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Growth of meromorphic solutions of linear difference equations ☆

Xiu-Min Zheng, Jin Tu *

School of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022, China

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ABSTRACT

In this paper, the authors continue to study the growth of meromorphic solutions of homogeneous or non-homogeneous linear difference equations with entire coefficients, and obtain some results which are improvement and extension of previous results in Chiang and Feng (2008) [7] and Laine and Yang (2007) [19]. Examples are also given to illustrate the sharpness of our results.

1. Introduction and results

Throughout this paper, we use standard notations in the Nevanlinna theory (see e.g. [8,13,18,20,21]). Let $f(z)$ be a meromorphic function. Here and in the following the word "meromorphic" means meromorphic in the whole complex plane. Moreover, we use notations $\sigma(f)$ and $\mu(f)$ for the order and the lower order of a meromorphic function $f(z)$ respectively.

Recently, there has been renewed interests in difference equations in the complex plane from the viewpoint of Nevanlinna theory (see e.g. [1,2,5–7,9–12,15–17,19,22]).

In particular, Chiang and Feng [7] investigated the proximity function and pointwise estimates of $\frac{f(z+\eta)}{f(z)}$, which are discrete versions of the classical logarithmic derivative estimates of $f(z)$. They also applied their results to obtain growth estimates of meromorphic solutions to higher order linear difference equations.

Theorem 1.A. *Let $A_j(z)$, $j=0,1,\ldots,n$, be entire functions such that there exists an integer l $(0\leqslant l\leqslant n)$ such that*

$$\max_{\substack{0\leqslant j\leqslant n\\ j\neq l}}\left\{\sigma(A_j)\right\}<\sigma(A_l).$$

If $f(z)$ is a meromorphic solution to

$$A_n(z)y(z+n)+\cdots+A_1(z)y(z+1)+A_0(z)y(z)=0, \tag{1.1}$$

then we have $\sigma(f)\geqslant\sigma(A_l)+1$.

When the coefficients in (1.1) are polynomials, they also obtain the following result.

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* Corresponding author.
E-mail addresses: zhengxiumin2008@sina.com (X.-M. Zheng), tujin2008@sina.com (J. Tu).

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Theorem 1.B. *Let $P_j(z)$, $j = 0, 1, \dots, n$, be polynomials such that there exists an integer l $(0 \leqslant l \leqslant n)$ such that*

$$\max_{\substack{0\leqslant j\leqslant n\\ j\neq l}} \{\deg(P_j)\} < \deg(P_l).$$

If $f(z)$ is a meromorphic solution to

$$P_n(z)y(z+n) + \cdots + P_1(z)y(z+1) + P_0(z)y(z) = 0, \tag{1.2}$$

then we have $\sigma(f) \geqslant 1$.

Note that the above results occur when there exists only one dominant coefficient. In the case that there are more than one dominant coefficients, Laine and Yang [19] obtained the following result.

Theorem 1.C. *Let $A_j(z)$, $j = 0, 1, \dots, n$, be entire functions of finite order such that among those having the maximal order $\sigma = \max_{0\leqslant j\leqslant n}\{\sigma(A_j)\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution of (1.1), we have $\sigma(f) \geqslant \sigma + 1$.*

In the following, we continue to consider growth estimates of meromorphic solutions to higher order linear difference equations. Firstly, we consider the lower order of meromorphic solutions of homogeneous linear difference equations.

