ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 199 (2007) 95–112
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Laguerre functions on symmetric cones and recursion relations in the real case

Michael Aristidou, Mark Davidson, Gestur Ólafsson*

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

Received 3 December 2004; received in revised form 12 July 2005

Abstract

In this article we derive differential recursion relations for the Laguerre functions on the cone Ω of positive definite real matrices. The highest weight representations of the group $\mathrm{Sp}(n, \mathbb{R})$ play a fundamental role. Each such representation acts on a Hilbert space of holomorphic functions on the tube domain $\Omega + i\,\mathrm{Sym}(n, \mathbb{R})$. We then use the Laplace transform to carry the Lie algebra action over to $L^2(\Omega, \mathrm{d}\mu_\nu)$. The differential recursion relations result by restricting to a distinguished three-dimensional subalgebra, which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

MSC: Primary 33C45; secondary 43A85

Keywords: Lagurre functions and polynomials; Symmetric matrics; Tube type domains; Highest weight representations; Recursion relations

0. Introduction

The theory of special functions has its origins in the late 18th and early 19th centuries when it was seen that the algebraic, exponential, and trigonometric functions (and their inverses) were not adequate to express results to differential equations that arose in the context of some important physical problems. New functions arose to which we have associated names like Bessel, Hermite, Jacobi, Laguerre, and Legendre. Then there are the Gamma, Beta, Hypergeometric and many other families of special functions. By the latter half of the 19th century these same functions arose in different contexts and their name ‘special’ began to take on greater meaning. Their functional properties were explored and included functional relations, differential and difference recursion relations, orthogonality relations, integral relations, and others.

The preface of Vilenkin’s book [27] notes that the connection between special functions and group representations was first discovered by É. Cartan in the early part of the 20th century. By the time Vilenkin’s book appeared in the 1960s that interplay had been well established. The texts in [19,27,28], for example, well document the general philosophy. In short, group representation theory made it possible to express the classical special functions as matrix entries of a representation and to unify many of the disparate relationships mentioned above. The representation can then be used to

* Corresponding author.
E-mail addresses: aristido@math.lsu.edu (M. Aristidou), davidson@math.lsu.edu (M. Davidson), olafsson@math.lsu.edu (G. Ólafsson).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.12.002

derive differential equations and differential recursion relations for those functions. One of the problems then becomes to *find explicit expressions for the differential operators given abstractly by the representation theory*.

Generalizations of the standard Laguerre functions on polynomials showed up as early as 1955 in the work of Herz [11] on special functions on the space of complex $m \times m$-matrices. Here the Laguerre polynomials were defined on the cone of positive definite complex matrices in terms of generalized hypergeometric functions introduced in the same article.

In 1964, Simon Gindikin published his paper: ‘Analysis on Homogeneous Domains’, cf. [10]. This important paper developed special functions as part of analysis on homogeneous convex cones and built upon the earlier work of Siegel [26] on the cone of positive definite matrices. The Siegel integral of the first and second kind generalize to become the Beta and Gamma functions for the cone, respectively. Generalized hypergeometric functions are extended to homogeneous cones. Many important differential properties also extend.

Around the same time Koecher [15,16] began to develop his analysis on symmetric cones and the complex tube domains associated with them. Jordan algebras proved to be a decisive tool for framing and obtaining many important fundamental results. The outstanding text in [9] documents this interaction (see also its extensive bibliography). Nevertheless, the representation theory of Hermitian groups, which are naturally associated with tube domains, is not used in any outstanding way.

In a series of papers [6,7,4,5] the second and third authors (with Genkai Zhang in the first two referenced articles) use the representation theory of Hermitian groups in a decisive way to obtain differential and difference recursion relations on series of special functions. Motivated by the results in [4] and this article, the authors were recently able to derive the differential equations and recursions relations in general, cf. [1]. In the context of bounded symmetric domains the relevant special functions are generalized Meixner polynomials and in the context of tube domains over a symmetric cone the relevant special functions are Laguerre functions. These special functions exist in distinguished L^2-spaces, which are unitarily isomorphic to Hilbert spaces of holomorphic functions on either a bounded symmetric domain or a tube domain $T(\Omega)$. The well-known representation theory that exists there then transfers to the corresponding L^2-space to produce differential and difference relations that exist among the special functions. In particular, abstract representation theory shows that there exists three differential operators related to the Laguerre functions such that one of them has the Laguerre functions as eigenfunctions, one of them raises the indices, and the third one lowers the indices. Furthermore, the complex linear combinations of these three differential operators form a complex Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. It should be noted that such a three-dimensional Lie algebra of differential operators has shown up in several places in the literature. We would like to mention its important role in the study of the Huygens’ principle [3,12,13], in representation theory [14] and the reference therein, and in the theory of special functions [23].

One cannot downplay the essential role that Jordan algebras play in establishing and expressing many of the fundamental results obtained about orthogonal families of special functions defined on symmetric cones. Nevertheless, the theory of highest weight representations adds fundamental new results not otherwise easily obtained. In short, our philosophy is that there is a strong interplay between Jordan algebras, highest weight representations, and special functions which has not been fully exploited.

The starting point in this project has been the representation theory, wherein the Laguerre polynomials form an orthogonal family of functions invariant under a group action. However, the Laguerre polynomials have also been introduced in the literature using several variable Jack polynomials [2,8,18]. To explain the connection a little more notation is needed. Let J be an irreducible Euclidean Jordan algebra of rank r. Let $c_1, \dots, c_r \in J$ be a Jordan frame, $\mathfrak{a} = \bigoplus_{j=1}^{r} \mathbb{R}c_j$ and $e = c_1 + \cdots + c_r$. Let $\Omega = \{x^2 | x \in J \text{ and } x \text{ regular}\}$ be the standard symmetric cone in J. Let $H = \{g \in \mathrm{GL}(J) | g\Omega = \Omega\}_0$ and L the maximal compact subgroup of H fixing e. Then the Laguerre functions and polynomials are L-invariant functions on Ω. Let

$$\Omega_1 = \mathfrak{a} \cap \Omega \simeq (\mathbb{R}^+)^r.$$

Then $\Omega = L \cdot \Omega_1$ and therefore the Laguerre polynomial and functions are uniquely determined by their restriction to Ω_1. Thus, the Laguerre functions can also be defined as polynomials on Ω_1 or the vector space $\mathfrak{a}$, invariant under the Weyl group $W_H = N_L(\mathfrak{a})/Z_L(\mathfrak{a})$. This is the way the Laguerre polynomials are defined in the above references.

In the case of symmetric matrices, this boils down to the fact that each symmetric matrix can be diagonalized. Thus

$$\Omega_1 = \{\mathrm{d}(\lambda_1, \dots, \lambda_n) \mid \lambda_j > 0\}$$

and the Laguerre polynomials can be viewed as polynomials in the eigenvalues, invariant under permutations of the eigenvalues.

In this present paper, we will continue the themes outlined in [6,7,4,5] for the Laguerre functions defined of the cone of positive definite real symmetric matrices. The underlying group is $\mathrm{Sp}(n, \mathbb{R})$ and its representation theory establishes new differential recursion relations that Laguerre functions satisfy. The case $n = 1$ reduces to the Laguerre functions defined on $\mathbb{R}^+$. Briefly, the Laguerre polynomials on $\mathbb{R}$ are defined by the formula

$$L_m^\nu(x) = \sum_{k=0}^{m} \frac{\Gamma(m+\nu)}{\Gamma(k+\nu)} \binom{m}{k} (-x)^k.$$

These are up to multiplication by $m!$ and a shift by one in the ν parameter the standard Laguerre polynomials on $\mathbb{R}$. The Laguerre functions are defined by

$$\ell_m^\nu(x) = \mathrm{e}^{-x} L_m^\nu(2x).$$

The differential recursion relations are then expressed by the following three formulas:

(1) $(xD^2 + \nu D - x)\ell_m^\nu(x) = -(2m+\nu)\ell_m^\nu(x)$,
(2) $(xD^2 + (2x+\nu)D + (x+\nu))\ell_m^\nu(x) = -2m(\nu+m-1)\ell_{m-1}^\nu(x)$,
(3) $(xD^2 - (2x-\nu)D + (x-\nu))\ell_m^\nu(x) = -2\ell_{m+1}^\nu(x)$.

It is these three formulas that we generalize via the representation theory of $\mathrm{Sp}(n, \mathbb{R})$ to Laguerre functions defined on the cone of positive definite real symmetric matrices. (In [6], but not [7], a factor of $1/m!$ is included in the definition of L_m^ν. The inclusion of this factor changes the differential recursion relations slightly.) A key calculation is an explicit formula for the Lie algebraic action of $\mathfrak{sp}(n, \mathbb{C})$ on $L^2(\Omega, \mathrm{d}\mu_\nu)$. In [1] we use the triangular decomposition of the Lie algebra $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, as coordinates to derive this action. Here, however, we use the Harish–Chandra decomposition of $\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^-$.

