



Generalized μ - τ symmetry and discrete subgroups of $O(3)$



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ABSTRACT

The generalized μ - τ interchange symmetry in the leptonic mixing matrix U corresponds to the relations: $|U_{\mu i}| = |U_{\tau i}|$ with $i = 1, 2, 3$. It predicts maximal atmospheric mixing and maximal Dirac CP violation given $\theta_{13} \neq 0$. We show that the generalized μ - τ symmetry can arise if the charged lepton and neutrino mass matrices are invariant under specific residual symmetries contained in the finite discrete subgroups of $O(3)$. The groups A_4 , S_4 and A_5 are the only such groups which can entirely fix U at the leading order. The neutrinos can be (a) non-degenerate or (b) partially degenerate depending on the choice of their residual symmetries. One obtains either vanishing or very large θ_{13} in case of (a) while only A_5 can provide θ_{13} close to its experimental value in the case (b). We provide an explicit model based on A_5 and discuss a class of perturbations which can generate fully realistic neutrino masses and mixing maintaining the generalized μ - τ symmetry in U . Our approach provides generalization of some of the ideas proposed earlier in order to obtain the predictions, $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$.

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1. Introduction

The data from various neutrino oscillation experiments analyzed in the context of three neutrino oscillations have revealed five fundamental parameters by now [1–3]. These include two squared differences of neutrino masses and three mixing angles in the Pontecorvo–Maki–Nakagawa–Sakata mixing matrix U_{PMNS} . For any of the normal or inverted ordering in the neutrino masses, their 3σ ranges can be summarized as [1]:

$$0.270 < \sin^2 \theta_{12} < 0.344, \quad 0.385 < \sin^2 \theta_{23} < 0.644,$$

$$0.0188 < \sin^2 \theta_{13} < 0.0251$$

$$7.02 < \frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2} < 8.09,$$

$$2.325 < \frac{\Delta m_{31}^2}{10^{-3} \text{ eV}^2} < 2.599 \quad \text{or} \quad -2.259 < \frac{\Delta m_{32}^2}{10^{-3} \text{ eV}^2} < -2.307$$

Here $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$. Discrete symmetry based approaches have been quite widely used in order to explain the special values of lepton mixing angles, see for example recent reviews [4–8]. One assumes that the global symmetry group G_f of the leptons is

spontaneously broken to the smaller symmetries G_ν and G_l of the neutrino and the charged lepton mass matrices respectively. The leptonic mixing can solely be fixed from the choice of G_l and G_ν in a given G_f [9–13]. Possible choices of G_f leading to three non-degenerate neutrinos are extensively studied in [14–21] and mixing patterns are analyzed. In a novel approach, it is shown that suitable choices of G_f can also lead to the cases with one massless neutrino [22,23], two or three degenerate neutrinos [24,25] and two degenerate and one massless neutrino [25]. In an alternate approach, it is shown recently in [26] that a massless neutrino with/without a degenerate pair of neutrinos can arise if neutrino mass matrix is assumed to be anti-symmetric under G_ν .

After the clear evidence of nonzero θ_{13} and with the most recent data, we now start to have an indirect indication of the sixth parameter, namely the Dirac CP phase δ_{CP} in the lepton sector. In fact the observed value of θ_{13} and measured combination of θ_{13} and δ_{CP} by T2K long-baseline experiment [27] are in good agreement if $\delta_{CP} \sim -\pi/2$ [28,29]. This however is a mere indication at present and more data will certainly provide clear picture in the near future. Nevertheless, such a special value of CP phase may be indicative signal of some hidden symmetries in the lepton sector. The current global fits of neutrino oscillation data disfavor the maximal atmospheric mixing angle at 1σ however it is in accordance with the data at 3σ in case of both normal and inverted ordering in the neutrino masses. The ansatz and symmetries of neutrino mass matrix predicting $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$ have been proposed earlier in [30–33]. In the simplest case, the above

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prediction can be obtained if the Majorana neutrino mass matrix in the diagonal basis of the charged leptons, namely $\mathcal{M}_{\nu f}$, satisfies

$$S_{23}^T \mathcal{M}_{\nu f} S_{23} = \mathcal{M}_{\nu f}^*, \quad (1)$$

where

$$S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2)$$

The symmetry transformation is a discrete Z_2 symmetry corresponding to μ - τ interchange together with CP conjugation [33–43]. Such an $\mathcal{M}_{\nu f}$ leads to the relations among the elements of PMNS matrix, $U \equiv U_{\text{PMNS}}$:

$$|U_{\mu i}| = |U_{\tau i}| \quad \text{for } i = 1, 2, 3 \quad (3)$$

and predicts $\theta_{23} = \pi/4$ and $\sin\theta_{13} \cos\delta_{\text{CP}} = 0$, equivalently $\delta_{\text{CP}} = \pm\pi/2$ if $\theta_{13} \neq 0$. The relations in Eq. (3) were first proposed in [31] and we refer them as the results of “generalized μ - τ symmetry” in the leptonic mixing matrix.¹