Theorem 1.1. *Let $A_j(z)$, $j = 0, 1, \dots, n$, be entire functions such that there exists an integer l $(0 \leqslant l \leqslant n)$ such that*

$$\max\{\sigma(A_j),\ j = 0, \dots, n,\ j \neq l\} \leqslant \mu(A_l) < \infty, \tag{1.3}$$

and

$$\max\{\tau(A_j)\colon \sigma(A_j) = \mu(A_l),\ j = 0, \dots, n,\ j \neq l\} < \underline{\tau}(A_l), \tag{1.4}$$

where

$$\underline{\tau}(A_l) = \varliminf_{r\to\infty} \frac{\log M(r, A_l)}{r^{\mu(A_l)}} \quad \text{and} \quad \tau(A_j) = \varlimsup_{r\to\infty} \frac{\log M(r, A_j)}{r^{\sigma(A_j)}}$$

denote the lower type of $A_l(z)$ and the type of $A_j(z)$ respectively. If $f(z)$ is a meromorphic solution to (1.1), then we have $\mu(f) \geqslant \mu(A_l) + 1$.

When the coefficients in (1.1) are polynomials, we obtain a similar result as Theorem 1.1, which is also a refinement of Theorem 1.B.

Theorem 1.2. *Let $P_j(z)$, $j = 0, 1, \dots, n$, be polynomials such that there exists an integer l $(0 \leqslant l \leqslant n)$ such that*

$$\max\{\deg(P_j),\ j = 0, \dots, n,\ j \neq l\} \leqslant \deg(P_l), \tag{1.5}$$

and

$$\sum_{j\in J} |a_j| < |a_l|, \tag{1.6}$$

where $J = \{j \in \{0, \dots, n\}\backslash\{l\}\colon \deg(P_j) = \deg(P_l)\}$, and a_j, $j = 0, \dots, n$, are the leading coefficients of $P_j(z)$, $j = 0, \dots, n$, respectively. If $f(z)$ is a meromorphic solution to (1.2), then we have $\mu(f) \geqslant 1$.

The following two examples illustrate the sharpness of Theorems 1.1 and 1.2.

Example 1.1. The function $f(z) = e^{z^2}$ satisfies the equations

$$e^{-4z-4} f(z+2) + e^{-2z-1} f(z+1) - 2f(z) = 0$$

and

$$e^{-4z-4} f(z+2) - f(z) = 0,$$

where the coefficients satisfy the assumptions (1.3) and (1.4). Therefore, we have $\mu(f) = 2 = \mu(A_2) + 1$, showing that Theorem 1.1 may occur.

Example 1.2. The function $f_1(z) = e^{z\log 2}$ satisfies the equation

$$(z+1)f(z+2) - 2f(z+1) - 4zf(z) = 0,$$

and the function $f_2(z) = \Gamma(z)$ satisfies the equation

$$f(z+2) + (z+1)f(z+1) - 2z(z+1)f(z) = 0.$$

It is clear that the assumptions (1.5) and (1.6) hold. Therefore, we have $\mu(f_1) = \mu(f_2) = 1$, showing that Theorem 1.2 may occur.

The following theorems investigate the order of meromorphic solutions of (1.1) in the case when there are more than one coefficients which have the maximal orders.

Theorem 1.3. *Let H be a complex set satisfying $\overline{\log\operatorname{dens}}\{r = |z|: z \in H\} > 0$, and let $A_j(z)$, $j = 0, 1, \ldots, n$, be entire functions satisfying $\max\{\sigma(A_j),\ j = 0, \ldots, n\} \leqslant \alpha_1$. If there exist a positive constant α_2 ($\alpha_2 < \alpha_1$) and an integer l ($0 \leqslant l \leqslant n$) such that for any given ε ($0 < \varepsilon < \alpha_1 - \alpha_2$),*

$$\bigl|A_l(z)\bigr| \geqslant \exp\bigl\{r^{\alpha_1-\varepsilon}\bigr\}, \quad z \in H, \tag{1.7}$$

$$\bigl|A_j(z)\bigr| \leqslant \exp\bigl\{r^{\alpha_2}\bigr\}, \quad z \in H,\ j = 0, \ldots, n,\ j \neq l, \tag{1.8}$$

then every meromorphic solution $f(z)$ of (1.1) satisfies $\sigma(f) \geqslant \alpha_1 + 1 = \sigma(A_l) + 1$.