This article is organized as follows: In the first section we introduce some standard Jordan algebra notation. In particular, we introduce the Laguerre functions and polynomials. Even though this material and most of the material in Sections 2 and 3 hold in general for simple Euclidean Jordan algebras, we specialize to the case $J = \mathrm{Sym}(n, \mathbb{R})$, the Jordan algebra of symmetric real $n \times n$-matrices. In Section 2 we introduce the tube domain $T(\Omega) = \Omega + i\,\mathrm{Sym}(n, \mathbb{R})$, where Ω is the open self dual cone of positive definite matrices. We also discuss the structure of the group $\mathrm{Sp}(n, \mathbb{R})$ and its Lie algebra $\mathfrak{sp}(n, \mathbb{R})$. Some important subalgebras of $\mathfrak{sp}(n, \mathbb{C})$ are introduced. This structure is later used to construct the differential operators that give rise to the differential equations satisfied by the Laguerre functions.

Section 3 is devoted to the discussion of the highest weight representations $(\pi_\nu, \mathscr{H}_\nu(T(\Omega)))$. We also introduce the Laplace transform as a special case of the *restriction principle* introduced in [22]. In Section 4 we describe how the Lie algebra, $\mathfrak{sp}(n, \mathbb{C})$ acts on $\mathscr{H}_\nu(T(\Omega))$. In particular, Proposition 4.2 gives an explicit formula for the action of the derived representation for each of the three subalgebras $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$, whose direct sum is $\mathfrak{sp}(n, \mathbb{C})$. It should be noted, however, that not all of this information is needed to establish the differential recursion relations for the Laguerre functions. In fact, only the action of three elements are needed. The action of $\mathfrak{sp}(n, \mathbb{C})$ on $L^2(\Omega, \mathrm{d}\mu_\nu)$ is described in Section 5. The result is the following theorem:

Theorem 5.2. *For $f \in L^2_\nu(\Omega)$ a smooth vector we have*

1. $\lambda_\nu(X)f(x) = \mathrm{tr}[(bx + (ax - xa - \nu b)\nabla - x\nabla b\nabla]f(x)$, $X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_\mathbb{C}$.
2. $\lambda_\nu(X)f(x) = \mathrm{tr}[(\nu a + ax + (ax + xa + \nu a)\nabla + x\nabla a\nabla]f(x)$, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$.
3. $\lambda_\nu(X)f(x) = \mathrm{tr}[(\nu a - ax + (ax + xa - \nu a)\nabla - x\nabla a\nabla]f(x)$, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$.

Here we use ∇ to denote the gradient, $\mathfrak{k}_\mathbb{C}$ is the complexification of the Lie subalgebra $\mathfrak{u}(n) \subset \mathfrak{sp}(n, \mathbb{R})$, and $\mathfrak{p}^\pm$ are certain Abelian subalgebras of $\mathfrak{sp}(n, \mathbb{C})$ on which $\mathfrak{k}_\mathbb{C}$ acts. We explain the main ideas for the special case of $\mathrm{Sp}(1, \mathbb{R}) \simeq \mathrm{SL}(2, \mathbb{R})$.

Specializing the above results to the elements

$$X=\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\in\mathfrak{p}^+,\quad Y=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\in\mathfrak{p}^-\quad\text{and}\quad Z=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in\mathfrak{k}_{\mathbb{C}},$$

where 1 stands for the $n\times n$ identity matrix, using properties of highest weight representations, and employing Lemma 5.5 of [5] we get our main result.

Theorem 6.3. *The Laguerre functions are related by the following differential recursion relations*:

(1) $\operatorname{tr}(-x\nabla\nabla-\nu\nabla+x)\ell^\nu_{\mathbf{m}}(x)=(n\nu+2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}(x)$.
(2) $\operatorname{tr}(x\nabla\nabla+(\nu I+2x)\nabla+(\nu I+x))\ell^\nu_{\mathbf{m}}(x)=-2\sum_{j=1}^r\binom{\mathbf{m}}{\mathbf{m}-\mathbf{e}_j}(m_j-1+\nu-(j-1))\ell^\nu_{\mathbf{m}-\mathbf{e}_j}(x)$.
(3) $\operatorname{tr}(-x\nabla\nabla+(-\nu I+2x)\nabla+(\nu I-x))\ell^\nu_{\mathbf{m}}(x)=2\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\mathbf{e}_j}(x)$.

1. The Jordan algebra of real symmetric matrices

In this section we introduce the Jordan algebra J of real symmetric matrices. We then discuss the space of L-invariant polynomial functions on J and the Γ-function associated to the cone of symmetric positive matrices. Finally we introduce the generalized Laguerre functions and polynomials. We refer to [9] for more information on Jordan algebras.

1.1. *The Jordan algebra* $J=\mathrm{Sym}(n,\mathbb{R})$

We denote by J the vector space of all real symmetric $n\times n$ matrices. The multiplication $x\circ y=\frac{1}{2}(xy+yx)$ and the inner product $(x\mid y)=\operatorname{tr}(xy)$ turn J into a real Euclidean simple Jordan algebra. The determinant and trace functions for J are the usual determinant and trace of an $n\times n$ matrix and will be denoted det and tr, respectively. Observe that $\dim J:=d=n(n+1)/2$. Let Ω denote the interior of the cone of squares: $\{x^2\mid x\in J\}$. Then Ω is the set of all positive definite matrices in J. Let

$$H(\Omega)=\{g\in\mathrm{GL}(J)\mid g\Omega=\Omega\}$$

and let H be the connected component of the identity of $H(\Omega)$. Then H can be identified with $\mathrm{GL}(n,\mathbb{R})_+$ (where $+$ indicates positive determinant) acting on Ω by the formula

$$g\cdot x=gxg^{\mathrm{t}},\quad g\in H,\ x\in\Omega.$$

This action is transitive and, since Ω is self-dual, it follows that Ω is a symmetric cone. Let L be the stability subgroup of the identity $e\in\Omega$. Then $L=\mathrm{SO}(n,\mathbb{R})$ and

$$\Omega\simeq H/L. \tag{1.1}$$

Let $E_{i,i}$ be the diagonal $n\times n$ matrix with 1 in the (i,i)-position and zeros elsewhere. Then $(E_{1,1},\dots,E_{n,n})$ is a Jordan frame for J. Let $J^{(k)}$ be the $+1$-eigenspace of the idempotent $E_{1,1}+\cdots+E_{k,k}$ acting on J by multiplication. Each $J^{(k)}$ is a Jordan subalgebra and we have

$$J^{(1)}\subset J^{(2)}\subset\cdots\subset J^{(n)}=J.$$

If $\det_k$ is the determinant function for J_k and P_k is orthogonal projection of J onto $J^{(k)}$ then the function $\Delta_k(x)=\det_k P_k(x)$ is the usual kth principal minor for an $n\times n$ symmetric matrix; it is homogeneous of degree k. In particular $\Delta(x):=\Delta_n(x)=\det(x)$. Note also that

$$\Delta(h\cdot x)=\det(h)^2\Delta(x)\quad\forall h\in H. \tag{1.2}$$

For $\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{C}^n$ we write $\mathbf{m}\geqslant 0$ if each m_i is a nonnegative integer and $m_1\geqslant m_2\geqslant\cdots\geqslant m_n\geqslant 0$. We let $\Lambda=\{\mathbf{m}\mid\mathbf{m}\geqslant 0\}$. For each $\mathbf{m}\in\Lambda$ define

$$\Delta_{\mathbf{m}}=\Delta_1^{m_1-m_2}\Delta_2^{m_2-m_3}\cdots\Delta_{n-1}^{m_{n-1}-m_n}\Delta_n^{m_n}.$$

These are the *generalized power functions*. It is not hard to see that the degree of $\Delta_{\mathbf{m}}$ is $|\mathbf{m}| := m_1 + \cdots + m_n$. Observe that each generalized power function extends to a holomorphic polynomial on $J_{\mathbb{C}} = \mathrm{Sym}(n, \mathbb{C})$ in a unique way.

1.2. The L-invariant polynomials

For each $\mathbf{m} \in \Lambda$ we define an L-invariant polynomial, $\psi_{\mathbf{m}}$, on $J_{\mathbb{C}}$ by

$$\psi_{\mathbf{m}}(z) = \int_L \Delta_{\mathbf{m}}(lz)\,\mathrm{d}l, \quad z \in J_{\mathbb{C}},$$

where $\mathrm{d}l$ is normalized Haar measure on L. A well-known theorem of Schmid (cf. [25]) gives the following.