We show that predictions in Eq. (3) or generalized μ - τ symmetry arise on more general grounds and can follow without invoking CP and/or the μ - τ symmetry and can follow even if Eq. (1) is not satisfied by $\mathcal{M}_{\nu f}$. As we shall see, Eq. (3) arises if neutrinos and the charged lepton mass matrices are invariant under specific residual symmetries contained in some discrete subgroups (DSG) of $O(3)$. The residual symmetries contained in DSG of $O(3)$ can be used to get a neutrino mass matrix with non-degenerate or partially degenerate spectrum with two of the masses being equal. The generalized μ - τ symmetry follows in both the cases. While the general result that we derive holds for any DSG of $O(3)$, we shall discuss specific examples of groups having three dimensional irreducible representation (irreps). There are only three such groups, namely A_4 , S_4 and A_5 . All of which have been widely discussed in the literature [4–8] and we shall recapitulate some of the known results and present new examples specifically in case of the partially degenerate neutrino mass spectrum.

The A_5 symmetry together with CP transformation has been studied recently in [44–47] in order to predict the neutrino mixing angles and CP phases in the case of three massive Majorana neutrinos. Our approach is different from these works as we do not impose CP explicitly but discuss situations under which the generalized CP predictions arise automatically. Also the choice of residual symmetry G_ν leading to degenerate solar pair is not considered in the quoted works.

In the next section, we present our main result and discuss the emergence of generalized μ - τ symmetry from the DSG of $O(3)$. We then discuss specific examples of the general result in Section 3. An explicit model based on the A_5 group is constructed in Section 4. Finally, we summarize in the last section.

2. Discrete subgroups of $O(3)$ and maximal θ_{23} and δ_{CP}

We discuss the sufficiency conditions leading to generalized μ - τ symmetry predictions, Eq. (3). Let T_l , T_ν and S_{av} with $a = 1, 2$ denote 3×3 real orthogonal matrices with the property

$$T_l^n = T_\nu^m = S_{av}^2 = 1 \quad \text{with } n, m \geq 3 \quad (4)$$

and $[S_{1\nu}, S_{2\nu}] = 0$, $[T_l, T_\nu] \neq 0$, $[T_l, S_{av}] \neq 0$. Let the Hermitian combination $M_l M_l^\dagger$ of the charged lepton mass matrix M_l satisfy

$$T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger \quad (5)$$

and neutrino mass matrix be invariant under either S_{av} or T_ν :

$$(a) \quad S_{av}^T M_\nu S_{av} = M_\nu \quad \text{or} \quad (b) \quad T_\nu^T M_\nu T_\nu = M_\nu. \quad (6)$$

Then the resulting U_{PMNS} displays the exact generalized μ - τ symmetry with elements satisfying Eq. (3). It is clear that if (T_l, S_{av}) or (T_l, T_ν) close to a finite group, then the minimal such group would be a DSG of $O(3)$. Thus the DSG of $O(3)$ can naturally lead to the generalized μ - τ symmetry.

The case (a) in Eq. (6) corresponds to three non-degenerate neutrino masses and (b) to partially degenerate spectrum with two equal neutrino masses. The neutrino mass matrix is invariant under a $Z_2 \times Z_2$ symmetry in the case (a). This symmetry corresponds to changing the signs of any two of the three neutrino fields in their mass basis. Such a symmetry is always present if all three neutrinos are massive Majorana particles and non-degenerate. If two of the neutrinos are degenerate then the residual symmetry is bigger since one can multiply the corresponding fields ν_1 and ν_2 by complex phase η and η^* respectively leaving their combined mass term invariant. The residual symmetry in this case is Z_m with $m \geq 3$ and implies a partially degenerate spectrum which has been considered in detail in [24,25].