Theorem 1.4. *Let $A_j(z)$, $j = 0, 1, \ldots, n$, be entire functions. If there exists an integer l ($0 \leqslant l \leqslant n$) such that $\max_{0\leqslant j\leqslant n}\{\sigma(A_j)\} \leqslant \sigma(A_l)$ and*

$$\varlimsup_{r\to\infty} \frac{\sum_{j\neq l} m(r, A_j)}{m(r, A_l)} < 1, \tag{1.9}$$

then every meromorphic solution $f(z)$ of (1.1) satisfies $\sigma(f) \geqslant \sigma(A_l) + 1$.

The following example illustrate the sharpness of Theorems 1.3 and 1.4.

Example 1.3. The function $f(z) = e^{z^2-3z}$ satisfies the equation

$$e^{-z}f(z+2) + e^{z}f(z+1) - 2e^{3z-2}f(z) = 0,$$

where $A_2(z) = e^{-z}$, $A_1(z) = e^{z}$, $A_0(z) = -2e^{3z-2}$, satisfying $\sigma(A_2) = \sigma(A_1) = \sigma(A_0) = 1$.

(i) Set $H = \{z: \arg z = \pi\}$ and $l = 2$, it is clear that $\overline{\operatorname{dens}}\{r = |z|: z \in H\} = 1 > 0$. Moreover, $A_i(z)$, $i = 0, 1, 2$, satisfy the assumptions (1.7) and (1.8). Therefore, we have $\sigma(f) = 2 = \sigma(A_2) + 1$.

(ii) Set $l = 0$, it is clear that $A_i(z)$, $i = 0, 1, 2$, satisfy the assumption (1.9). Therefore, we also have $\sigma(f) = 2 = \sigma(A_0) + 1$.

Secondly, we consider the growth of entire solutions of non-homogeneous linear difference equations. Note that the above results may not be applicable to the equation

$$A_n(z)y(z+n) + \cdots + A_1(z)y(z+1) + A_0(z)y(z) = F(z), \tag{1.10}$$

to which (1.1) is the corresponding homogeneous equation (see the following Example 1.4). But we can obtain similar results with some additional conditions.

Theorem 1.5. *Let $A_j(z)$, $j = 0, 1, \ldots, n$, $F(z)$ be entire functions such that there exists an integer l ($0 \leqslant l \leqslant n$) such that*

$$b = \max\bigl\{\sigma(A_j),\ j = 0, \ldots, n,\ j \neq l,\ \sigma(F)\bigr\} < \sigma(A_l) < \frac{1}{2}, \tag{1.11}$$

then every nontrivial entire solution $f(z)$ of (1.10) satisfies $\sigma(f) \geqslant \sigma(A_l) + 1$.

Theorem 1.6. *Let $A_j(z)$, $j = 0, 1, \ldots, n$, $F(z)$ be entire functions such that there exists an integer l ($0 \leqslant l \leqslant n$) such that*

$$b = \max\bigl\{\sigma(A_j),\ j = 0, \ldots, n,\ j \neq l,\ \sigma(F)\bigr\} < \sigma(A_l) < \infty. \tag{1.12}$$

Suppose also that $A_l(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ satisfies that the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition

$$\frac{\lambda_n}{n} \to \infty, \tag{1.13}$$

then every nontrivial entire solution $f(z)$ of (1.10) satisfies $\sigma(f) \geqslant \sigma(A_l) + 1$.

Theorem 1.7. *Let $P_j(z)$, $j = 0, 1, \dots, n$, $F(z)$ be polynomials such that there exists an integer l $(0 \leqslant l \leqslant n)$ such that* (1.5) *and* (1.6) *hold. If $f(z)$ is a transcendental entire solution to*

$$P_n(z)y(z+n) + \cdots + P_1(z)y(z+1) + P_0(z)y(z) = F(z), \tag{1.14}$$

then we have $\mu(f) \geqslant 1$.

