Higher-Order Generalized Invexity and Duality in Mathematical Programming

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In this paper, we introduce the concepts of higher-order type-I, pseudo-type-I, and quasi-type-I functions and establish various higher-order duality results involving these functions.

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1. INTRODUCTION

Consider the nonlinear programming problem

\[
\begin{align*}
(P) \quad \text{Minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \geq 0,
\end{align*}
\]

where \( f: \mathbb{R}^n \to \mathbb{R} \) and \( g: \mathbb{R}^n \to \mathbb{R}^m \) are twice differentiable functions.

The Mangasarian second-order dual [3] is

\[
\begin{align*}
(MD) \quad \text{Maximize} & \quad (u - y^T g(u) - \frac{1}{2} p^T \nabla^2 [f(u) - y^T g(u)] p \\
\text{subject to} & \quad \nabla [f(u) - y^T g(u)] + \nabla^2 [f(u) - y^T g(u)] y = 0, \\
& \quad y \geq 0.
\end{align*}
\]
Mangasarian [3] formulated the following higher-order dual by introducing two differentiable functions \( h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( k: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \):

\[
\text{(HD1)} \quad \text{Maximize} \quad f(u) + h(u, p) - y^T g(u) - y^T k(u, p) \\
\text{subject to} \quad \nabla_p h(u, p) = \nabla_p (y^T k(u, p)), \quad y \geq 0. \quad (1.2)
\]

\[
\nabla_p h(u, p) \text{ denotes the } n \times 1 \text{ gradient of } h \text{ with respect to } p \text{ and } \\
\nabla_p (y^T k(u, p)) \text{ denotes the } n \times 1 \text{ gradients of } y^T k \text{ with respect to } p. 
\]

If
\[
h(u, p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p
\]

and
\[
k(u, p) = p^T \nabla g(u) + \frac{1}{2} p^T \nabla^2 g(u) p
\]

then (HD1) becomes (MD).

Mond and Zhang [5] obtained duality results for various higher-order dual programming problems under higher-order invexity assumptions. They considered the following dual to (P):

\[
\text{(HD)} \quad \text{Maximize} \quad f(u) + h(u, p) - p^T \nabla_p h(u, p) \\
\text{subject to} \quad \nabla_p h(u, p) = \nabla_p (y^T k(u, p)), \quad y \geq 0. \quad (1.4)
\]

\[
y_i g_i(u) + y_i k_i(u, p) - p^T \nabla_p (y_i k_i(u, p)) \leq 0, \quad i = 1, 2, \ldots, m, \quad (1.5)
\]

\[y \geq 0. \quad (1.6)
\]

In this paper, we will give more general invexity-type conditions, such as higher-order type-I, higher-order pseudo-type-I, and higher-order quasi-type-I conditions, and establish various duality results under these conditions.

Mond and Zhang [5] proved duality results between (P) and (HD) assuming that there exists a function \( \eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
f(x) - f(u) \geq \alpha(x, u) \nabla_p h(u, p) \eta(x, u) + h(u, p) - p^T (\nabla_p h(u, p)) \quad (1.7)
\]

and
\[
g_i(x) - g_i(u) \leq \beta_i(x, u) \nabla_p k_i(u, p) \eta(x, u) + k_i(u, p) - p^T (\nabla_p k_i(u, p)), \quad i = 1, 2, \ldots, m, \quad (1.8)
\]

where \( \alpha: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\} \) and \( \beta_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}, \ i = 1, 2, \ldots, m, \) are positive functions.
Combining the concept of type-I functions [2] and conditions (1.7) and
(1.8) when
\( h(u, p) = p^T \nabla f(u) \) and \( k_i(u, p) = p^T \nabla g_i(u), i = 1, 2, \ldots, m, \)
we say that \( (f, g), i = 1, 2, \ldots, m, \) is V-type I at the point \( u \) with respect
to functions \( \eta, \alpha, \) and \( \beta; \)
\[
 f(x) - f(u) \geq \alpha(x, u) \nabla f(u) \eta(x, u)
\]
and
\[
 -g_i(u) \leq \beta_i(x, u) \nabla g_i(u) \eta(x, u), \quad i = 1, 2, \ldots, m.
\]
Mond and Zhang [6] extended the notion of V-invexity to second-order
and established duality theorems under generalized second-order V-invex-
ity conditions. If \( (f, -g_i), i = 1, 2, \ldots, m, \) satisfies conditions (1.7) and
(1.8) with \( h(u, p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p + k_i(u, p) = p^T \nabla g_i(u)
+ \frac{1}{2} p^T \nabla^2 g_i(u) p \) then \( (f, -g_i) \) is said to be second-order V-type I.