The proof of the above uses an important and well known result that matrices diagonalizing symmetry operators of the mass matrices also diagonalize the corresponding mass matrices themselves [9–13]. Specifically, let V_l (V_ν) be 3×3 unitary matrix diagonalizing the symmetry operators T_l (S_{av} or T_ν). Then the matrices U_l and U_ν , diagonalizing $M_l M_l^\dagger$ and M_ν respectively, are given by $U_l = V_l P_l$ and $U_\nu = V_\nu P_\nu$, where P_l and P_ν are arbitrary diagonal phase matrices. As a result, the elements of the $U \equiv U_{\text{PMNS}}$ matrix satisfy

$$|U_{ij}| = |(U_l^\dagger U_\nu)_{ij}| = |(V_l^\dagger V_\nu)_{ij}|. \quad (7)$$

Eqs. (5), (6) allow us to determine the general form of V_l and V_ν . For this, we note that eigenvalues of any unitary unimodular matrix satisfy

$$\lambda^3 - \chi \lambda^2 + \chi^* \lambda - 1 = 0, \quad (8)$$

where χ denotes the trace of the matrix (or character) and all the eigenvalues λ satisfy $|\lambda| = 1$. If χ is real then one of the roots of the above equation is $\lambda_1 = 1$ and the other two are given by $\lambda_{2,3} = \frac{1}{2} (\chi - 1 \pm \sqrt{(\chi - 1)^2 - 4})$. This has only two real solutions of modulus one corresponding to $\chi = 3$ and $\chi = -1$. These respectively correspond to an identity element and elements of order 2. The remaining solutions are non-real and complex conjugate to each other. Such elements necessarily have order ≥ 3 . It follows that the matrices T_l , T_ν satisfying Eq. (4) have eigenvalues $\lambda_i = (1, \eta, \eta^*)$ with $\eta \neq \pm 1$ and $|\eta|^2 = 1$ while S_{av} have eigenvalues $(1, -1, -1)$.

Any T_l with a pair of complex conjugate eigenvalues is necessarily non-diagonal in the basis in which it is real and its eigenvalue equation is given by

$$T_l v_i = \lambda_i v_i, \quad (9)$$

where v_i are eigenvectors. It follows from the eigenvalues of T_l that v_1 can be chosen real and $v_2 = v_3^*$. Thus, V_l diagonalizing T_l can be chosen to have a general form

$$V_l = \begin{pmatrix} x_1 & z_1 & z_1^* \\ x_2 & z_2 & z_2^* \\ x_3 & z_3 & z_3^* \end{pmatrix}, \quad (10)$$

with real x_i and complex z_i . The corresponding matrix diagonalizing $M_l M_l^\dagger$ would be given by $U_l = V_l P_l$. Next, we show that the matrix U_ν diagonalizing M_ν has the form

¹ This is also referred as “ μ - τ reflection symmetry” in some literature [31,35].

$$U_\nu = O_\nu Q_\nu \quad (11)$$

in both the cases (a) and (b), where O_ν is a real orthogonal matrix and Q_ν is a diagonal phase matrix. Since $[S_{1\nu}, S_{2\nu}] = 0$, both $S_{a\nu}$ are diagonalized by a common unitary matrix and since $S_{a\nu}$ and their eigenvalues are real, the eigenvectors of $S_{a\nu}$ can also be chosen real. The same O_ν would diagonalize the neutrino mass matrix also due to symmetry relation Eq. (6). But the neutrino masses can be complex and Q_ν in Eq. (11) corresponds to their phases. For the case (b), the matrix V_ν that diagonalizes T_ν is formally the same as Eq. (10) which diagonalizes T_l . This follows from the fact that both T_l and T_ν are real and have a pair of complex conjugate eigenvalues. Thus we can write

$$V_\nu = \begin{pmatrix} u_1 & u_1^* & w_1 \\ u_2 & u_2^* & w_2 \\ u_3 & u_3^* & w_3 \end{pmatrix} \quad (12)$$

with w_i real. Note that the ordering of eigenvectors is not determined from the symmetry arguments and we have chosen an ordering in Eq. (10) which would give generalized μ - τ symmetry. Other choices would correspond to e - τ or e - μ symmetries leading to the predictions $|U_{ei}| = |U_{\mu i}|$ or $|U_{ei}| = |U_{\tau i}|$ respectively in U_{PMNS} . The ordering in V_ν in Eq. (12) is however chosen requiring that the degenerate pair of neutrinos corresponds to the solar neutrinos pair. While V_ν diagonalizing T_ν is given above, the diagonalizing matrix U_ν does not differ from it merely by a phase matrix as in the case of non-degenerate neutrinos. The degeneracy in the first two masses implies [25]

$$U_\nu = V_\nu U_{12} R_{12}(\theta_X) P_{\beta_2}, \quad (13)$$

with

$$U_{12} = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

R_{12} denoting arbitrary rotation in the 1–2 plane by an angle θ_X and $P_{\beta_2} = \text{Diag.}(1, 1, e^{i\beta_2/2})$. It then follows from Eqs. (13), (14) that U_ν also has the same form as given by Eq. (11). It is then straightforward to verify that $U_l = V_l P_l$ with V_l as in Eq. (10) and U_ν as in Eq. (11) lead to U_{PMNS} matrix satisfying Eq. (3).