Example 1.4. The function $f(z) = e^z$ satisfies the equation

$$f(z+2) - ef(z+1) + f(z) = e^z,$$

and

$$f(z+2) - ef(z+1) + e^{-z}f(z) = 1.$$

Though there is only one dominant coefficient such that the assumptions in Theorems 1.1, 1.3–1.4 hold, we cannot get similar results in the non-homogeneous equation case.

Example 1.5. The function $f(z) = \frac{1}{\Gamma(z)} + 1$ satisfies the equation

$$z(z+1)f(z+2) - zf(z+1) = z^2,$$

where the polynomial coefficients satisfy the assumption (1.5). Therefore, we have $\mu(f) = \sigma(f) = 1$, showing that Theorem 1.7 may occur.

Example 1.6. The function $f(z) = z$ satisfies the equation

$$zf(z+2) - (z+1)f(z+1) + z^2 f(z) = z^3 - 1,$$

where the polynomial coefficients satisfy the assumption (1.5), showing Eq. (1.14) may have non-transcendental solutions.

2. Lemmas for Proofs of the theorems

We introduce some results of the proximity function and pointwise estimates of $\frac{f(z+\eta)}{f(z)}$ as following:

Lemma 2.1. *(See [7].) Let $f(z)$ be a meromorphic function, $\eta(\neq 0)$, η_1, η_2 $(\eta_1 \neq \eta_2)$ be complex numbers, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants, then there exists a subset $E_1 \subset (1, +\infty)$ of finite logarithmic measure,*

(a) *and a constant A depending only on γ and η, such that for all $|z| = r \notin (E_1 \cup [0, 1])$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leqslant A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^\gamma r \log^+ n(\gamma r)\right); \tag{2.1}$$

(b) *and if in addition that $f(z)$ has finite order σ, and such that for all $|z| = r \notin (E_1 \cup [0, 1])$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leqslant \left|\frac{f(z+\eta)}{f(z)}\right| \leqslant \exp\{r^{\sigma-1+\varepsilon}\}$$

or

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leqslant \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leqslant \exp\{r^{\sigma-1+\varepsilon}\}. \tag{2.2}$$

The following Lemmas 2.2 and 2.7 are essentially known in [8,13,20]. For the convenience of readers, we give their proof.

Lemma 2.2. *Let $f(z)$ be a meromorphic function with $\mu(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a subset $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have that*

$$T(r, f) < r^{\mu(f)+\varepsilon}. \tag{2.3}$$

Proof. By the definition of the lower order, there exists a sequence $\{r_n\}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f).$$

Then for any given ε (> 0), there exists an n_1 such that for $n \geqslant n_1$, we have

$$T(r_n, f) \leqslant r_n^{\mu(f)+\frac{\varepsilon}{2}}.$$

Let $E_2 = \bigcup_{n=n_1}^{\infty} [(\frac{n}{n+1})r_n, r_n]$, then for any $r \in E_2$, we have

$$T(r, f) \leqslant T(r_n, f) \leqslant r_n^{\mu(f)+\frac{\varepsilon}{2}} \leqslant \left(\frac{n+1}{n} r\right)^{\mu(f)+\frac{\varepsilon}{2}} < r^{\mu(f)+\varepsilon},$$

and $m_l E_2 = \sum_{n=n_1}^{\infty} \int_{\frac{n}{n+1} r_n}^{r_n} = \sum_{n=n_1}^{\infty} \log(1 + \frac{1}{n}) = \infty$. Thus, Lemma 2.2 is proved. □

By substituting (2.3) into (2.1), we can generalize (2.2) in Lemma 2.1(b) into finite lower order case as following.

