwise. Then $g \in F_1$ so there is a finite $q \subseteq g$ such that $q \in B_q \subseteq V$. Then $M \cap B_q \cap B_{p_\beta} = \emptyset$ for all $\beta$. By elementarity, $B_q \cap B_{p_\beta} = \emptyset$. Claim dom$(q) \cap \{d_\beta \setminus d\} = \emptyset$ for all $\beta \in T$ such that $\beta \geq \eta$. If so, dom$(q)$ is uncountable, contradiction. To get this, suppose $\epsilon \in d \cap \text{dom}(q)$, $h(\delta) = (\alpha, \gamma)$. Then $q(\delta) = g(\delta) = f(\delta)$. On the other hand, since $h(\delta) \in \eta \times c \subseteq \beta \times c$, $p_\beta(\delta) = g_\beta(\delta) = f_{\alpha,\gamma}(\delta) = f(\delta)$. So $q$ and $p_\beta$ must disagree off $d$. \( \square \)
In conclusion we would like to state two problems that are interesting and appear to be difficult, especially the second.

**Problem 7.21.** Find a compact $T_2$ first countable $X$ and an $M$ such that $X_M$ is not normal.

**Problem 7.22.** Find a consistent example of a compact $T_2$ $X$ with countable tightness and a countably closed $M$ such that $X_M$ is not normal.

**Acknowledgments**

We thank Winfried Just and the referee for catching a number of errors in earlier versions.

**References**