A neutrino mass matrix which is $Z_2 \times Z_2$ symmetric can in general possess non-trivial phases represented by Q_ν in Eq. (11). If these phases are trivial and if U_l is in the form of Eq. (10) then the Majorana neutrino mass matrix in the diagonal basis of the charged leptons is given by

$$\mathcal{M}_{\nu f} \equiv U_l^T M_\nu U_l = \begin{pmatrix} X & A & A^* \\ A & B & C \\ A^* & C & B^* \end{pmatrix}, \quad (15)$$

where X and C are real parameters. This provides the most general solution of Eq. (1). The above $\mathcal{M}_{\nu f}$ was first obtained [30,32] in the context of A_4 model with quasidegenerate neutrinos. It was then argued in [33] that this form can result from a combined operation of the μ - τ and CP symmetry and leads to prediction of the maximal δ_{CP} .

If M_ν is $Z_2 \times Z_2$ symmetric but Majorana phases are non-trivial then even with U_l as in Eq. (10) one does not get the above specific form of Eq. (15) but Eq. (3) still holds. Thus the combined operation of CP and μ - τ symmetry is sufficient but not necessary to get the maximal θ_{23} and δ_{CP} .

It has been noticed before [48–50] that the form given in Eq. (15) follows if V_l is given by

$$V_l = U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (16)$$

with $\omega = e^{2\pi i/3}$ and if neutrino mass matrix is real. The above form of V_l is a special case of our general form, Eq. (10) and results when T_l is identified with a Z_3 group associated with cyclic permutations of three objects. A similar case is also studied recently in the contexts of type II seesaw [51–53].

We end this section with some important remarks connected with the above result.

- If one were to replace Z_n invariance of $M_l M_l^\dagger$ also by a $Z_2 \times Z_2$ symmetry then both U_l and U_ν would be real upto a diagonal phase multiplication on right and δ_{CP} would be zero. If $Z_2 \times Z_2$ invariance of M_ν in case of the non-degenerate neutrinos is replaced by a single Z_2 then reality of V_ν and hence the prediction of the generalized μ - τ symmetry does not hold. An example of this is found in a specific model [54] based on the A_5 group which uses a single Z_2 symmetry for neutrinos. As far as the degenerate neutrinos are concerned, the order of T_ν is necessarily > 2 . Thus all DSG of $O(3)$ giving degenerate neutrinos necessarily also give Eq. (3).
- If neutrinos are degenerate then both the solar angle and δ_{CP} are undefined. This is reflected by the presence of the unknown angle θ_X in Eq. (13). But note that the relations in Eq. (3) hold even if $U \rightarrow UR_{12}(\theta_X)P_{\beta_2}$ and therefore the arbitrariness in defining U_ν arising from the degeneracy of the solar pair does not affect the underlying generalized μ - τ symmetry. Equivalently, one finds [24,25] that the quantity $I_\alpha \equiv \text{Im}(U_{\alpha 1}^* U_{\alpha 2})$ remains invariant under $U \rightarrow UR_{12}(\theta_X)P_{\beta_2}$. These quantities can be written in the standard parameterization of U_{PMNS} as

$$\begin{aligned} c_{12}s_{12} \sin \frac{\beta_1}{2} &= \frac{1}{c_{13}^2} I_e, \\ c_{12}^2 \sin \left(\delta_{\text{CP}} - \frac{\beta_1}{2} \right) + s_{12}^2 \sin \left(\delta_{\text{CP}} + \frac{\beta_1}{2} \right) \\ &= \frac{1}{s_{23}c_{23}s_{13}} \left(I_\mu - \frac{s_{23}^2 s_{13}^2 - c_{23}^2}{c_{13}^2} I_e \right), \end{aligned} \quad (17)$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$. Using the form of U_{PMNS} obtained in the degenerate case above, one finds that $I_e = 0$ and $I_\mu = -I_\tau = \pm \frac{1}{2} \sin \theta_{13}$. Since these invariants are independent of θ_X , one can use the leading order values of θ_{12} to obtain information on δ_{CP} . These are determined by the choice of T_l and T_ν . If $c_{12}s_{12} \neq 0$ at the leading order, then the above equations predict $\beta_1 = 0$ and $\delta_{\text{CP}} = \pm \frac{\pi}{2}$. On the other hand if $c_{12}s_{12} = 0$ at the leading order than one gets $\sin(\delta_{\text{CP}} \pm \frac{\beta_1}{2}) = \pm 1$. It is thus expected that small perturbations will stabilize δ_{CP} around the values obtained in these two cases depending on the choice of the residual symmetries. Examples of specific perturbations doing this have been considered in [24]. Also general perturbations to the U_{PMNS} matrix obtained in case of the A_5 group were numerically analyzed in [25] and δ_{CP} was found to be near $\pm \frac{\pi}{2}$ for the choices of T_l and T_ν made there. We shall give here an explicit model where one gets the same result after perturbations.