Lemma 2.3. *Let $f(z)$ be a meromorphic function with $\mu(f) < \infty$, η_1, η_2 be distinct complex numbers, and let ε (> 0) be given real constant, then there exists a subset $E_3 \subset (1, +\infty)$ of infinite logarithmic measure such that for all $|z| = r \in E_3$, we have*

$$\exp\left\{-r^{\mu(f)-1+\varepsilon}\right\} \leqslant \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leqslant \exp\left\{r^{\mu(f)-1+\varepsilon}\right\}.$$

Lemma 2.4. *(See [7].) Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$, and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

We also make use of the following minimal moduli theorems for entire functions of slow growth.

Lemma 2.5. *(See [3].) Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < \frac{1}{2}$ and denote $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$. If $\sigma < \alpha < 1$, then*

$$\underline{\log \operatorname{dens}}\left\{r\colon\ A(r) > (\cos \pi\alpha) B(r)\right\} \geqslant 1 - \frac{\sigma}{\alpha}.$$

Lemma 2.6. *(See [4].) Let $f(z)$ be entire with $\mu(f) = \mu < \frac{1}{2}$ and $\mu < \sigma = \sigma(f)$. If $\mu \leqslant \delta < \min\{\sigma, \frac{1}{2}\}$ and $\delta < \alpha < \frac{1}{2}$, then*

$$\overline{\log \operatorname{dens}}\left\{r\colon\ A(r) > (\cos \pi\alpha) B(r) > r^{\delta}\right\} > C(\sigma, \delta, \alpha),$$

where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ, α.

Lemma 2.7. *Let $f(z)$ be an entire function of order $0 < \sigma(f) = \sigma < \infty$, then for any $\beta < \sigma$, there exists a set E_4 with positive upper logarithmic density such that for all $|z| = r \in E_4$, we have that*

$$\log M(r, f) > r^{\beta},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Proof. By the definition of the order, there exists a sequence $\{r_n\}$ tending to ∞ such that for any given $\varepsilon > 0$, we have

$$\log M(r_n, f) > r_n^{\sigma-\varepsilon}.$$

Since $\beta < \sigma$, we can choose ε (sufficiently small) and α to satisfy $1 < \alpha < \frac{\sigma-\varepsilon}{\beta}$. Then for all $r \in [r_n, r_n^{\alpha}]$ ($n \geqslant 1$), we have

$$\log M(r, f) \geqslant \log M(r_n, f) > r_n^{\sigma-\varepsilon} \geqslant r^{\frac{\sigma-\varepsilon}{\alpha}} > r^{\beta}.$$

Setting $E_4 = \bigcup_{n=1}^{\infty} [r_n, r_n^{\alpha}]$, we have

$$\overline{\log \operatorname{dens}}\, E_4 \geqslant \overline{\lim_{n\to\infty}} \frac{m_l(E_4 \cap [1, r])}{\log r} \geqslant \overline{\lim_{n\to\infty}} \frac{m_l(E_4 \cap [1, r_n^{\alpha}])}{\log r_n^{\alpha}} \geqslant \lim_{n\to\infty} \frac{m_l([r_n, r_n^{\alpha}])}{\log r_n^{\alpha}} = \frac{\alpha-1}{\alpha} > 0.$$

Thus, Lemma 2.7 is proved. □

Lemma 2.8. *(See [14].) Let $f(z)=\sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of order $0<\sigma(f)<\infty$. If the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap condition* (1.13)*, then for any $\beta<\sigma(f)$, there exists a set E_6 with positive upper logarithmic density such that for all $|z|=r\in E_6$, we have*

$$\log L(r,f)>r^{\beta},$$

where $L(r,f)=\min_{|z|=r}|f(z)|$.