2. MANGASARIAN HIGHER-ORDER DUALITY

**Theorem 2.1** (weak duality). Let \( x \) be feasible for \( (P) \), and let \((u, y, p)\)
be feasible for \( (HD1) \). If, for all feasible \((x, u, y, p)\), there exists a function \( \eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that
\[
f(x) - f(u) \geq \eta(x, u)^T \nabla_p h(u, p) + h(u, p) - p^T (\nabla_p h(u, p)) \quad (2.1)
\]
and
\[
 -g_i(u) \leq \eta(x, u)^T \nabla_p k_i(u, p) + k_i(u, p) - p^T (\nabla_p k_i(u, p)),
\]
\[
i = 1, 2, \ldots, m, \quad (2.2)
\]
then infimum \((P) \geq \supremum \,(HD1)\).

**Proof.**
\[
f(x) - f(u) - h(u, p) + y^T g(u) + y^T k(u, p)
\]
\[
\geq \eta(x, u)^T \nabla_p h(u, p) - p^T (\nabla_p h(u, p))
\]
\[
+ y^T g(u) + y^T k(u, p), \quad \text{by } (2.1),
\]
\[
= \eta(x, u)^T \nabla_p (y^T k(u, p)) - p^T (\nabla_p y^T k(u, p))
\]
\[
+ y^T g(u) + y^T k(u, p), \quad \text{by } (1.2),
\]
\[
\geq 0, \quad \text{by } (1.3) \text{ and } (2.2).
\]

The following strong duality theorem is similar to [5, Theorem 2].
THEOREM 2.2 (strong duality). Let $x_0$ be a local or global optimal solution of $(P)$ at which a constraint qualification is satisfied, and let

$$
h(x_0, 0) = 0, \quad k(x_0, 0) = 0, \quad \nabla_p h(x_0, 0) = \nabla f(x_0), \quad \nabla_p k(x_0, 0) = \nabla g(x_0). \quad (2.3)
$$

Then there exists $y \in \mathbb{R}^n$ such that $(x_0, y, p = 0)$ is feasible for $(HD_1)$, and the corresponding values of $(P)$ and $(HD_1)$ are equal. If (2.1) and (2.2) are satisfied for all feasible $(x, u, y, p)$, then $x_0$ and $(x_0, y, p = 0)$ are global optimal solutions for $(P)$ and $(HD_1)$, respectively.

Remark 2.1. If $\begin{align*} h(u, p) &= p^T \nabla f(u) \quad \text{and} \quad k_i(u, p) = p^T \nabla g_i(u), \quad i = 1, 2, \ldots, m, \end{align*}$
then (2.1) and (2.2) become the conditions given by Hanson and Mond [2] to define a type-I function. If

$$
h(u, p) = p^T f(u) + \frac{1}{2} p^T \nabla^2 f(u) p,
$$

and

$$
k_i(u, p) = p^T \nabla g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p, \quad i = 1, 2, \ldots, m,
$$

then (2.1) and (2.2) become the second-order type-I conditions given by Hanson [1] when $p = q = r$.

3. MOND–WEIR HIGHER-ORDER DUALITY

THEOREM 3.1 (weak duality). Let $x$ be feasible for $(P)$ and let $(u, y, p)$ be feasible for $(HD)$. If, for all feasible $(x, u, y, p)$, there exists a function $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$
\begin{align*}
f(x) - f(u) &\geq \alpha(x, u) \nabla_p h(u, p) \eta(x, u) + h(u, p) - p^T \left( \nabla_p h(u, p) \right) \\
\end{align*}
$$

and

$$
\begin{align*}
-g_i(u) &\leq \beta_i(x, u) \nabla_p k_i(u, p) \eta(x, u) + k_i(u, p) - p^T \left( \nabla_p k_i(u, p) \right), \quad i = 1, 2, \ldots, m, \quad (3.1)
\end{align*}
$$

where $\alpha: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\beta_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}, i = 1, 2, \ldots, m$, are positive functions, then infimum $(P) \geq$ supremum $(HD)$. 