Lemma 1.1. *If $\mathscr{P}(J_{\mathbb{C}})$ is the space of all polynomial functions on $J_{\mathbb{C}}$ and $\mathscr{P}(J_{\mathbb{C}})^L$ denotes the space of L-invariant polynomials then $\{\psi_{\mathbf{m}} \mid \mathbf{m} \geqslant 0\}$ is a basis of $\mathscr{P}(J_{\mathbb{C}})^L$. Furthermore, if $\mathscr{P}_k(J_{\mathbb{C}})^L$ denotes the space of L-invariant polynomials of degree less than or equal to k then $\{\psi_{\mathbf{m}} \mid |\mathbf{m}| \leqslant k\}$ is a basis of $\mathscr{P}_k(J_{\mathbb{C}})^L$.*

This lemma implies among other things that $\psi_{\mathbf{m}}(e + x)$ is a linear combination of $\psi_{\mathbf{n}}$, $|\mathbf{n}| \leqslant |\mathbf{m}|$. The *generalized binomial coefficients*, $\binom{\mathbf{m}}{\mathbf{n}}$, are defined by the equation

$$\psi_{\mathbf{m}}(e + x) = \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(x).$$

1.3. The generalized Gamma function

For $\mathbf{m} \in \mathbb{C}^n$, we define $\Delta_{\mathbf{m}}(x)$, $x \in \Omega$, by the same formula given above for $\mathbf{m} \in \Lambda$. The *generalized Gamma function* is defined by

$$\Gamma_\Omega(\mathbf{m}) = \int_\Omega \mathrm{e}^{-\mathrm{tr}x} \Delta_{\mathbf{m}}(x) \Delta(x)^{-(n+1)/2}\,\mathrm{d}x.$$

Conditions for convergence of this integral are given in the proposition below. If λ is a real number we will associate the multi-index $(\lambda, \ldots, \lambda)$ and denote it by λ as well. The context of use should not cause confusion. Thus we define

$$(\lambda)_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}.$$

For later reference we note the following facts about the generalized Gamma function:

Proposition 1.2. *Let the notation be as above. Then the following holds*:

(1) *If $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n$ then the integral defining the generalized Gamma function converges if* $\mathrm{Re}(m_j) > (j-1)\frac{1}{2}$, *for $j = 1, \ldots, n$, and in this case*

$$\Gamma_\Omega(\mathbf{m}) = (2\pi)^{n(n-1)/4} \prod_{i=1}^{n} \Gamma\left(m_i - (i-1)\frac{1}{2}\right),$$

where Γ is the usual Gamma function. In particular it follows, that the Γ-function has a meromorphic continuation to all of $\mathbb{C}^n$.

(2) *If $\mathbf{e}_k$ is an n*-vector *with* 1 *in the kth position and* 0*'s elsewhere then the following holds for all $\mathbf{m} \in \mathbb{C}^n$*:

(a) $\dfrac{\Gamma_\Omega(\mathbf{m})}{\Gamma_\Omega(\mathbf{m} - \mathbf{e}_k)} = m_k - 1 - (k-1)\dfrac{1}{2}$,

(b) $\dfrac{\Gamma_\Omega(\mathbf{m} + \mathbf{e}_k)}{\Gamma_\Omega(\mathbf{m})} = m_k - (k-1)\dfrac{1}{2}$.

Proof. Part (2) follows immediately from (1) and part (1) is Theorem 7.1.1 of Faraut and Koranyi [9]. □

1.4. The generalized Laguerre functions and polynomials

Let $\nu > 0$ and $\mathbf{m} \in \Lambda$. The *generalized Laguerre polynomial* is defined (cf. [9], p. 242) by the formula

$$L^{\nu}_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x), \quad x \in J$$

and the *generalized Laguerre function* is defined by

$$\ell^{\nu}_{\mathbf{m}}(x) = \mathrm{e}^{-\operatorname{tr} x} L^{\nu}_{\mathbf{m}}(2x).$$

Remark 1.3. In the case $n = 1$, i.e. in the case $G \simeq \mathrm{Sp}(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, the generalized Laguerre polynomials and functions defined above are precisely the classical Laguerre polynomials and functions defined on $\mathbb{R}^+$. We refer to [6] for the discussion of that case.

The determinant $\mathrm{Det}(h)$ of $h \in H$ acting on J is

$$\mathrm{Det}(h) = \det(h)^{n+1}.$$

It follows from (1.2) that the measure

$$\mathrm{d}\mu_0(x) = \Delta(x)^{-(n+1)/2}\, \mathrm{d}x$$

is H-invariant. Here $\mathrm{d}x$ is the Lebesgue measure on J. More generally, we set $\mathrm{d}\mu_\nu(x) = \Delta(x)^{\nu-(n+1)/2}\, \mathrm{d}x$ and define

$$L^2_\nu(\Omega) = L^2(\Omega, \mathrm{d}\mu_\nu).$$

We observe that by (1.2) it follows that H acts unitarily on $L^2_\nu(\Omega)$ be the formula

$$\lambda_\nu(h) f(x) = \det(h)^\nu f(h^{\mathrm{t}} \cdot x).$$

Theorem 1.4 (*Davidson, Ólafsson and Zhang [7], Faraut and Koranyi [9]*). *The set* $\{\ell^\nu_{\mathbf{m}} \mid \mathbf{m} \geqslant 0\}$ *forms a complete orthogonal system in* $L^2_\nu(\Omega)^L$, *the Hilbert space of* L*-invariant functions in* $L^2_\nu(\Omega)$.

In [6] it was shown, that the classical differential recursion relations and differential equations for the Laguerre functions on $\mathbb{R}^+$ follows from the representation theory of $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. In [4] this was generalized to the space of complex Hermitian matrices. It is a goal of this article to extend this result to the generalized Laguerre functions defined of the cone of symmetric matrices. This indicates nicely what the more general results should be. Here we use heavily the structure of $\mathrm{Sp}(n, \mathbb{R})$ and its Lie algebra, but the proof of the general results should be more in the line of Jordan algebras.

2. The tube domain $T(\Omega)$, the group $Sp(n, \mathbb{R})$ and its lie algebra

In this section we introduce the tube domain $T(\Omega) = \Omega + i\,\mathrm{Sym}(n, \mathbb{R})$ and discuss the action of the group $\mathrm{Sp}(n, \mathbb{R})$ on this domain. We then discuss some important Lie subalgebras of $\mathfrak{sp}(n, \mathbb{C})$, the complexification of the Lie algebra of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$. These subalgebras will show up again in Section 4 where we compute their action on Hilbert spaces of holomorphic functions on $T(\Omega)$ introduced in the next section. We then use that information to construct the Laguerre differential operators.

2.1. The group $Sp(n, \mathbb{R})$

Let $T(\Omega) = \Omega + iJ$ be the tube over Ω in $J_{\mathbb{C}}$, which we identify with the space of complex $n \times n$ symmetric matrices. Let G_0 be the group of biholomorphic diffeomorphisms on $T(\Omega)$. Then G_0 is a Lie group with Lie algebra isomorphic to $\mathfrak{sp}(n, \mathbb{R})$ and acts homogeneously on $T(\Omega)$. The group $\mathrm{Sp}(n, \mathbb{R})$ is isomorphic to a finite covering group of G_0 in the following precise way. The usual definition of $\mathrm{Sp}(n, \mathbb{R})$ is

$$\mathrm{Sp}(n, \mathbb{R}) = \{g \in \mathrm{SL}(2n, \mathbb{R}) \mid gjg^{\mathrm{t}} = j\},$$

where $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2n, \mathbb{R})$. Defined in this way $\mathrm{Sp}(n, \mathbb{R})$ acts by linear fractional transformations on the *upper half plane* $J + i\Omega$. Let G be the group defined by

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}) \right\}.$$

This means then that the $(1, 2)$ and $(2, 1)$ entries of an element of G are purely imaginary matrices. For a $2n \times 2n$-matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have $g \in G$ if and only if the following relations among A, B, C, and D hold:

$$\begin{array}{ll} A^{\mathrm{t}}C - C^{\mathrm{t}}A = 0, & AB^{\mathrm{t}} - BA^{\mathrm{t}} = 0, \\ A^{\mathrm{t}}D - C^{\mathrm{t}}B = I, & AD^{\mathrm{t}} - BC^{\mathrm{t}} = I, \\ B^{\mathrm{t}}D - D^{\mathrm{t}}B = 0, & CD^{\mathrm{t}} - DC^{\mathrm{t}} = 0, \\ B^{\mathrm{t}}C - D^{\mathrm{t}}A = -I, & AD^{\mathrm{t}} - BC^{\mathrm{t}} = -I. \end{array}$$

Clearly G is isomorphic to $\mathrm{Sp}(n, \mathbb{R})$. It acts on the *right-half plane* $T(\Omega) = \Omega + iJ$ by linear fractional transformations: if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $z \in T(\Omega)$ then

$$g \cdot z = (Az + B)(Cz + D)^{-1}.$$

It is also a finite covering group of G_0.

2.2. Some subgroups of G

Let e be the $n \times n$-identity matrix. Then $e \in \Omega \subset T(\Omega)$. Let K be the stability subgroup of e in G. Then

$$K = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G \mid A \pm B \in U(n) \right\} \simeq U(n).$$

Here the isomorphism is given by

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto A + B.$$

The subgroup K is a maximal compact subgroup of G and G/K is naturally isomorphic to $T(\Omega)$ by the map $gK \to g \cdot e$. The connected component of the identity of the subgroup of G that leaves Ω invariant is isomorphic to H via the map

$$h \to \begin{pmatrix} h & 0 \\ 0 & (h^{\mathrm{t}})^{-1} \end{pmatrix}.$$

This map realizes L as a subgroup of G as well. In fact, we have

$$L = H \cap K,$$

via the above isomorphism.