3. Proofs of Theorems 1.1–1.7

Proof of Theorem 1.1. Suppose that $f(z)$ is a meromorphic solution to (1.1) satisfying

$$\mu(f)<\mu(A_l)+1<\infty. \tag{3.1}$$

In relation to (1.3) and (1.4), we set

$$\sigma=\max\bigl\{\sigma(A_j)\colon \sigma(A_j)<\mu(A_l),\ j=0,\dots,n,\ j\neq l\bigr\}$$

and

$$\tau=\max\bigl\{\tau(A_j)\colon \sigma(A_j)=\mu(A_l),\ j=0,\dots,n,\ j\neq l\bigr\}.$$

Then for any given ε (>0) and sufficiently large r, we have that

$$\bigl|A_j(z)\bigr|\leqslant\exp\bigl\{r^{\sigma+\varepsilon}\bigr\}, \tag{3.2}$$

if $\sigma(A_j)<\mu(A_l)$, and

$$\bigl|A_j(z)\bigr|\leqslant\exp\bigl\{(\tau+\varepsilon)r^{\mu(A_l)}\bigr\}, \tag{3.3}$$

if $\sigma(A_j)=\mu(A_l)$. Moreover, by Lemma 2.3, there exists a subset $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that for all z satisfying $|z|=r\in E_3$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right|\leqslant\exp\bigl\{r^{\mu(f)-1+\varepsilon}\bigr\},\quad j=0,\dots,n,\ j\neq l. \tag{3.4}$$

Then we can choose ε (>0) sufficiently small to satisfy

$$\max\bigl\{\sigma,\mu(f)-1\bigr\}+2\varepsilon<\mu(A_l)\quad\text{and}\quad\tau+2\varepsilon<\underline{\tau}(A_l). \tag{3.5}$$

Now, we divide Eq. (1.1) by $f(z+l)$ to get

$$-A_l(z)=A_n(z)\frac{f(z+n)}{f(z+l)}+\cdots+A_{l+1}(z)\frac{f(z+l+1)}{f(z+l)}+A_{l-1}(z)\frac{f(z+l-1)}{f(z+l)}+\cdots+A_0(z)\frac{f(z)}{f(z+l)}. \tag{3.6}$$

Substituting (3.2)–(3.4) into (3.6), we have that

$$M(r,A_l)\leqslant\exp\bigl\{r^{\mu(f)-1+\varepsilon}\bigr\}O\bigl(\exp\bigl\{r^{\sigma+\varepsilon}\bigr\}+\exp\bigl\{(\tau+\varepsilon)r^{\mu(A_l)}\bigr\}\bigr),\quad r\in E_3.$$

Consequently, we have by (3.5) that

$$\underline{\tau}(A_l)\leqslant\varliminf_{\substack{r\to\infty\\ r\in E_3}}\frac{\log M(r,A_l)}{r^{\mu(A_l)}}\leqslant\tau+\varepsilon<\underline{\tau}(A_l)-\varepsilon,$$

a contradiction. Therefore, we have $\mu(f)\geqslant\mu(A_l)+1$. □

Proof of Theorem 1.2. Suppose that $f(z)$ is a meromorphic solution to (1.2) satisfying $\mu(f)<1$. We divide through Eq. (1.2) by $f(z+l)$ to get

$$-P_l(z)=P_n(z)\frac{f(z+n)}{f(z+l)}+\cdots+P_{l+1}(z)\frac{f(z+l+1)}{f(z+l)}+P_{l-1}(z)\frac{f(z+l-1)}{f(z+l)}+\cdots+P_0(z)\frac{f(z)}{f(z+l)}. \tag{3.7}$$

Since $\mu(f)<1$, we may choose ε (>0) sufficiently small to satisfy $\mu(f)+\varepsilon<1$. Then by Lemma 2.3, there exists a subset $E_3\subset(1,+\infty)$ having infinite logarithmic measure such that for all z satisfying $|z|=r\in E_3$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right|\leqslant\exp\bigl\{r^{\mu(f)-1+\varepsilon}\bigr\}=\exp\bigl\{o(1)\bigr\}=1+o(1),\quad j=0,\dots,n,\ j\neq l. \tag{3.8}$$