Proof. Since \((u, y, p)\) is feasible for (HD), we have
\[-y_i g_i(u) - y_i k_i(u, p) + p^T \nabla_p (y_i k_i(u, p)) \geq 0, \quad i = 1, 2, \ldots, m.\]
Since \(x\) is feasible for (P), then by (3.2) and \(y_i \geq 0\), we obtain
\[\beta_i(x, u) \nabla_p (y_i k_i(u, p)) \eta(x, u) \geq 0, \quad i = 1, 2, \ldots, m.\]
Since \(\beta_i(x, u) > 0\), we have
\[\nabla_p (y_i k_i(u, p)) \eta(x, u) \geq 0, \quad i = 1, 2, \ldots, m,
\]
hence
\[\nabla_p (y^T k(u, p)) \eta(x, u) \geq 0. \quad (3.3)\]
By (3.1), it follows that
\[f(x) - f(u) - h(u, p) + p^T \nabla_p h(u, p)\]
\[\geq \alpha(x, u) \nabla_p h(u, p) \eta(x, u)\]
\[= \alpha(x, u) \nabla_p (y^T k(u, p)) \eta(x, u), \quad \text{by (1.4)},\]
\[\geq 0, \quad \text{by (3.3) and } \alpha(x, u) > 0.\]

Theorem 3.2 (strong duality). Let \(x_0\) be a local or global optimal solution of (P) at which a constraint qualification is satisfied, and let conditions (2.3) be satisfied. Then there exists \(y \in \mathbb{R}^m\) such that \((x_0, y, p = 0)\) is feasible for (HD) and the corresponding values of (P) and (HD) are equal. If also (3.1) and (3.2) are satisfied for all feasible \((x, u, y, p)\), then \(x_0\) and \((x_0, y, p = 0)\) are global optimal solutions for (P) and (HD), respectively.

Proof. It follows on the linear of [5, proof of Theorem 5].

Remark 3.1. If \(h(u, p) = p^T \nabla f(u)\) and \(k_i(u, p) = p^T \nabla g_i(u), \quad i = 1, 2, \ldots, m,\) then \((f, -g), \quad i = 1, 2, \ldots, m,\) satisfying conditions (3.1) and (3.2), is V-type I, and the higher-order dual (HD) reduces to the Mond–Weir dual:

\[(D) \quad \text{Maximize } f(u) \]
subject to \(\nabla f(u) - y^T g(u) = 0\)
\[y_i g_i(u) \leq 0, \quad i = 1, 2, \ldots, m,\]
\[y \geq 0.\]

If
\[h(u, p) = p^T (u) + \frac{1}{2} p^T \nabla^2 f(u) p\]
and
\[ k_i(u, p) = p^T \nabla g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p, \quad i = 1, 2, \ldots, m, \]
then \((f, -g_i), i = 1, 2, \ldots, m\), satisfying conditions (3.1) and (3.2), is second-order V-type I, and the higher-order dual (HD) reduces to the second-order Mond–Weir dual:

\[(2D) \quad \text{Maximize} \quad f(u) - \frac{1}{2} p^T \nabla^2 f(u) p
\]
\[ \text{subject to} \quad \nabla f(u) + \nabla^2 f(u) p = \nabla y^T g(u) + \nabla^2 y^T g(u) p,
\]
\[ y_i g_i(u) - \frac{1}{2} p^T \nabla^2 y_i g_i(u) p \leq 0, \quad i = 1, 2, \ldots, m,
\]
\[ y \geq 0. \]

The conditions (2.1) and (2.2) are special cases of the conditions (3.1) and (3.2), where \(a(x, u) = 1\) and \(b(x, u) = 1\), \(i = 1, 2, \ldots, m\).

We can also show that (HD) is a dual to (P) under weaker conditions.

**Theorem 3.3** (weak duality). Let \(x\) be feasible for (P), and let \((u, y, p)\) be feasible for (HD). If, for all feasible \((x, u, y, p)\), there exists a function \(\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) such that
\[ \eta(x, u)^T \nabla_p h(u, p) \geq 0 \]
\[ \Rightarrow \quad f(x) - f(u) - h(u, p) + p^T \nabla_p h(u, p) \geq 0 \quad (3.4) \]
and
\[ \sum_{i=1}^m \phi_i(x, u) \left[ y_i g_i(u) + y_i k_i(u, p) - p^T \nabla_p (y_i k_i(u, p)) \right] \geq 0 \]
\[ \Rightarrow \quad \eta(x, u)^T \nabla_p (y^T k(u, p)) \geq 0, \quad (3.5) \]
where \(\phi_i: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}, i = 1, 2, \ldots, m\), are positive functions, then infimum (P) \(\geq\) supremum (HD).

