Note

Partitioning Permutations into Increasing and Decreasing Subsequences

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A permutation is an \((r, s)\)-permutation if it can be partitioned into \(r\) increasing and \(s\) decreasing, possibly empty subsequences. For any fixed non-negative integers \(r\) and \(s\), the family of \((r, s)\)-permutations is characterized by a finite list of forbidden subsequences. This is derived from a more general graph-theoretic proof showing that, for any fixed non-negative integers \(r\) and \(s\), the family of perfect graphs whose vertex set admits a partition into \(r\) cliques and \(s\) independent sets is characterized by a finite list of forbidden induced subgraphs. © 1996 Academic Press, Inc.

1. Introduction

A permutation \(\pi\) of \(\{n\} = \{1, \ldots, n\}\) is considered as a sequence \(\pi(1), \pi(2), \ldots, \pi(n)\). We shall omit commas when doing so produces no ambiguity. The length of \(\pi\) is \(n\).

Let \(\tau\) and \(\pi\) be two permutations of lengths \(m\) and \(n\), respectively. Suppose that \(\tau(i) = b_i\), for \(1 \leq i \leq m\), and \(\pi(i) = a_i\), for \(1 \leq i \leq n\). We say that \(\pi\) contains \(\tau\) if \(m\) of the \(a_i\)'s exist, \(a_{i_1}, a_{i_2}, \ldots, a_{i_m}\) such that, for all \(1 \leq j < k \leq m\), the inequality \(a_{i_j} < a_{i_k}\) holds if and only if \(b_j < b_k\). For example, 532687941 contains 2143 because of its subsequence 5387. If a permutation \(\pi\) does not contain \(\tau\), we shall say that \(\pi\) avoids \(\tau\).
A permutation is an \((r, s)\)-permutation if it can be partitioned into \(r\) increasing and \(s\) decreasing, possibly empty subsequences. For example, the permutation 2143 is not a \((1, 1)\)-permutation, but is a \((0, 2)\)-permutation and a \((2, 0)\)-permutation. Note that if a permutation \(\tau\) is not an \((r, s)\)-permutation, then neither is any permutation that contains \(\tau\). For example, as noted above, 2143 is not a \((1, 1)\)-permutation and 532687941 contains 2143, so 532687941 is not a \((1, 1)\)-permutation.

Recently, West [12] and Stankova [10] investigated enumerative problems involving sets of permutations that avoid certain forbidden subsequences. They also described how, in computer science, forbidden subsequences of permutations arise naturally from sorting and substring problems. Stankova observed that the \((1, 1)\)-permutations are precisely the permutations avoiding 2143 and 3412; i.e., 2143 and 3412 are the forbidden subsequences characterizing the \((1, 1)\)-permutations. Prompted by this observation, we prove in the next section that, for any fixed non-negative integers \(r\) and \(s\), the family of \((r, s)\)-permutations is precisely the permutations avoiding some finite list of forbidden subsequences. We prove this in a more general graph-theoretic context that we now describe.

To any permutation \(\pi\) of \([n]\) we associate a graph, called the permutation graph of \(\pi\), with vertices \(1, \ldots, n\) and edge \(ij\) if and only if \((\pi(i) - \pi(j))/(i - j) < 0\). Permutation graphs have a long history [3,6]. The permutation graph of 3412 is a four-vertex cycle, denoted \(C_4\), and the permutation graph of 2143 is a pair of disjoint edges, denoted \(2K_2\). Observe that two distinct permutations may produce isomorphic permutation graphs, e.g., 436512 and 562143. Also observe that increasing subsequences of a permutation correspond to independent sets—sets of mutually non-adjacent vertices—of its permutation graph and decreasing subsequences correspond to cliques—sets of mutually adjacent vertices. With this terminology, Stankova's observation about \((1, 1)\)-permutations becomes: The permutation graphs whose vertex set can be partitioned into an independent set and a clique are precisely the permutation graphs with no induced \(C_4\) or \(2K_2\). This observation follows from Földes and Hammer's characterization of split graphs (see Chap. 6 of [6]).