Substituting (3.8) into (3.7), we have that

$$\left|P_l(z)\right| \leqslant \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} \left|P_j(z)\right|\left|\frac{f(z+j)}{f(z+l)}\right| \leqslant \bigl(1+o(1)\bigr) \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} \left|P_j(z)\right|, \quad r\in E_3,$$

which is a contradiction to the assumptions (1.5) and (1.6). Therefore, we have $\mu(f)\geqslant 1$. □

Proof of Theorem 1.3. If $\sigma(f)=\infty$, then the result is trivial. Next, we suppose that $\sigma(f)<\infty$. Denote $H_1=\{r=|z|:\ z\in H\}$. Since $\overline{\log\text{dens}}\,H_1>0$, then H_1 is a set of r of infinite logarithmic measure. By the assumptions that $\sigma(A_l)\leqslant\alpha_1$ and (1.7), it is easy to obtain $\sigma(A_l)=\alpha_1$. Moreover, by Lemma 2.1(b), there exists a subset $E_1\subset(1,+\infty)$ of finite logarithmic measure such that for any given ε (>0) and for all z satisfying $|z|=r\notin([0,1]\cup E_1)$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right| \leqslant \exp\bigl\{r^{\sigma(f)-1+\varepsilon}\bigr\}, \quad j=0,\dots,n,\ j\neq l. \tag{3.9}$$

Substituting (3.9) and (1.7)–(1.8) into (3.6), we have that

$$\exp\bigl\{r^{\alpha_1-\varepsilon}\bigr\} \leqslant \left|A_l(z)\right| \leqslant n\exp\bigl\{r^{\alpha_2}\bigr\}\exp\bigl\{r^{\sigma(f)-1+\varepsilon}\bigr\}, \qquad |z|=r\in H_1\backslash\bigl([0,1]\cup E_1\bigr). \tag{3.10}$$

By (3.10) and the assumption that $\alpha_2+\varepsilon<\alpha_1$, we have that $\sigma(f)\geqslant\alpha_1+1=\sigma(A_l)+1$. □

Proof of Theorem 1.4. If $\sigma(f)=\infty$, then the result is trivial. Next, we suppose that $\sigma(f)<\infty$. By Lemma 2.4, we have that for sufficiently large r and any given ε (>0),

$$m\left(r, \frac{f(z+j)}{f(z+l)}\right) = O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr), \quad j=0,\dots,n,\ j\neq l. \tag{3.11}$$

Substituting (3.11) into (3.6), we have that for sufficiently large r,

$$m(r,A_l) \leqslant \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} m\left(r,\frac{f(z+j)}{f(z+l)}\right) + \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} m(r,A_j) = O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} m(r,A_j). \tag{3.12}$$

By (3.12) and the assumption (1.9), we have that

$$\sigma(A_l)\leqslant\sigma(f)-1+\varepsilon.$$

Since ε (>0) is arbitrary, we have that $\sigma(f)\geqslant\sigma(A_l)+1$. □

Proof of Theorem 1.5. If $\sigma(f)=\infty$, then the result is trivial. Next, we suppose that $\sigma(f)<\infty$. We divide through Eq. (1.10) by $f(z+l)$ to get

$$\begin{aligned}-A_l(z) = {} & A_n(z)\frac{f(z+n)}{f(z+l)} + \cdots + A_{l+1}(z)\frac{f(z+l+1)}{f(z+l)} + A_{l-1}(z)\frac{f(z+l-1)}{f(z+l)} + \cdots \\ & + A_0(z)\frac{f(z)}{f(z+l)} - \frac{F(z)}{f(z)}\frac{f(z)}{f(z+l)}.\end{aligned} \tag{3.13}$$