- The third column of U is not affected by arbitrariness in the choice of θ_{12} and the values of θ_{13} is uniquely fixed by the choice of T_ν and T_l . We consider leading order prediction of θ_{13} for DSG of $O(3)$ in the next section concentrating mainly on A_5 .

3. Examples of generalized μ - τ symmetry and A_5

The groups S_3 , D_N , A_4 , S_4 and A_5 are the only finite DSG of $O(3)$. Of these only A_4 , S_4 and A_5 possess faithful three dimensional irreducible representations. Any choice of residual symmetries within them consistent with the previous discussion would lead to prediction Eq. (3). The mixing angle predictions for A_5 group have already been studied [55–58] in case of the non-degenerate neutrinos. One gets either vanishing or large θ_{13} at the leading order in this case. The same holds for the groups A_4 and S_4 even in case of the partially degenerate spectrum. The group A_5 provides only non-trivial example which gives a non-zero θ_{13} close to its experimental value if two of the neutrinos are degenerate. We discuss this case explicitly and enumerate all the residual symmetries within A_5 giving generalized μ - τ symmetry.

The A_5 group has sixty elements which are generated using E , F and H where

$$H = 1/2 \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}; \quad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (18)$$

with $\mu_{\pm} = 1/2(-1 \pm \sqrt{5})$. We list all the sixty elements in terms of E , F , H defined above in Appendix A. Properties of A_5 group has been studied earlier in [55–57] and Ref. [58] also gives list of all elements using different matrices. We have defined them in a way which makes the appearance of the generalized μ - τ symmetry for A_5 explicit.

We divide the sixty elements into four categories: (i) An identity element, (ii) the 15 elements of order 2 to be collectively called O_2 . The character χ of these elements is -1 , (iii) the 20 elements of order 3 to be called O_3 , all with $\chi = 0$ and (iv) 24 elements of order 5 collectively called O_5 . The 12 of these have $\chi = -\mu_+$ and another 12 have $\chi = -\mu_-$. All these elements and their diagonalizing matrices are listed in Table 1 in Appendix A. Following Eq. (8), we find that all the elements in category O_3 and O_5 have one real and two complex conjugate eigenvalues. Thus there are 44 elements belonging to O_3 and O_5 which qualify to be the residual symmetry T_ν , T_l of neutrinos and the charged leptons respectively. The 15 elements in O_2 contain five distinct $Z_2 \times Z_2$ subgroups which can be used as residual symmetry of M_ν in case of the non-degenerate spectrum. For each of these five choices, there exists 44 T_l giving generalized μ - τ or e - τ or e - μ symmetry. The last two can be converted to U_{PMNS} satisfying Eq. (3) after proper reordering in the columns of T_l . If any of the five $Z_2 \times Z_2$ group is used as residual symmetry of M_ν and any of 24 elements in class O_5 as T_l then one gets the following $|U_{\text{PMNS}}|$:

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.8507 & 0.5257 & 0 \\ 0.3717 & 0.6015 & 0.7071 \\ 0.3717 & 0.6015 & 0.7071 \end{pmatrix} \quad (19)$$

or matrix which differs from above by reordering of row and columns. This matrix has the property of golden ratio prediction [59] for the solar mixing angle $\sin^2 \theta_{12} = 0.276$. It however predicts $\sin^2 \theta_{13} = 0$. This case provides a good zeroth order approximation and it has already been discussed in [55–59]. If one chooses any of 20 elements in $O(3)$ as T_l then one gets generalized μ - τ symmetry but the resulting form of $|U_{\text{PMNS}}|$ differs significantly from the observed one.