2.3. Lie algebras

If $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ then $G = P^{-1}\mathrm{Sp}(n, \mathbb{R})P$. From this it follows that the Lie algebra, $\mathfrak{g}$, of G is given by $\mathfrak{g} = P^{-1}\mathfrak{sp}(n, \mathbb{R})P$, and hence

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & ib \\ -ic & -a^{\mathrm{t}} \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b, c \text{ real}, b = b^{\mathrm{t}}, c = c^{\mathrm{t}} \right\}. \tag{2.1}$$

We define a *Cartan involution* on $\mathfrak{g}$ by

$$\theta(X) = -X^*.$$

It induces a decomposition of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ into ± 1-eigenspaces, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$. The $+1$-eigenspace $\mathfrak{k}$ is the Lie algebra of K. These spaces are given by

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\} = \left\{ \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = -a^{\mathrm{t}}, b = b^{\mathrm{t}}, \mathrm{tr}(a) = 0 \right\}$$

and

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\} = \left\{ \begin{pmatrix} a & ib \\ -ib & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = a^{\mathrm{t}}, b = b^{\mathrm{t}} \right\}.$$

Their complexifications are given by

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = -a^{\mathrm{t}}, b = b^{\mathrm{t}}, \mathrm{tr}(a) = 0 \right\}$$

and

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = a^{\mathrm{t}}, b = b^{\mathrm{t}} \right\}.$$

It is clear that $K_{\mathbb{C}}$ acts on $\mathfrak{p}_{\mathbb{C}}$. This representation decomposes into two parts. For that let $Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $Z \in \mathscr{Z}(\mathfrak{k}_{\mathbb{C}})$ and $\mathrm{ad}(Z) : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ has eigenvalues 0, 2, -2. The 0-eigenspace is $\mathfrak{k}_{\mathbb{C}}$, the $+2$-eigenspace is denoted by $\mathfrak{p}^-$ and is given by

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}} \mid a = a^{\mathrm{t}} \right\} \subset \mathfrak{p}_{\mathbb{C}}$$

and the -2-eigenspace is denoted by $\mathfrak{p}^+$ and is given by

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{g}_{\mathbb{C}} \mid a = a^{\mathrm{t}} \right\} \subset \mathfrak{p}_{\mathbb{C}}.$$

Each of the spaces $\mathfrak{p}^{\pm}$ are invariant under $K_{\mathbb{C}}$ and irreducible as $K_{\mathbb{C}}$ representation. Note that this is not necessarily the standard notation. In our notation the eigenvectors (in $\mathfrak{p}^+$) with -2-eigenvalue correspond to *annihilation* operators while the eigenvectors (in $\mathfrak{p}^-$) with $+2$-eigenvalue correspond to *creation* operators. These operators will be described is Section 4 below.

3. The highest weight representations $(\pi_\nu, \mathscr{H}_\nu(T(\Omega)))$

In this section we introduce the highest weight representations $(\pi_\nu, \mathscr{H}_\nu(T(\Omega)))$ and state the main results needed. We also introduce the Laplace transform as a special case of the *restriction principle* introduced in [22].

3.1. Unitary representations of G in $\mathscr{O}(T(\Omega))$

In this subsection we define a series of unitary representations of G on a Hilbert space of holomorphic functions on $T(\Omega)$. These representations are well known. Let $\tilde{G}$ be the universal covering group of G. Then $\tilde{G}$ acts on $T(\Omega)$ by $(g, z) \mapsto \kappa(g) \cdot z$ where $\kappa : \tilde{G} \to G$ is the canonical projection. For $\nu > n$ let $\mathscr{H}_\nu(T(\Omega))$ be the space of holomorphic functions $F : T(\Omega) \to \mathbb{C}$ such that

$$\|F\|_v^2 := \alpha_v \int_{T(\Omega)} |F(x+iy)|^2 \Delta(x)^{v-(n+1)}\, dx\, dy < \infty, \tag{3.1}$$

where

$$\alpha_v = \frac{2^{nv}}{(4\pi)^d \Gamma_\Omega\left(v - \frac{n+1}{2}\right)}.$$

Then $\mathscr{H}_v(T(\Omega))$ is a non-trivial Hilbert space with inner product

$$(F \mid G) = \alpha_v \int_{T(\Omega)} F(x+iy)\overline{G(x+iy)}\, \Delta(x)^{v-(n+1)}\, dx\, dy. \tag{3.2}$$

For $v \leqslant n$ this space reduces to $\{0\}$. If $v = n+1$ this is the *Bergman space*. The space $\mathscr{H}_v(T(\Omega))$ is a *reproducing kernel Hilbert space*. This means that point evaluation

$$E_z : \mathscr{H}_v((T(\Omega)) \to \mathbb{C},$$

given by $E_z F = F(z)$, is continuous for every $z \in T(\Omega)$. This implies the existence of a kernel function $K_z \in \mathscr{H}_v(T(\Omega))$, such that $F(z) = (F \mid K_z)$ for all $F \in \mathscr{H}_v(T(\Omega))$ and $z \in T(\Omega)$. Set $K(z, w) = K_w(z)$. Then $K(z, w)$ is holomorphic in the first variable and antiholomorphic in the second variable. The function $K(z, w)$ is called *the reproducing kernel* for $\mathscr{H}_v(T(\Omega))$. We note that the Hilbert space is completely determined by the function $K(z, w)$. In particular, we have:

(1) The space of finite linear combinations $\mathscr{H}_v(T(\Omega))^0 := \{\sum c_j K_{w_j} \mid c_j \in \mathbb{C},\ w_j \in T(\Omega)\}$ is dense in $\mathscr{H}_v(T(\Omega))$;
(2) The inner product in $\mathscr{H}_v(T(\Omega))^0$ is given by

$$\left(\sum_j c_j K_{w_j} \,\middle|\, \sum_k d_k K_{z_k}\right) = \sum_{j,k} c_j \overline{d_k} K(z_k, w_j).$$

We refer to [7,17] for more details.

3.2. The unitary representations $(\pi_v, \mathscr{H}_v(T(\Omega)))$

For $g \in \tilde{G}$ and $z \in T(\Omega)$, let $J(g, z)$ be the *complex* Jacobian determinant of the action of $g \in \tilde{G}$ on $T(\Omega)$ at the point z. We will use the same notation for elements $g \in G$. A straightforward calculation gives

$$J(g, z) = \det(Cz + D)^{-n-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \text{ and } z \in T(\Omega).$$

We also have the *cocycle relation*

$$J(ab, z) = J(a, b \cdot z) J(b, z)$$

for all $a, b \in \tilde{G}$ and $z \in T(\Omega)$. It is well known that for $v > n$ the formula

$$\pi_v(g) f(z) = J(g^{-1}, z)^{v/(n+1)} f(g^{-1} \cdot z) = \det(A - zC)^{-v} f(g^{-1} \cdot z), \tag{3.3}$$

defines a unitary irreducible representation of $\tilde{G}$. In [9,24,29] it was shown that this unitary representation $(\pi_v, \mathscr{H}_v(T(\Omega)))$ has an analytic continuation to the half-interval $v > (n-1)\frac{1}{2}$. Here the representation π_v is given by the same formula (3.3) but the formula for the *norm* in (3.1) is no longer valid. There are also finitely many equidistant values of v that give rise to unitary representations, but they will not be of concern to us here.

In the following theorem we summarize what we have discussed and collect additional information from [7,9] (cf. p. 260, in particular, Theorem XIII.1.1 and Proposition XIII.1.2).

Theorem 3.1. *Let the notation be as above and assume that $\nu > n$. Then the following hold*:

(1) *The space $\mathscr{H}_\nu(T(\Omega))$ is a reproducing Hilbert space.*
(2) *The reproducing kernel of $\mathscr{H}_\nu(T(\Omega))$ is given by*

$$K_\nu(z, w) = \Gamma_\Omega(\nu)\Delta(z + \bar{w})^{-\nu}.$$

(3) *If $\nu > \frac{1}{2}(n-1)$ then there exists a Hilbert space $\mathscr{H}_\nu(T(\Omega))$ of holomorphic functions on $T(\Omega)$ such that $K_\nu(z, w)$ defined in* (2) *is the reproducing kernel of that Hilbert space. The representation π_ν defines a unitary representation of $\tilde{G}$ on $\mathscr{H}_\nu$.*
(4) *If $\nu > \frac{1}{2}(n-1)$ then the functions*

$$q^\nu_{\mathbf{m}}(z) := \Delta(z+e)^{-\nu}\psi_{\mathbf{m}}\left(\frac{z-e}{z+e}\right), \quad \mathbf{m} \in \Lambda,$$

form an orthogonal basis of $\mathscr{H}_\nu(T(\Omega))^L$, the space of L-invariant functions in $\mathscr{H}_\nu(T(\Omega))$.

3.3. The restriction principle and the Laplace transform

The restriction principle [21,20] is a general recipe to construct unitary maps between a reproducing kernel Hilbert space of holomorphic functions and L^2-spaces on a totally real submanifold. Suppose M is a complex manifold and $\mathbb{H}(M)$ is a reproducing kernel Hilbert space of holomorphic functions on M with kernel K. Suppose X is a totally real submanifold of M and a measure space for some measure μ.