For our purposes, it suffices to note that permutation graphs are perfect graphs. To define perfect graphs, we need to define four parameters. In a graph, a set of vertices is called a complete subgraph or a clique if all of its vertices are mutually adjacent. A clique with \(p\) vertices is denoted \(K_p\). A set of vertices is called an independent subgraph or a stable set if all of its vertices are mutually non-adjacent. The clique number of a graph \(G\), denoted \(\omega(G)\), is the maximum size of a complete subgraph of \(G\). The independence number of a graph \(G\), denoted \(\alpha(G)\), is the maximum size of an independent subgraph of \(G\). The chromatic number of a graph \(G\), denoted \(\chi(G)\), is the minimum number of independent subgraphs needed to cover the vertices of
The clique cover number of a graph $G$, denoted $\theta(G)$, is the minimum number of complete subgraphs needed to cover the vertices of $G$. A graph $G$ is perfect if $\omega(H) = \chi(H)$ for every induced subgraph $H$ of $G$. Equivalently, because of the perfect graph theorem, a graph $G$ is perfect if $\alpha(H) = \theta(H)$ for every induced subgraph $H$ of $G$.

Suppose that $G = (V, E)$ is a finite perfect graph with vertex set $V$ and edge set $E$. If $G$ contains a collection of $r$ disjoint cliques $C_1, C_2, \ldots, C_r$ such that $G - \bigcup_{i=1}^{r} C_i$ is $K_{s+1}$-free, then $G$ is called an $(r, s)$-split graph. The empty set is considered both as an independent set and as a clique. If $G$ is not an $(r, s)$-split graph but $G - v$ is for all $v \in V$, then $G$ is called $(r, s)$-critical, or simply critical when $r$ and $s$ are understood. Observe that, because $G$ is perfect, if $r$-cliques $C_1, \ldots, C_r$ exist such that $G - \bigcup_{i=1}^{r} C_i$ is $K_{s+1}$-free, then the vertex set of $G$ can be partitioned into $V_1, \ldots, V_{r+s}$ with $r$ of the $V_i$'s cliques and $s$ of the $V_i$'s independent sets. In particular, if $G_{\pi}$ is the permutation graph corresponding to the permutation $\pi$, then $G_{\pi}$ is an $(r, s)$-split graph if and only if $\pi$ is an $(r, s)$-permutation. Hence, to show that the $(r, s)$-permutations are characterized by a finite family of forbidden subsequences, it suffices to prove that there are only a finite number of perfect $(r, s)$-critical graphs (see Example 1 following the proof of the main theorem).

Decomposing graphs into cliques and independent sets is related to a parameter called the cochromatic number $[1, 4, 5]$. Lesniak and Straight [8] were the first to define the cochromatic number of a graph $G$, denoted $\chi_c(G)$, as the minimum positive integer $k$ such that a partition of the vertex set into $k$ sets exists so that each set induces complete or independent graph. For a perfect graph $G$, $\chi_c(G)$ is the minimum value of $r + s$ such that $G$ is an $(r, s)$-split graph. Wagner [11] has shown that this parameter is NP-complete for permutation graphs. Hence determining the least $r + s$ such that a permutation is an $(r, s)$-permutation is NP-complete.

2. THE MAIN RESULT

To prove that there are only a finite number of perfect $(r, s)$-critical graphs we require a few preliminary observations that we now present as lemmas.

Suppose that $G$ is $(r, s)$-critical for some positive integers $r$ and $s$. For each vertex $v$ in $G$ choose and fix a collection $\{C^v_i\}_{i=1}^{r}$ of cliques such that $G - v - \bigcup_{i=1}^{r} C^v_i$ is $K_{s+1}$-free. Let $Q_v$ denote $\bigcup_{i=1}^{r} C^v_i$. Similarly, choose and fix some $(s+1)$-clique $R_v$ containing $v$ in $G - Q_v$. The $Q_v$'s are called critical disjunctions and the $R_v$'s are critical cliques. The vertex $v'$ is called the corresponding vertex to $Q_v$ and $R_v$. 

Lemma 2.1. Suppose $G$ is an $(r, s)$-critical graph. If $x_1, x_2, ..., x_{s+1}$ are distinct vertices and $Q_{x_1} \subseteq Q_{x_2} \subseteq \cdots \subseteq Q_{x_{s+1}}$, then $R_{x_{s+1}} = \{x_1, x_2, ..., x_{s+1}\}$.

Proof. First note that if $x$ and $y$ are distinct vertices and $Q_x \not\subseteq Q_y$, then $x \notin R_y$ because $R_y \not\subseteq G - Q_x \subseteq G - Q_y$ and $G - Q_x - x$ is $K_{s+1}$-free. Hence \{x_1, ..., x_s\} $\subset R_{x_{s+1}}$. By definition, $x_{s+1} \in R_{x_{s+1}}$. The result follows now because $x_1, ..., x_{s+1}$ are distinct vertices.

Observe that $2K_{s+1}$ is a $(1, s)$-critical graph in which $s + 1$ vertices of one $K_{s+1}$ all have mutually comparable critical disjunctions, so Lemma 2.1 is tight.