In case of the partially degenerate neutrino spectrum, one has the choice of 44 elements as residual symmetries of M_ν and M_l

consistent with generalized μ - τ . The structure of the PMNS matrix follows from the basic structure of U_l , U_ν . In particular, one gets from Eq. (10) and Eq. (12)

$$\sin^2 \theta_{13} = \left| \sum_i x_i u_i \right|^2,$$

where x_i (u_i) denotes the eigenvector of T_l (T_ν) corresponding to the eigenvalue 1. This can be determined from the structure of the elements O_2 and O_5 as given in Appendix A. All possible values of θ_{13} obtained in this way are given by

$$\sin^2 \theta_{13} = \{0.035, 0.111, 0.2, 0.556, 0.632\}.$$

Similar exercise in case of the A_4 and S_4 groups gives:

$$A_4 : \sin^2 \theta_{13} = 0.111;$$

$$S_4 : \sin^2 \theta_{13} = \{0, 0.111, 0.333\}. \quad (20)$$

The same results also follow from [25] in which an extensive analysis was performed on several discrete subgroups of $SU(3)$ which can lead to the appropriate symmetries for degenerate solar pair. The numerical results presented in Table I in [25] show that among all the analyzed groups, the only group with prediction maximal θ_{23} and δ_{CP} for $0 < \sin^2 \theta_{13} < 0.05$ is A_5 or the group which contains it as a subgroup, for example $\Sigma(1080)$.

Of all the predicted values, $\sin^2 \theta_{13} = 0.035$ can be considered close to experiments which can be brought within 3σ limit of the experimental value with relatively small corrections. This value is obtained if T_l belongs to O_5 and T_ν to $O(3)$ or vice versa. There exists more than one structures of $|U_{\text{PMNS}}|$ corresponding to the same value of s_{13}^2 . We note here two qualitatively different cases.

If $T_l = T$ and $T_\nu = E^{-1}AE$ then one gets

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.8507 & 0.4911 & 0.1876 \\ 0.3717 & 0.616 & 0.6946 \\ 0.3717 & 0.616 & 0.6946 \end{pmatrix}$$

upto a rotation by an angle θ_X in the 12 plane, where A , T are defined in Appendix A. The same T_l but $T_\nu = AEA^{-1}$ gives instead

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.9822 & 0 & 0.1876 \\ 0.1326 & 0.7071 & 0.6946 \\ 0.1326 & 0.7071 & 0.6946 \end{pmatrix}$$

In the first case, $c_{12}s_{12}$ is non-zero at the leading order. Then invariants given in Eq. (17) lead to $\beta_1 = 0$, $\delta_{\text{CP}} = \pm \frac{\pi}{2}$. In the second case, $c_{12}s_{12} = 0$ and one gets $\sin(\delta_{\text{CP}} \pm \beta_1/2) = \pm 1$. The small perturbations are then required to fix θ_{12} to its experimental value and to generate splittings in the solar pair. Such perturbations would also fix δ_{CP} close to the values around this.

4. An A_5 model

We now provide explicit model in which the results of previous section can be realized. The model is very similar to the one presented in [54]. Major difference being a different vacuum alignment and the form of the charged lepton mass matrix. The group A_5 has **1**, **3₁**, **3₂**, **4** and **5** dimensional irreps where **3₁** and **3₂** are non-equivalent irreps. The model is supersymmetric with the three generations of leptons l_l and l^c both transforming as **3₁** under A_5 as in [54]. It follows from the product

$$\mathbf{3}_1 \times \mathbf{3}_1 = (\mathbf{1} + \mathbf{5})_{\text{sym.}} + \mathbf{3}_{1\text{antisym.}}$$

that symmetric neutrino masses can arise from **1** + **5** and the charged lepton masses can arise from all three irreps. Accordingly,

we introduce two flavons, a 5-plet ϕ_ν and a singlet s_ν to generate neutrino masses. The Higgs doublets of the minimal supersymmetric standard model, H_u and H_d , are singlet of A_5 . We introduce a weak triplet Δ as an A_5 singlet. The relevant superpotential is:

$$W_\nu = \frac{1}{2\Lambda} l_L^T \Delta_L (h_{5\nu} s_\nu + h_{5\nu} \phi_\nu). \quad (21)$$

The charged lepton masses are generated by three additional flavons, a singlet s_l , a 5-plet ϕ_l and a triplet χ_l . The corresponding superpotential is

$$W_l = \frac{1}{\Lambda} l_L H_d l^c (h_{s_l} s_l + h_{5l} \phi_l + h_{3l} \chi_l). \quad (22)$$