Assume we have a holomorphic function D on M, such that D is positive on X, and such that the map

$$R : \mathbb{H}(M) \to L^2(X, \mu),$$

given by $Rf(x) = D(x)f(x)$, is densely defined. As each f is holomorphic, its restriction to X is injective. It follows that R is an injective map. We call R a *restriction map*. Assume R is closed and has dense range. If $K(z, w) = K_w(z)$ is the reproducing kernel for $\mathbb{H}(M)$, and $f \in L^2(X, \mathrm{d}\mu)$, then

$$\begin{aligned} R^* f(z) &= (R^* f \mid K_z) \\ &= (f \mid RK_z) \\ &= \int_X f(x)D(x)K(z, x)\,\mathrm{d}\mu(x). \end{aligned}$$

In particular, if we set $\Psi(x, y) = D(y)D(x)K(y, x)$, then RR^* is given by

$$RR^* f(y) = \int_X f(x)\Psi(x, y)\,\mathrm{d}\mu(x)$$

and thus is an integral operator. Consider the polar decomposition of the operator R^*. We can write

$$R^* = U\sqrt{RR^*},$$

where U is a unitary operator

$$U : L^2(X, \mu) \to \mathbb{H}(M).$$

The unitary map U is sometimes called the *generalized Segal–Bargmann transform*. In many applications of the restriction principle, M and X will be homogeneous spaces with a group H acting on both. When the restriction map R is H-intertwining so will the unitary operator U. This is exactly what happens in the situation at hand. Here we can take $D = 1$ and define $R : \mathscr{H}_\nu(T(\Omega)) \to L^2_\nu(\Omega)$ by

$$Rf(x) = f(x)\,.$$

Then we obtain the following:

Theorem 3.2. *The map R is injective, densely defined and has dense range. The unitary part, U, of the polar decomposition of R^*: $R^* = U\sqrt{RR^*}$, is the Laplace transform given by*

$$Uf(z) = \mathscr{L}_\nu f(z) = \int_\Omega \mathrm{e}^{-(z|x)} f(x)\,\mathrm{d}\mu_\nu(x).$$

Furthermore,

$$\mathscr{L}_\nu(\lambda_\nu(h)f) = \pi_\nu(h)\mathscr{L}_\nu(f)$$

for all $h \in H$. In particular, $\mathscr{L}_\nu$ induces an isomorphism $\mathscr{L}_\nu : L^2_\nu(\Omega)^L \to \mathscr{H}_\nu(T(\Omega))^L$. Moreover

$$\mathscr{L}_\nu(\ell^\nu_{\mathbf{m}}) = \Gamma_\Omega(\mathbf{m}+\nu)q^\nu_{\mathbf{m}}.$$

Proof. The first proof of this theorem was done for SL(2, $\mathbb{R}$) in [6]. The general case is on pp. 187–190 of Davidson and Ólafsson [7]. □

Remark 3.3. Rossi and Vergne [24] obtained the unitarity of the Laplace transform using a result of Nussbaum.

The unitarity of the Laplace transform allows us to transfer the representation, π_ν, of G on $\mathscr{H}_\nu(T(\Omega))$ to an equivalent representation of G on $L^2_\nu(\Omega)$, which extends λ_ν by the above theorem. We will denote the extension by λ_ν as well. It is possible to describe λ_ν on various subgroups of G whose product is dense in G. However, it is a difficult problem at best to describe a global realization of λ_ν on all of G. However, part of the point of this paper is to give a formula for the derived representation of λ_ν on the Lie algebra of G and its complexification. It is from the derived representation that new differential recursion relations arise that relate the generalized Laguerre functions.

4. The action of $\mathfrak{g}_\mathbb{C}$

In this section we introduce some subalgebras of $\mathfrak{sp}(n,\mathbb{C})$, the complexification of the Lie algebra of G, and explain how they act in the Hilbert space $\mathscr{H}_\nu(T(\Omega))$.

4.1. The derived representation on $\mathscr{H}_\nu(T(\Omega))$

Denote by $\mathscr{H}_\nu(T(\Omega))^\infty$ the space of functions $F \in \mathscr{H}_\nu(T(\Omega))$ such that the map

$$\mathbb{R} \ni t \mapsto \pi_\nu(\exp tX)F \in \mathscr{H}_\nu(T(\Omega))$$

is smooth for all $X \in \mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$. If $f \in C^\infty_c(G)$, then $\pi_\nu(f)F = \int_G f(g)\pi_\nu(g)F\,\mathrm{d}g$ is in $\mathscr{H}_\nu(T(\Omega))^\infty$ and it follows, that $\mathscr{H}_\nu(T(\Omega))^\infty$ is dense in $\mathscr{H}_\nu(T(\Omega))$. The Lie algebra representation, denoted also by π_ν, of $\mathfrak{g}$ on $\mathscr{H}_\nu(T(\Omega))^\infty$ is given, by differentiation as follows:

$$\begin{aligned}\pi_\nu(X)F &= \lim_{t\to 0}\frac{\pi_\nu(\exp tX)F - F}{t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\pi_\nu(\exp(tX))F|_{t=0}.\end{aligned}$$

Note that the limit is taken in the Hilbert space norm in $\mathscr{H}_\nu(T(\Omega))$, but it is easy to see that if $F \in \mathscr{H}_\nu(T(\Omega))^\infty$, then in fact for $X \in \mathfrak{g}$:

$$\pi_\nu(X)F(z) = \frac{\mathrm{d}}{\mathrm{d}t}J(\exp(-tX), z)^{\nu/(n+1)}F(\exp(-tX)\cdot z)|_{t=0}, \tag{4.1}$$

for all $z \in T(\Omega)$. We extend this by complex linearity to $\mathfrak{g}_\mathbb{C}$.

Define D_w by

$$D_w F(z) = \frac{\mathrm{d}}{\mathrm{d}t} F(z + tw)|_{t=0} = F'(z)w,$$

where F' denotes the derivative of F.

Lemma 4.1. *Suppose z, w are $n \times n$ matrices over* $\mathbb{C}$ and *z is invertible. Then*

$$D_w \det(z)^n = n \det(z)^n \operatorname{tr}(z^{-1}w).$$

Proof. This follows from the chain rule and the fact that

$$D_w \det(z) = \frac{\mathrm{d}}{\mathrm{d}t} \det(z + tw)|_{t=0} = \det z \frac{\mathrm{d}}{\mathrm{d}t} \det(1 + tz^{-1}w)|_{t=0} = \det(z) \operatorname{tr}(z^{-1}w). \qquad \square$$

The following proposition expresses the relevant formulas on $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$. Its proof is a straight forward calculation using Lemma 4.1.

Proposition 4.2. *For each piece of the Lie algebra of $\mathfrak{g}_{\mathbb{C}}$ introduced in Subsection* 2.3, *we have*

(1) $\pi_\nu(X)F(z) = \nu \operatorname{tr}(bz)F(z) + D_{za-az-b+zbz}F(z)$, $X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$.

(2) $\pi_\nu(X)F(z) = -\nu \operatorname{tr}(az + a)F(z) - D_{(za+az)+zaz+a}F(z)$, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$.

(3) $\pi_\nu(X)F(z) = -\nu \operatorname{tr}(-az + a))F(z) + D_{-(za+az)+zaz+a}F(z)$, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$.

4.2. Highest weight representations

The fact that π_ν is a highest weight representation plays a decisive role in the recursion relations that we obtain. At this point we explain what this notion means.

We assume G is a Hermitian group, which means that G is simple and the maximal compact subgroup K has a one dimensional center. The Hermitian groups have been classified in terms of their Lie algebras. They are $\mathfrak{su}(p, q)$, $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(2, n)$, and two exceptional Lie algebras. The assumption that K has a one dimensional center implies that G/K is a bounded symmetric domain. In particular, there is a G-invariant complex structure on G/K. It also implies that the complexification of the Lie algebra, $\mathfrak{g}_{\mathbb{C}}$, has a decomposition of the form $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$, Specifically, $\mathfrak{p}^+$, $\mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{p}^-$ are the $-2, 0, 2$-eigenspaces of $\operatorname{ad}(Z)$, respectively, where Z is in the center of $\mathfrak{k}_{\mathbb{C}}$.

Lemma 4.3. *We have the following inclusions*:

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm,$$
$$[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}_{\mathbb{C}}.$$

Suppose that π is an irreducible representation of G on a Hilbert space $\mathbb{H}$. We say π is a *highest weight representation* if there is a nonzero vector $v \in \mathbb{H}$ such that

$$\pi(X)v = 0,$$

for all $X \in \mathfrak{p}^+$. Let $\mathbb{H}_0$ be the set of all such vectors. The following theorem is well known.