**Proof.** Since \((u, y, p)\) is feasible for (HD), then by (1.5),
\[ -y_i g_i(u) - y_i k_i(u, p) + p^T \nabla_p (y_i k_i(u, p)) \geq 0, \quad i = 1, 2, \ldots, m. \]
Since \(x\) is feasible for (P) and \(\phi_i(x, u) > 0\), it follows that
\[ \sum_{i=1}^m \phi_i(x, u) \left[ y_i g_i(u) + y_i k_i(u, p) - p^T \nabla_p (y_i k_i(u, p)) \right] \geq 0. \]
By (3.5), we obtain
\[ \eta(x, u)^T \nabla_p (y^T k(u, p)) \geq 0. \]
Using (1.4), it follows that
\[ \eta(x, u)^T \nabla_p h(u, p) \geq 0. \]
Therefore by (3.4), we have
\[ f(x) \geq f(u) + h(u, p) - p^T \nabla_p h(u, p). \]

**Remark 3.2.** If \( h(u, p) = p^T \nabla f(u) \) and \( k_i(u, p) = p^T \nabla g_i(u) \), \( i = 1, 2, \ldots, m \), then (3.4) becomes the condition for \( f \) to be pseudo-type I (see [7]), and if \( \phi = 1 \), (3.5) becomes the condition for \(-g\) to be quasi-type I [7]. If
\[ h(u, p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p \]
and
\[ k_i(u, p) = p^T \nabla g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p, \quad i = 1, 2, \ldots, m, \]
than (3.4) becomes the condition for \( f \) to be second-order pseudo-type I [4] and if \( \phi = 1 \), (3.5) becomes the condition for \(-y^2 g\) to be second-order quasi-type I [4].

Strong duality between (P) and (HD) holds if (3.1) and (3.2) are replaced by (3.4) and (3.5), respectively.

**Theorem 3.4 (strict converse duality).** Let \( x^0 \) be an optimal solution of (P) at which a constraint qualification is satisfied. Let condition (2.3) be satisfied at \( x^0 \), and let conditions (3.4) and (3.5) be satisfied for all feasible \((x, u, y, p)\). If \((x^*, y^*, p^*)\) is an optimal solution of (HD), and if, for all \( x \neq x^* \)
\[ \eta(x, x^*)^T \nabla_p h(x^*, p^*) \geq 0 \]
\[ \Rightarrow \quad f(x) - f(x^*) - h(x^*, p^*) + p^*^T \nabla_p h(x^*, p^*) > 0, \quad (3.6) \]
then \( x^0 = x^* \); i.e., \( x^* \) solves (P) and
\[ f(x^0) = f(x^*) + h(x^*, p^*) - p^*^T \nabla_p h(x^*, p^*). \]

**Proof.** We suppose that \( x^0 \neq x^* \) and exhibit a contradiction. Since \( x^0 \) is a solution of (P) at which a constraint qualification is satisfied, it follows by strong duality that there exists \( y^0 \in R^m \) such that \((x^0, y^0, p = 0)\) solves (HD) and the corresponding values of (P) and (HD) are equal. Therefore,
\[ f(x^0) = f(x^*) + h(x^*, p^*) - p^*^T \nabla_p h(x^*, p^*). \quad (3.7) \]
Since \((x^*, y^*, p^*)\) is feasible for (HD), we have that
\[-y_i g_i(x^*) - y^*_i k_i(x^*, p^*) + p^* T \nabla_p (y^*_i k_i(x^*, p^*)) \geq 0\]
for \(i = 1, 2, \ldots, m\).

Since \(x^0\) is feasible for (P) and \(\phi_i(x^0, x^*) > 0\), it follows that
\[-\sum_{i=1}^{m} \phi_i(x^0, x^*) \left\{ y^*_i g_i(x^*) + y^*_i k_i(x^*, p^*) - p^* T \nabla_p (y^*_i k_i(x^*, p^*)) \right\} \geq 0.\]

By (3.5), we obtain
\[\eta(x^0, x^*) T \nabla_p (y^* T k(x^*, p^*)) \geq 0,\]
and then, by (1.4),
\[\eta(x^0, x^*) T \nabla_p h(x^*, p^*) \geq 0.\]

From (3.6), it follows that
\[f(x^0) - f(x^*) - h(x^*, p^*) + p^* T \nabla_p h(x^*, p^*) > 0,\]
which is a contradiction to (3.7).