Among the various possible choices of the residual symmetries given in Appendix A, we specialize to a particular choice with $T_l = E$ and $T_\nu = f_2 T f_2$. A hermitian combination of the charged lepton mass matrix $M_l M_l^\dagger$ invariant under T_l results if the vacuum expectation values (VEV) $\langle \chi_l \rangle$ and $\langle \phi_l \rangle$ satisfy

$$T_l(3) \langle \chi_l \rangle = \langle \chi_l \rangle, \quad w T_l(5) \langle \phi_l \rangle = \langle \phi_l \rangle, \quad (23)$$

where $T_l(3)$ ($T_l(5)$) denotes the matrices corresponding to the $\mathbf{3}_1$ ($\mathbf{5}$) representation. The $T_l(3) = E$ and $T_l(5)$ is given [54] by:

$$T_l(5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \quad (24)$$

Denoting $\langle \chi_l \rangle = (\chi_1, \chi_2, \chi_3)^T$ and $\langle \phi_l \rangle = (q_1, q_2, q_3, q_4, q_5)^T$, Eqs. (23) are solved by

$$v_1 = v_2 = v_3 \equiv v_l, \quad q_1 = q_2 = q_3 \equiv q_l \quad \text{and} \quad q_4 = q_5 = 0. \quad (25)$$

Inserting this solution in the superpotential in Eq. (22) leads to a charged lepton mass matrix

$$M_l = \begin{pmatrix} m_0 & m_1 - m_2 & m_1 + m_2 \\ m_1 + m_2 & m_0 & m_1 - m_2 \\ m_1 - m_2 & m_1 + m_2 & m_0 \end{pmatrix}. \quad (26)$$

The $M_l M_l^\dagger$ is diagonalized by the matrix U_ω which also diagonalizes the corresponding symmetry generator $T_l(3) = E$. Explicitly,

$$U_\omega^\dagger M_l M_l^\dagger U_\omega = \text{Diag.}(m_e^2, m_\mu^2, m_\tau^2), \quad (27)$$

with

$$\begin{aligned} m_e^2 &= |m_0 + 2m_1|^2, \\ m_\mu^2 &= |m_0 - m_1 - \sqrt{3}i m_2|^2, \\ m_\tau^2 &= |m_0 - m_1 + \sqrt{3}i m_2|^2 \end{aligned} \quad (28)$$

Here m_0 can be taken real without loss of generality. Note that the electron mass given above corresponds to the eigenvector $(1, 1, 1)^T$ of U_ω . This has to be identified as the first column of U_l in order to get the μ - τ symmetry as already mentioned. The remaining two eigenvalues can be identified with muon and tau lepton masses and can be interchanged. The contributions labeled by m_0 , m_2 , m_1 arise from the VEVs of singlet, triplet and the 5-plet. The M_l is symmetric in the absence of triplet. In this case, T_l invariance implies two degenerate charged leptons. Thus a large triplet contribution m_2 is essential to split the muon and tau lepton masses. Moreover, simultaneous presence of m_0 and m_1 is also required to suppress the electron mass. But given all the three contributions,

one can fit the charged lepton masses with appropriate choice of parameters.

Neutrino masses follow analogously from Eq. (21). In order to get degeneracy, we impose the residual symmetry $T_\nu = f_2 T f_2$ and require that

$$T_\nu(5) \langle \phi_\nu \rangle = \langle \phi_\nu \rangle, \quad (29)$$

where $T_\nu(5)$ can be shown to be²

$$T_\nu(5) = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}. \quad (30)$$

Let $\langle \phi_\nu \rangle = (p_1, p_2, p_3, p_4, p_5)^T$. A solution for Eq. (29) is given by

$$p_1 = p_3 = 0, \quad p_2 = -\sqrt{2} p_4, \quad p_5 = -\frac{p_4}{\sqrt{3}}. \quad (31)$$

Inserting these in the neutrino superpotential leads to a neutrino mass matrix

$$M_{0\nu} = \begin{pmatrix} m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-) & 0 & 0 \\ 0 & m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_- - 1) & -m_{1\nu} \\ 0 & -m_{1\nu} & m_{0\nu} - \frac{m_{1\nu}}{3}(1 - \mu_+) \end{pmatrix}. \quad (32)$$