Theorem 4.4. *Suppose π is an irreducible unitary highest weight representation of G on $\mathbb{H}$ and $\mathbb{H}_0$ is defined as above. Then $(\pi|_K, \mathbb{H}_0)$ is irreducible. Furthermore, there is a scalar λ such that*

$$\pi(Z)v = \lambda v,$$

for all $v \in \mathbb{H}_0$. If

$$\mathbb{H}_n = \{v \in \mathbb{H} \mid \pi(Z)v = (\lambda + 2n)v\},$$

then

$$\mathbb{H} = \bigoplus_{n \geqslant 0} \mathbb{H}_n.$$

Additionally,

$$\begin{aligned} \pi(Z) &: \mathbb{H}_n \longrightarrow \mathbb{H}_n, \quad Z \in \mathfrak{k}_{\mathbb{C}}, \\ \pi(X) &: \mathbb{H}_n \longrightarrow \mathbb{H}_{n-1}, \quad X \in \mathfrak{p}^+, \\ \pi(Y) &: \mathbb{H}_n \longrightarrow \mathbb{H}_{n+1}, \quad Y \in \mathfrak{p}^-, \end{aligned}$$

where in the case $n = 0$, $\mathbb{H}_{-1}$ *is understood to be the* $\{0\}$ *space*.

Proof. By Lemma 4.3, $\mathbb{H}_0$ is an invariant K-space. Suppose $\mathbb{V}_0$ is a nonzero invariant subspace of $\mathbb{H}_0$ and $\mathbb{W}_0$ is its orthogonal complement in $\mathbb{H}_0$. Define $\mathbb{V}_n$ inductively as follows:

$$\mathbb{V}_n = \operatorname{span}\{\pi(Y)v \mid Y \in \mathfrak{p}^-, v \in \mathbb{V}_{n-1}\}.$$

Let $\mathbb{V} = \oplus \mathbb{V}_n$. Define $\mathbb{W}_n$ in the same way as $\mathbb{V}_n$ and let $\mathbb{W} = \oplus \mathbb{W}_n$. Then, by Lemma 4.3, $\mathbb{V}$ and $\mathbb{W}$ are invariant $\mathfrak{g}_{\mathbb{C}}$ subspaces of $\mathbb{H}$. Since π is unitary $\mathbb{V}$ and $\mathbb{W}$ are orthogonal. However, since π is irreducible and $\mathbb{V}$ is nonzero, it follows that $\mathbb{V} = \mathbb{H}$ and hence $\mathbb{W} = 0$. This implies $\mathbb{W}_0 = 0$ and thus $\pi|_K$ is irreducible. Since $\pi(Z)$ commutes with $\pi(K)$ Schur's lemma implies that $\pi(Z) = \lambda$ on $\mathbb{H}_0$ for some scalar λ. Since $\mathbb{V}_0 = \mathbb{H}_0$, induction, Lemma 4.3, and irreducibility of π implies that $\mathbb{V}_n = \mathbb{H}_n$. The remaining claims follow from Lemma 4.3. $\square$

Remark 4.5. The operators $\pi(X)$, $X \in \mathfrak{p}^+$, are called *annihilation operators* because, for v in the algebraic direct sum $\oplus \mathbb{H}_n$, sufficiently many applications of $\pi(X)$ annihilates v. For $Y \in \mathfrak{p}^-$ the operators $\pi(Y)$ are called *creation operators*.

Remark 4.6. A straightforward calculation gives

$$\pi_\nu(X) q_0^\nu = 0,$$

for all $X \in \mathfrak{p}^+$ and that $\mathscr{H}_\nu(T(\Omega))_0 = \mathbb{C} q_0^\nu$. Thus $(\pi_\nu, \mathscr{H}_\nu(T(\Omega)))$ is an irreducible unitary highest weight representation of G and by unitary equivalence so is $(\lambda_\nu, L^2_\nu(\Omega))$.

5. The realization of λ_ν acting on $L^2(\Omega, \mathrm{d}\mu_\nu)$

In this section we determine explicitly the action of $\mathfrak{g}_{\mathbb{C}}$ on $L^2(\Omega, \mathrm{d}\mu_\nu)$. More specifically, we define λ_ν via the Laplace transform by the following formula

$$\lambda_\nu(X) = \mathscr{L}_\nu^{-1} \pi_\nu(X) \mathscr{L}_\nu$$

and will determine explicit formulas for $\lambda_\nu(X)$, for $X \in \mathfrak{p}^+$, $X \in \mathfrak{k}_{\mathbb{C}}$, and $X \in \mathfrak{p}^-$.

5.1. Preliminaries

Let E_{ij} be the $n \times n$ matrix with a 1 in the (i, j) position and 0's elsewhere. Define $\tilde{E}_{i,j} = \frac{1}{2}(E_{i,j} + E_{j,i})$. Then the collection $\{\tilde{E}_{i,j} \mid 1 \leqslant i \leqslant j \leqslant n\}$ is a basis of J and $J_{\mathbb{C}}$, the real and complex symmetric matrices. Furthermore, $(\tilde{E}_{i,j} \mid \tilde{E}_{k,l}) = \frac{1}{2}(\delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik})$, which implies $\{\tilde{E}_{i,j} \mid 1 \leqslant i \leqslant j \leqslant n\}$ is an orthogonal basis. Set $D_{i,j} = D_{\tilde{E}_{i,j}}$ and observe that $D_{i,j} = D_{j,i}$. The *gradient* of f, ∇f, is defined by

$$(\nabla f(x) \mid u) = D_u f(x).$$

Proposition 5.1. *Suppose $f, g \in L^2(\Omega, \mathrm{d}\mu_\nu)$ are smooth and f vanishes on the boundary of the cone Ω. Let $1 \leqslant i, j \leqslant n$. Then*

(1) $\int_\Omega D_{i,j} f(s) g(s)\,\mathrm{d}s = -\int_\Omega f(s) D_{i,j} g(s)\,\mathrm{d}s$,
(2) $z_{i,j} \int_\Omega \mathrm{e}^{-(z|s)} f(s)\,\mathrm{d}s = \int_\Omega \mathrm{e}^{-(z|s)} D_{i,j} f(s)\,\mathrm{d}s$.

Proof. Step (1) is Stokes Theorem and (2) follows from (1) and the fact that $D_{i,j}\mathrm{e}^{-(z|s)} = -\mathrm{e}^{-(z|s)} z_{i,j}$, $z \in J_{\mathbb{C}}$. □

5.2. The representation λ_ν

Recall that we determined the action of $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ on $\mathscr{H}_\nu(T(\Omega))^\infty$ in Proposition 4.2. We denote the subspace of smooth vectors in $L^2_\nu(\Omega)$ by $L^2_\nu(\Omega)^\infty$. Thus $f \in L^2_\nu(\Omega)^\infty$ if and only if the map

$$\mathbb{R} \ni t \mapsto \lambda_\nu(\exp tX) f \in L^2_\nu(\Omega)$$

is smooth for all $X \in \mathfrak{g}$. Thus

$$L^2_\nu(\Omega)^\infty = \mathscr{L}_\nu^{-1}(\mathscr{H}_\nu(T(\Omega))^\infty).$$

The action of $\mathfrak{g}$ on $L^2_\nu(\Omega)^\infty$ is, as usual, defined by

$$\lambda_\nu(X) f = \lim_{t \to 0} \frac{\lambda_\nu(\exp tX) f - f}{t},$$

for $X \in \mathfrak{g}$, and then by complex linearity the action extends to $\mathfrak{g}_{\mathbb{C}}$. The following theorem collects the corresponding equivalent action on the Hilbert space $L^2_\nu(\Omega)^\infty$. We remark again that these formulas can be stated in terms of the Jordan algebra structure of J indicating the extension of these results to other tube domains, cf. [1].

Theorem 5.2. *For $f \in L^2_\nu(\Omega)$ a smooth function we have*

(1) $\lambda_\nu(X) f(x) = \mathrm{tr}[(bx + (ax - xa - \nu b)\nabla - x\nabla b\nabla] f(x)$, $X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_{\mathbb{C}}$,
(2) $\lambda_\nu(X) f(x) = \mathrm{tr}[\nu a + ax + (ax + xa + \nu a)\nabla + x\nabla a\nabla] f(x)$, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+$,
(3) $\lambda_\nu(X) f(x) = \mathrm{tr}[(\nu a - ax + (ax + xa - \nu a)\nabla - x\nabla a\nabla] f(x)$, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^-$.

5.3. Idea of the proof

Our first proof was similar to the one given in [4] for the Jordan algebra of Hermitian symmetric matrices. After submitting this paper we extended this result to arbitrary irreducible Euclidean Jordan algebras. See [1] for a detailed proof. For the purpose of exposition we discuss here only the case of $\mathrm{Sp}(1, \mathbb{R})$. A detailed account of this case (modeled on the upper half plane) is found in [6].

Let $G = \left\{ \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \right\}$. The group G acts on the right half-plane $T(\mathbb{R}^+)$ by linear fractional transformations. The complexification, $\mathfrak{g}_{\mathbb{C}}$, of the Lie algebra of G is $\mathfrak{sl}(2, \mathbb{C})$, a three-dimensional Lie algebra spanned by

$$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Proposition 4.2 for this case reads as follows.