4. GENERAL HIGHER-ORDER MOND–WEIR DUALITY

In this section we consider the following general Mond–Weir type higher-order dual to (P) as in [5],
\[
\text{(M-WHD)} \quad \text{Max} \quad f(u) + h(u, p) - p^T \nabla_p h(u, p)
\]
\[- \sum_{i \in I_0} y_i g_i(u) - \sum_{i \in I_0} y_i k_i(u, p)
+ p^T \nabla_p \left[ \sum_{i \in I_0} y_i k_i(u, p) \right] \]
subject to
\[\nabla_p h(u, p) = \nabla_p (y^T k(u, p)) \]
\[
\sum_{i \in I_u} y_i g_i(u) + \sum_{i \in I_u} y_i k_i(u, p) - p^T \nabla_p \left[ \sum_{i \in I_u} y_i k_i(u, p) \right] \leq 0,\]
\[\alpha = 1, 2, \ldots, r,\]
\[y \geq 0,\]
where $I_a \subseteq \mathcal{M} = \{1, 2, \ldots, m\}, \alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{\alpha=0}^{r} I_a = \mathcal{M}$ and $I_a \cap I_\beta = \emptyset$, if $\alpha \neq \beta$.

In [5], it is shown that (M-WHD) is a dual to (P) under the conditions

$$
\eta(x, u)^T \left[ \nabla_p h(u, p) - \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \right] \geq 0
$$

$$
\Rightarrow \quad f(x) - \sum_{i \in I_0} y_i g_i(x) - \left( f(u) - \sum_{i \in I_0} y_i g_i(u) \right)
$$

$$
- \left( h(u, p) - \sum_{i \in I_0} y_i k_i(u, p) \right)
$$

$$
+ p^T \nabla_p h(u, p) - \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \geq 0 \quad (4.1)
$$

and

$$
\sum_{i \in I_a} y_i g_i(x) - \sum_{i \in I_a} y_i g_i(u) - \sum_{i \in I_a} y_i k_i(u, p) + p^T \nabla_p \left( \sum_{i \in I_a} y_i k_i(u, p) \right) \geq 0
$$

$$
\Rightarrow \quad \eta(x, u)^T \left[ \sum_{i \in I_a} y_i k_i(u, p) \right] \geq 0, \quad \alpha = 1, 2, \ldots, r. \quad (4.2)
$$

We can generalize (4.1) and (4.2) under which (M-WHD) is a dual to (P), to generalized type-I conditions, i.e., pseudo-type-I and quasi-type-I conditions. Since the proof follows along the lines of the one in [5], we state the theorem without proof.

**Theorem 4.1 (weak duality).** Let $x$ be feasible for (P), and let $(u, y, p)$ be feasible for (M-WHD). If for all feasible $(x, u, y, p)$

$$
\eta(x, u)^T \left[ \nabla_p h(u, p) - \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \right] \geq 0
$$

$$
\Rightarrow \quad f(x) - \sum_{i \in I_0} y_i g_i(x) - \left( f(u) - \sum_{i \in I_0} y_i g_i(u) \right)
$$

$$
- \left( h(u, p) - \sum_{i \in I_0} y_i k_i(u, p) \right)
$$

$$
+ p^T \nabla_p h(u, p) - \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \geq 0 \quad (4.3)
$$
and

$$- \sum_{i \in I_0} y_i g_i(u) - \sum_{i \in I_0} y_i k_i(u, p) + p^T \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \geq 0$$

$$\Rightarrow \eta(x, u)^T \nabla_p \left( \sum_{i \in I_0} y_i k_i(u, p) \right) \geq 0, \quad \alpha = 1, 2, \ldots, r, \quad (4.4)$$

then infimum (P) $\geq$ supremum (M-WHD).

**Remark 4.1.** If $I_0 = \emptyset$ and $I_i = \{i\}, \ i = 1, 2, \ldots, m \ (r = m)$, then (M-WHD) becomes (HD) and the conditions (4.3) and (4.4) reduce to the conditions (3.4) and (3.5), respectively.

**REFERENCES**