By Lemma 2.1(b), we have that (3.9) holds for any given ε (>0) and for all z satisfying $|z|=r\notin([0,1]\cup E_1)$, where $E_1\subset(1,+\infty)$ has finite logarithmic measure. By the assumption (1.11), we have that for sufficiently large $r=|z|$,

$$\left|A_j(z)\right| \leqslant \exp\bigl\{r^{b+\varepsilon}\bigr\}, \quad j=0,\dots,n,\ j\neq l, \tag{3.14}$$

and

$$\left|F(z)\right| \leqslant \exp\bigl\{r^{b+\varepsilon}\bigr\}.$$

Since $M(r,f)>1$ for sufficiently large $r=|z|$, we have that

$$\frac{|F(z)|}{M(r,f)} \leqslant \left|F(z)\right| \leqslant \exp\bigl\{r^{b+\varepsilon}\bigr\}. \tag{3.15}$$

By Lemma 2.5 (if $\mu(A_l)=\sigma(A_l)$) or Lemma 2.6 (if $\mu(A_l)<\sigma(A_l)$), there exists a subset $E_7\subset(1,+\infty)$ having infinite logarithmic measure such that for all z satisfying $|z|=r\in E_7$, we have that

$$\left|A_l(z)\right| \geqslant \exp\bigl\{r^{\sigma(A_l)-\varepsilon}\bigr\}. \tag{3.16}$$

Substituting (3.9), (3.14)–(3.16) into (3.13), for all z satisfying $|z| = r \in E_7 \setminus ([0,1] \cup E_1)$ and $|f(z)| = M(r, f)$, we have

$$\exp\{r^{\sigma(A_l)-\varepsilon}\} \leqslant |A_l(z)| \leqslant (n+1)\exp\{r^{b+\varepsilon}\}\exp\{r^{\sigma(f)-1+\varepsilon}\}. \tag{3.17}$$

Now, we may choose ε (> 0) sufficiently small to satisfies $b + 2\varepsilon < \sigma(A_l)$. Then (3.17) gives that $\sigma(f) \geqslant \sigma(A_l) + 1$. □

Proof of Theorem 1.6. By using Lemma 2.8 instead of Lemmas 2.5 and 2.6 in the proof of Theorem 1.5, we can prove Theorem 1.6 similarly. □

Proof of Theorem 1.7. Suppose that $f(z)$ is a transcendental entire solution to (1.14) satisfying $\mu(f) < 1$. We now divide through Eq. (1.14) by $f(z+l)$ to get

$$\begin{aligned} -P_l(z) &= P_n(z)\frac{f(z+n)}{f(z+l)} + \cdots + P_{l+1}(z)\frac{f(z+l+1)}{f(z+l)} + P_{l-1}(z)\frac{f(z+l-1)}{f(z+l)} + \cdots \\ &\quad + P_0(z)\frac{f(z)}{f(z+l)} - \frac{F(z)}{f(z)}\frac{f(z)}{f(z+l)}. \end{aligned} \tag{3.18}$$

Since $\mu(f) < 1$, we have that (3.8) holds for any given ε $(0 < \varepsilon < 1 - \mu(f))$ and for all z satisfying $|z| = r \in E_3$, where $E_3 \subset (1, +\infty)$ has infinite logarithmic measure. Since $f(z)$ is transcendental, then we have that for sufficiently large $r = |z|$,

$$\frac{|F(z)|}{M(r, f)} = o(1). \tag{3.19}$$

Substituting (3.8) and (3.19) into (3.18), we have that for all z satisfying $|z| = r \in E_3$, $r \to \infty$, and $|f(z)| = M(r, f)$,

$$|P_l(z)| \leqslant \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} |P_j(z)|\left|\frac{f(z+j)}{f(z+l)}\right| + \frac{|F(z)|}{|f(z)|}\frac{|f(z)|}{|f(z+l)|} \leqslant (1+o(1)) \sum_{\substack{0\leqslant j\leqslant n\\ j\neq l}} |P_j(z)|,$$

which is a contradiction to the assumptions (1.5) and (1.6). Therefore, we have $\mu(f) \geqslant 1$. □

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