Theorem 5.3. *The action of* $\mathfrak{sl}(2,\mathbb{R})$ *on the right half-plane is given by*:

(1) $\pi_\nu(Z)F(z)=\nu zF(z)+(z^2-1)F'(z)$,
(2) $\pi_\nu(X)F(z)=-\nu(z+1)F(z)-(z+1)^2F'(z)$,
(3) $\pi_\nu(Y)F(z)=\nu(z-1)F(z)+(z-1)^2F'(z)$.

To find the corresponding action on $L^2_\nu(\mathbb{R}^+)$ we must compute the operators that corresponds to $D_z, M_z, M_{z^2}, M_z\circ D_z$ and $M_{z^2}\circ D_z$ in $L^2_\nu(\mathbb{R}^+)^\infty$. Here M stands for "multiplication operator". To do this, requires several uses of integration by parts, a special case of Stokes theorem. It was exactly this kind of computation that was done in [6] and we repeat it here:

For D_z we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathscr{L}_\nu(f)(z)&=\int_0^\infty\frac{\mathrm{d}e^{-zt}}{\mathrm{d}z}f(t)t^{\nu-1}\,\mathrm{d}t\\&=\mathscr{L}_\nu(-tf(t)).\end{aligned}$$

Thus $D_z\longleftrightarrow M_{-t}$.

For M_z we have

$$\begin{aligned}z\mathscr{L}_\nu(f)(z)&=\int_0^\infty-\frac{\mathrm{d}e^{-zt}}{\mathrm{d}t}f(t)t^{\nu-1}\,\mathrm{d}t\\&=\int_0^\infty e^{-zt}\frac{\mathrm{d}}{\mathrm{d}t}(f(t)t^{\nu-1})\,\mathrm{d}t\\&=\int_0^\infty e^{-zt}\left(f'(t)+\frac{\nu-1}{t}f(t)\right)t^{\nu-1}\,\mathrm{d}t.\end{aligned}$$

Thus $M_z\longleftrightarrow D+M_{(\nu-1)/t}$.

We calculate M_{z^2} similarly and get

$$M_{z^2}\longleftrightarrow D^2+\frac{2(\nu-1)}{t}D+\frac{(\nu-1)(\nu-2)}{t^2}.$$

Thus,

Lemma 5.4. *Let the notation be as above. Then the following holds*:

(1) $D_z\circ\mathscr{L}_\nu=-\mathscr{L}_\nu\circ(M_t)$,
(2) $M_z\circ\mathscr{L}_\nu=\mathscr{L}_\nu\circ\left(D+\frac{\nu-1}{t}\right)$,
(3) $M_z\circ D_z\circ\mathscr{L}_\nu=\mathscr{L}_\nu\circ(-tD-\nu)$,
(4) $M_{z^2}\circ D_z\circ\mathscr{L}_\nu=\mathscr{L}_\nu\circ(-tD^2-2\nu D-\frac{\nu(\nu-1)}{t})$.

Combining Theorem 5.3 and Lemma 5.4 gives

Lemma 5.5. *Let the notation be as above. Then the following holds*:

(1) $\lambda_\nu(Z)=-tD^2-\nu D+t$,
(2) $\lambda_\nu(X)=tD^2+(\nu+2t)D+(\nu+t)$,
(3) $\lambda_\nu(Y)=-tD^2+(-\nu+2t)D+(\nu-t)$.

Note that this is Theorem 5.2 for this special case.

One more ingredient is necessary for determining the classical recursion relations. This is a direct calculation and given in the following lemma. We note at this point that such a direct calculation is not done in the general case; deeper properties of the representation theory must be used. (cf. Proposition 6.1.)

Lemma 5.6. *Let* $q_m^\nu(z) = (z+1)^{-\nu}(\frac{z-1}{z+1})^m$. *Then*

(1) $\pi_\nu(Z)q_m^\nu = (\nu + 2m)q_m^\nu$,
(2) $\pi_\nu(X)q_m^\nu = -2mq_{m-1}^\nu$,
(3) $\pi_\nu(Y)q_m^\nu = 2(\nu + m)q_{m+1}^\nu$.

The combination of Lemmas 5.5 and 5.6 gives the classical recursion relations stated in the introduction and proves Theorem 6.3 for the classical case.

6. Differential recursion relations for ℓ_m^ν

We now turn our attention to differential recursion relations that exist among the generalized Laguerre functions. These relations are obtained by way of the highest weight representation λ_ν and generalize the classical case mentioned in the introduction.

We begin with some preliminaries and a result found in [7]. First we notice that in general the Lie algebra $\mathfrak{g}_\mathbb{C}$ does not map $(L_\nu^2(\Omega)^\infty)^L$ into itself. For the Laguerre functions the full Lie algebra is too big; we will in fact only need the much smaller Lie algebra $\mathfrak{g}_\mathbb{C}^L$, which maps $(L_\nu^2(\Omega)^\infty)^L$ into itself. It is well known, that in case $\mathfrak{g}$ is simple, then $\mathfrak{g}_\mathbb{C}^L \simeq \mathfrak{sl}(2,\mathbb{C})$. We choose Z, X, Y so that the isomorphism, which we will denote by φ, is given by

$$Z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X \mapsto \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad Y \mapsto \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Furthermore, we can assume that $\varphi(X^\mathrm{t}) = \varphi(X)^\mathrm{t}$. This shows that several calculations can in fact be reduced directly to $\mathfrak{sl}(2,\mathbb{C})$. We will come back to that later.

Define $Z^0 := \frac{1}{2}(X+Y)$. Then $\varphi(Z^0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and Z^0 is in the center of $\mathfrak{h}$. For $\mathbf{m} \in \Lambda$ let

$$c_\mathbf{m}(j) = \prod_{j \neq k} \frac{m_j - m_k - \frac{1}{2}(j+1-k)}{m_j - m_k - \frac{1}{2}(j-k)}.$$

Then by Lemma 5.5 in [7] we have:

Proposition 6.1. *The action of Z and* Z^0 *is given by*:

(1) $\pi_\nu(Z)q_\mathbf{m}^\nu = (n\nu + 2\,|\mathbf{m}|)q_\mathbf{m}^\nu$.
(2) $\pi_\nu(-2Z^0)q_\mathbf{m}^\nu = \sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e}_j} q_{\mathbf{m}-\mathbf{e}_j}^\nu - \sum_{j=1}^r(\nu + m_j - \frac{1}{2}(j-1)))c_\mathbf{m}(j)q_{\mathbf{m}+\mathbf{e}_j}^\nu$.

Corollary 6.2. *Let the notation be as above. Then the following holds*:

(1) $\lambda_\nu(Z)\ell_\mathbf{m}^\nu = (n\nu + 2|\mathbf{m}|)\ell_\mathbf{m}^\nu$.
(2) $\lambda_\nu(-2Z^0)\ell_\mathbf{m}^\nu = \sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e}_j}(m_j - 1 + \nu - (j-1))\ell_{\mathbf{m}-\mathbf{e}_j}^\nu - \sum_{j=1}^r c_m(j)\ell_{\mathbf{m}+\mathbf{e}_j}^\nu$.

Proof. This statement follows from Proposition 6.1 and the following three facts: $\lambda_\nu(X) = \mathscr{L}_\nu^{-1}\pi_\nu(X)\mathscr{L}_\nu$, $\mathscr{L}_\nu(\ell_\mathbf{m}^\nu) = \Gamma_\Omega(\mathbf{m}+\nu)q_\mathbf{m}^\nu$, and Proposition 1.2. In each of these formulas if either index $\mathbf{m}+\mathbf{e}_j$ or $\mathbf{m}-\mathbf{e}_j$ is not in Λ then it should be understood that the corresponding function does not appear. □

Theorem 6.3. *The Laguerre functions are related by the following differential recursion relations*:

(1) $\operatorname{tr}(-x\nabla\nabla - \nu\nabla + x)\ell_\mathbf{m}^\nu(x) = (n\nu + 2|\mathbf{m}|)\ell_\mathbf{m}^\nu(x)$.
(2) $\operatorname{tr}(x\nabla\nabla + (\nu I + 2x)\nabla + (\nu I + x))\ell_\mathbf{m}^\nu(x) = -2\sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\mathbf{e}_j}\,(m_j - 1 + \nu - (j-1))\ell_{\mathbf{m}-\mathbf{e}_j}^\nu(x)$.
(3) $\operatorname{tr}(-x\nabla\nabla + (-\nu I + 2x)\nabla + (\nu I - x))\ell_\mathbf{m}^\nu(x) = 2\sum_{j=1}^r c_\mathbf{m}(j)\ell_{\mathbf{m}+\mathbf{e}_j}^\nu(x)$.

Proof. If $Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then substituting $a = 0$ and $b = 1$ into Proposition 5.2, part 1, gives

$$\lambda_\nu(Z) = \operatorname{tr}(-x\nabla\nabla - \nu\nabla + x).$$

Combining this with part 1 of Corollary 6.2 gives the first formula.