Let $A \triangle B$ denote the symmetric difference of the sets $A$ and $B$. Clearly $|A| + |B| - 2|A \cap B| = |A \triangle B|$, for any sets $A$ and $B$. In particular this implies that, if $|A| + |B|$ is even, then $|A \triangle B|$ is even. Two sets $A$ and $B$ have the same parity if $|A| + |B|$ is even. A collection of sets has the same parity if the sets pairwise have the same parity.

Lemma 2.2. Any $(r, s)$-critical graph on $n$ vertices has at least $\lfloor n/2(s + 1) \rfloor$ mutually incomparable critical disjunctions with the same parity.

Proof. Consider the $n$ critical disjunctions of an $(r, s)$-critical graph on $n$ vertices $Q_1, Q_2, ..., Q_n$. Construct a graph $H$ whose vertices are these $n$ critical disjunctions, and whose edge set consists of all pairs $Q_i, Q_j$ such that $Q_i \not\subseteq Q_j$ or $Q_j \not\subseteq Q_i$. This graph $H$ is a comparability graph (see Golumbic's book [6]); in particular, it is perfect. Lemma 2.1 shows that this graph has no clique of order $s + 2$. Let $\alpha$ be the independence number of $H$ and let $\omega$ be its clique number. Because $H$ is a perfect graph, its chromatic number is $\omega$; hence some independent set of $H$ has at least $n/\omega$ vertices. Therefore $\alpha \geq n/\omega \geq n/(s + 1)$. So at least $\lfloor n/(s + 1) \rfloor$ mutually incomparable critical disjunctions exist among the $Q_i$'s. At least half of these have the same parity.

Lemma 2.3. If $G$ is an $(r, s)$-critical graph, and $x, y$ are distinct vertices of $G$, then $|Q_x \triangle Q_y| \leq 2r(s + 1)$.

Proof. It suffices to show that $|Q_x - Q_y|$ is at most $r(s + 1)$. Suppose that $Q_x = C_1^x \cup \cdots \cup C_r^x$. Clearly,

$$|Q_x - Q_y| = \sum_{i=1}^{r} |C_i^x - Q_y| \leq r(s + 1)$$

because $|C_i^x - Q_y| \leq s + 1$, for all $1 \leq i \leq r$; otherwise a $K_{s+2}$ clique exists in $G - Q_y$ contradicting that $G - Q_y - y$ is $K_{s+1}$-free.
If \(F_1, F_2, \ldots, F_m\) is a family of finite sets, then the degree of a point \(x \in \bigcup_{i=1}^m F_i\) is the number of the \(F_i\)'s containing \(x\). We will need the following result due to Deza [2] (see also Problem 13.17 of [9]).

**Lemma 2.4 (Deza).** If \(F_1, F_2, \ldots, F_m\) are sets such that

\[
|F_i \triangle F_j| = 2k \quad (1 \leq i < j \leq m),
\]

then

1. The degree \(d\) of any point satisfies: \(d(m - d) \leq km\),
2. if there is a point \(x\) with degree 0, 1, \(m - 1\), \(m\), then \(m \leq k^2 + k + 2\).

The Ramsey number \(R(i; j)\) is the smallest positive integer \(p\) such that in any \(i\)-coloring of the edges of \(K_p\), there exists a monochromatic \(K_j\). The existence of \(T(i; j)\) is guaranteed by Ramsey's theorem; in particular, it is finite. For more on Ramsey theory, the reader is referred to the book by Graham, Rothschild, and Spencer [7].

**Theorem 2.5.** For any fixed non-negative integers \(r\) and \(s\), there are only a finite number of perfect \((r, s)\)-critical graphs.

**Proof.** Let \(G\) be a perfect \((r, s)\)-critical graph on \(n\) vertices, for some non-negative integers \(r\) and \(s\). In the case \(r = 0\), note that for perfect graphs, the only obstruction for having chromatic number \(s\) is \(K_{s+1}\), since the chromatic number and clique number are equal. A similar argument shows, if \(s = 0\), that an independent set of order \(r + 1\) is the only obstruction for a perfect graph to have a clique covering with at most \(r\) cliques. In summary there are exactly one perfect \((0, s)\)-critical and one perfect \((r, 0)\)-critical graphs, namely, \(K_{s+1}\) and \(I_{r+1}\). Hence we may assume that \(r, s > 0\).