Recall that $X \in \mathfrak{p}^+$ and $Y \in \mathfrak{p}^-$. According to Theorem 4.4 we have that $\lambda_\nu(X)\ell^\nu_{\mathbf{m}}$ has to be a linear combination of $\ell^\nu_{\mathbf{m}'}$, with $m'_j \leqslant m_j$ for all j. Similarly, $\lambda_\nu(Y)\ell^\nu_{\mathbf{m}}$ has to be linear combination of those $\ell^\nu_{\mathbf{m}'}$ with $m'_j \geqslant m_j$. The statement follows now from Corollary 6.2 and the fact that $2Z^0 = X + Y$. $\square$

Remark 6.4. Note that viewing x as a constant with respect to differentiation we can write $\operatorname{tr}(x\nabla\nabla) = \operatorname{tr}(\nabla x \nabla)$ and, in terms of Jordan algebras the last operator is just $(P(\nabla)x \mid e)$ where P is the quadratic representation, $(\cdot \mid \cdot)$ is the canonical inner product in the Jordan algebra, and e is the identity element. This is the operator used in the final version of the general case, cf. [1].

7. Some open problems

There are still several open question that require further work. We mention three of these. One is a relation to the classical Laguerre polynomials, the other two are natural generalizations of classical relations.

7.1. Relation to classical Laguerre functions

Every positive symmetric matrix A can be written as $A = kDk^{-1}$, where $k \in \mathrm{SO}(n)$ and $D = \mathrm{d}(t_1, \ldots, t_n)$ is a diagonal matrix with $t_j > 0$. Thus, if

$$\Omega_1 = \{\mathrm{d}(\mathbf{t}) \mid \mathbf{t} \in (\mathbb{R}^+)^n\} \simeq (\mathbb{R}^+)^n$$

then

$$\Omega = L \cdot \Omega_1.$$

As the Laguerre functions are L-invariant, it follows that they are uniquely determined by their restriction to Ω_1. Let $T(\Omega_1) := \{\mathrm{d}(\mathbf{x}) + i\,\mathrm{d}(\mathbf{y}) \mid \mathbf{x} \in (\mathbb{R}^+)^n, \mathbf{y} \in \mathbb{R}^n\}$. Then $T(\Omega_1) \simeq (\mathbb{R}^+ + i\mathbb{R})^n$, and the group $\mathrm{SL}(2,\mathbb{R})^n$ acts transitively on the right hand side. But it is well known, that $\mathrm{SL}(2,\mathbb{R})^n$ can be realized as a closed subgroup of $\mathrm{Sp}(n,\mathbb{R})$. It follows therefore, that the generalized Laguerre functions can be written as a finite linear combinations of products of classical Laguerre functions. It is a natural problem to derive an exact formula.

7.2. Relations in the λ-parameter

It is well known that the classical Laguerre polynomials satisfy the following relations:

$$\begin{aligned} xL_n^\lambda &= (n+\lambda+1)L_n^{\lambda-1} - (n+1)L_{n+1}^{\lambda-1}, \\ xL_n^\lambda &= (n+\lambda)L_{n-1}^{\lambda-1} - (n-x)L_n^{\lambda-1}, \\ xL_n^{\lambda-1} &= L_n^\lambda - L_{n-1}^\lambda. \end{aligned}$$

In [6] it was shown, that these relations follows directly from the representation theory of $\mathfrak{sl}(2,\mathbb{R})$. It is therefore natural to look for similar relations for the generalized Laguerre polynomials and functions.

7.3. Relations in the x, y parameters

Several other classical relations should be extended to the general case. We name here only the following

$$L_m^{\alpha+\beta+1}(x+y) = \sum_{n=0}^{m} L_n^\alpha(x) L_{m-n}^\beta(y).$$

This relation is closely related to the decomposition of the tensor product of two highest weight representations and we expect that a similar relation can be derived also for the general case. Notice, however, that for general Laguerre polynomials the right hand side is L-invariant in the x and y variable while that is not the case on the left hand side. As the function on the left hand side is not L-invariant in x and y separately, any generalization will have to include averaging over L, i.e., the projection onto the space of L-invariant functions.

Acknowledgements

Aristidou was partially supported by NSF Grant DMS-0139783. Both Aristidou and Ólafsson were partially supported by the GSF—National Research Center for Environment and Health. The research of Ólafsson was supported by NSF grants DMS-0139783 and DMS-0402068. The authors would like to thank the referees for valuable comments and pointing out to us the references [2,8]. We would like to thank M. Rösler for also pointing out to us the Ref. [2] and explaining the results therein.

References

[1] M. Aristidou, M. Davidson, G. Ólafsson, Differential Recursion Relations for Laguerre Functions on Symmetric Cones, Bull. Sci. Math., 2005, to appear.
[2] T.H. Baker, P.J. Forrester, The Calogero–Sutherland model and generalized classical polynomials, Comm. Math. Phys. 188 (1997) 175–216.
[3] S. Ben-Säid, B. Ørsted, Wave Equations for Dunkl Differential-Difference Operators, 2005, preprint.
[4] M. Davidson, G. Ólafsson, Differential recursion relations for Laguerre functions on Hermitian matrices, Integral Transforms Special Funct. 14 (4) (2003) 469–484.
[5] M. Davidson, G. Ólafsson, The generalized Segal–Bargmann transform and special functions, Acta Appl. Math. 81 (2004) 29–50.
[6] M. Davidson, G. Ólafsson, G. Zhang, Laguerre polynomials, restriction principle, and holomorphic representations of $SL(2, \mathbb{R})$, Acta Appl. Math. 71 (2002) 261–277.
[7] M. Davidson, G. Ólafsson, G. Zhang, Segal–Bargmann transform on Hermitian symmetric spaces and orthogonal polynomials, J. Funct. Anal. 204 (2003) 157–195.
[8] J.F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement, Comm. Math. Phys. 188 (1997) 467–497.
[9] J. Faraut, A. Koranyi, Analysis on Symmetric Cones, Clarendon Press, Oxford, 1994.
[10] S.G. Gindikin, Analysis on homogeneous domains, Uspekhi Math. Nauk 19 (1964) 3–92 (in Russian), Russian Math. Surveys 19 (4) 1–89 (in English).
[11] C.S. Hertz, Bessel functions of matrix arguments, Ann. Math. 61 (1955) 474–523.
[12] R. Howe, On the role of the Heisenberg group in harmonic analysis, Bull. Amer. Math. Soc. 3 (1980) 821–843.
[13] R. Howe, C.B. Zhu, Nonabelian harmonic analysis, Applications of $SL(2, \mathbb{R})$, Springer, New York, 1992.
[14] T. Kobayashi, G. Mano, Integral formulas for the minimal representation of $O(p, 2)$, Acta Appl. Math. 83 (2005) 103–113.
[15] M. Koecher, Analysis in reellen Jordan Algebren, Nach. Akad. Wiss. Göttingen Math.-Phys., K1 II a, 67–74.
[16] M. Koecher, Jordan algebras and their applications, Lecture Notes, University of Missesota, Minneapolis, 1962.
[17] R. Kunze, Positive definite operator-valued kernels and unitary representations, in: Proceedings of the Conference on Functional Analysis at Irvine, Thompson Book Company, California, 1966, 235–247.
[18] M. Lassalle, Polynomes de Laguerre généralisés, C. R. Acad. Sci. Paris, t. Séries I 312 (1991) 725–728.
[19] W. Miller Jr., Lie Theorey and Special Functions, Academic Press, New York, 1968.
[20] G. Ólafsson, Analytic continuation in representation theory and harmonic analysis, in: J. P. Bourguignon, T. Branson, O. Hijazi (Eds.), Global Analysis and Harmonic Analysis, Seminares et Congres, vol. 4, The French Mathematical Society, 2000, pp. 201–2333.
[21] G. Ólafsson, B. Ørsted, The holomorphic discrete series for affine symmetric spaces, I, J. Funct. Anal. 81 (1988) 126–159.
[22] G. Ólafsson, B. Ørsted, Generalizations of the Bargmann transform. Lie theory and its applications in physics, in: H.-D. Doebner, et al. (Ed.), Proceedings of the International Workshop, Clausthal, Germany, August 14–17, 1995, World Scientific, Singapore, 1996, pp. 3–14, MR. 99c:22017.
[23] M. Rösler, Generalized Hermite polynomials and the heat equation, Comm. Math. Phys. 192 (1998) 519–542.
[24] H. Rossi, M. Vergne, Analytic continuation of the holomorphic discrete series of a semisimple Lie group, Acta Math. 136 (1976) 1–59.
[25] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent. Math. 9 (1969) 61–80.
[26] C.L. Siegel, Über die analytische Theorie der quadratishen Formen, Ann. of Math. 36 (1935) 527–606.
[27] N. J. Vilenkin, Special functions and the theory of group representations, Translations of Mathematical Monographs, vol. 32, Providence, RI, 1968.
[28] N.J. Vilenkin, A.U. Klimyk, Representations of Lie Groups and Special Functions, vol. 1, Kluwer Academic Publishers, Dorderecht, 1991.
[29] N. Wallach, The analytic continuation of the discrete series, I, II, Trans. Amer. Math. Soc. 251 (1979) 1–17, 19–37.