We shall prove that

\[
\left\lfloor \frac{n}{2(s + 1)} \right\rfloor < R(r(s + 1); (r(s + 1))^2 + r(s + 1) + 3)
\]

which implies that there are only a finite number of perfect \((r, s)\)-critical graphs. For convenience, let \(q = \lfloor n/2(s + 1) \rfloor\), and \(m = (r(s + 1))^2 + r(s + 1) + 3\). Suppose, for a contradiction, that \(q \geq R(r(s + 1); m)\). By Lemma 2.2, there are at least \(q\) mutually incomparable critical disjunctions \(Q_1, Q_2, \ldots, Q_q\) in \(G\) all with the same parity. Thus \(|Q_i \triangle Q_j|\) is even, for all \(1 \leq i < j \leq q\). Furthermore, Lemma 2.3 guarantees that \(|Q_i \triangle Q_j| \leq 2r(s + 1)\), for all \(1 \leq i < j \leq q\).
Color the edges of $K_q$ using the $r(s + 1)$ even numbers between 2 and $2r(s + 1)$ giving the edge $ij$ the color $|Q_i \triangle Q_j|$. Because $q \geq R(s + 1); m)$, this coloring has a monochromatic $K_m$. Let $F_1, F_2, ..., F_m$ be critical disjunctions forming a monochromatic $K_m$ in this coloring of $K_q$. Set $F = \bigcup_{i=1}^{m} F_i$. There is some $k$ with $1 \leq k \leq r(s + 1)$ such that $|F_i \triangle F_j| = 2k$, for all $1 \leq i < j \leq m$. Because $m > k^2 + k + 2$, Lemma 2.4 implies that every vertex in $F$ appears in 1, $m - 1$, or $m$ of the $F_i$'s. Let $S$ be the vertices in $F$ appearing in either $m - 1$ or $m$ of the $F_i$'s ($S$ may be empty).

We claim that $G[S]$, the graph induced by the vertices in $S$, can be covered by at most $r$ cliques (i.e., has clique cover number at most $r$). Suppose that $G[S]$ cannot be covered by at most $r$ cliques, then $G[S]$ must contain a set of $r + 1$ independent vertices $v_1, v_2, ..., v_{r+1}$ ($G[S]$ is perfect so its independence number is equal to its clique cover number). Each $v_i$ is absent from at most one of the $F_i$'s by definition of $S$. Therefore, since $m > r + 1$, one of the $F_i$'s contains all of the $v_i$'s, say $\{v_1, ..., v_{r+1}\} \subseteq F_j$, contradicting that $F_j$ can be covered by at most $r$ cliques.

Since $G$ is critical and $G[S]$ can be covered by at most $r$ cliques, it follows that $G - S$ contains an $(s + 1)$-clique; call it $K$. Each vertex in $K$ appears in at most one of the $F_i$'s because of our definition of $F$ and choice of $S$. Therefore at least $m - (s + 1)$ of the $F_i$'s do not intersect $K$. Each $F_i$ is the critical disjunction for some vertex $v_i$ (i.e., $G - F_i - v_i$ is $K_{s+1}$-free). Because $m - (s + 1)$ of the $F_i$'s do not intersect $K$, the vertices corresponding to these critical disjunctions must appear in $K$. Hence $s + 1 = |K| \geq m - (s + 1)$, a contradiction.

Theorem 2.5 can now be interpreted in the context of any hereditary family of perfect graphs. Here are two examples.

**Example 1 (Permutations).** As noted earlier, a permutation is an $(r, s)$-permutation if and only if its permutation graph is an $(r, s)$-split graph. Consider a permutation $\pi$ that is not an $(r, s)$-permutation, and its corresponding permutation graph $G_\pi$. Delete vertices of $G_\pi$ repeatedly so that the resulting graph remains a non $(r, s)$-split graph. Eventually a perfect $(r, s)$-critical graph emerges as an induced subgraph of $G_\pi$. Every perfect $(r, s)$-critical graph evidently corresponds to at most finitely many permutations. Thus Theorem 2.5 guarantees that any non-$(r, s)$-permutation contains one of the finitely many forbidden $(r, s)$-permutations.

The $(1, 1)$-permutations are characterized by the forbidden subsequences 2143 and 3412. We have tried to find the list of forbidden subsequences for the $(2, 1)$-permutations. We believe that our list of 102 forbidden subsequences for this family is complete, but a proof of this has eluded us.
Example 2 (Partial orders). Interpreting partial orders as comparability graphs (which are perfect), Theorem 2.5 implies that, for fixed \( r \) and \( s \), the partial orders whose element set can be partitioned into \( r \) antichains and \( s \) chains are characterized by a finite list of forbidden suborders.

References