As a consequence of the residual symmetry, one gets two degenerate neutrinos with a mass $m_{0\nu} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-)$ and the third mass is given by $m_{0\nu} + \frac{2m_{1\nu}}{3}(\mu_+ - \mu_-)$. The lower 2×2 block of M_ν is diagonalized by a rotation matrix with an angle θ given by:

$$\tan \theta = -\mu_-.$$

The full PMNS matrix at the leading order is thus given by

$$U_0 \equiv U_\omega^\dagger R_{23}(\theta) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & c_\theta + s_\theta & c_\theta - s_\theta \\ 1 & c_\theta \omega^2 + s_\theta \omega & c_\theta \omega - s_\theta \omega^2 \\ 1 & c_\theta \omega + s_\theta \omega^2 & c_\theta \omega^2 - s_\theta \omega \end{pmatrix}, \quad (33)$$

where $c_\theta = \cos \theta$, $s_\theta = \sin \theta$. The generalized μ - τ symmetry is apparent from the above. Moreover,

$$s_{13}^2 = \frac{1}{3}(c_\theta - s_\theta)^2 = \frac{1}{3} \left(1 + \frac{2\mu_-}{1 + \mu_-^2} \right) \approx 0.035 \quad (34)$$

as would be expected from the specific choice of the residual symmetry made in this example.

The above zeroth order prediction would get modified from the perturbations which are required to split the degenerate states, fix the solar angle and to change the zeroth order predictions for the mixing angles θ_{13} and θ_{23} . Effects of general perturbations were studied in [25] in the context of A_5 symmetry with a slightly different choice of the residual symmetry which also leads to the same zeroth order predictions as here. It was found that small perturbations can cause significant changes in θ_{13} as required experimentally and relatively small perturbations in the zeroth order values of θ_{23} and the maximal CP violating phase. Moreover, all three neutrinos are required to be quasidegenerate in order to reproduce all the mixing angles correctly as long as perturbations

² This is determined by noting that the presentations a, b, c introduced in [54] are given in terms of our presentation as $f_3 = a$, $E = b$, $f_1 = b^2 ab$ and $H = bc$.

are smaller than $\leq 5\%$. The analysis in [25] was for the most general possible perturbations. In the context of specific models, such perturbations can arise from the non-leading higher order terms in the Yukawa superpotential which directly correct the leptonic mass matrices and/or from the Higgs potential which may perturb the Higgs vacuum expectation values from the symmetric choice. Let us consider effect of a simple but interesting perturbation in the latter category. Assume that the perturbations change one of the VEVs given in Eq. (31), namely $p_2 \rightarrow p_2(1 + \epsilon)$. Similar perturbations in the VEV of other component would also arise in general but as we discuss here, this perturbation alone has interesting consequences. The zeroth order mass matrix in Eq. (32) now gets changed to

$$M_\nu = \begin{pmatrix} m_{\nu 0} - \frac{m_{1\nu}}{3}(\mu_+ - \mu_-) & 0 & 0 \\ 0 & m_{\nu 0} - \frac{m_{1\nu}}{3}(\mu_- - 1) & -m_{1\nu}(1 + \epsilon) \\ 0 & -m_{1\nu}(1 + \epsilon) & m_{\nu 0} - \frac{m_{1\nu}}{3}(1 - \mu_+) \end{pmatrix}. \quad (35)$$

The above perturbed matrix is also diagonalized by a rotation in the 2–3 plane but with a slightly different θ which is now given by

$$\tan \theta \approx -\mu_- \left(1 - \frac{\epsilon}{\sqrt{5}} \right) + \mathcal{O}(\epsilon^2).$$

This changes the zeroth order prediction of the mixing angle θ_{13} and Eq. (34) gets replaced by

$$s_{13}^2 = \frac{1}{3}(c_\theta - s_\theta)^2 = \frac{\mu_+^2}{3(1 + \mu_-^2)} - \frac{2\epsilon}{3\sqrt{5}} \frac{\mu_-^2}{(1 + \mu_-^2)^2} + \mathcal{O}(\epsilon^2). \quad (36)$$

Thus the appropriate choice of perturbation can be used to get agreement with experiments. The other major effect of ϵ is to split the degenerate pair and induce the solar scale:

$$\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = \frac{4\epsilon}{5} \left(\frac{3\sqrt{5}m_{0\nu} - 5m_{1\nu}}{6\sqrt{5}m_{0\nu} + 5m_{1\nu}} \right) + \mathcal{O}(\epsilon^2)$$

The overall effect of the perturbation is best appreciated by going to the flavor basis with $M_l M_l^\dagger$ diagonal. In this basis

$$M_{\nu f} \equiv U_\omega^T M_\nu U_\omega = \begin{pmatrix} m_{0\nu} - \frac{2}{3}m_{1\nu}(1 + \epsilon) & -\frac{m_{1\nu}}{3}(\mu_+ + \omega^2 - \epsilon) & -\frac{m_{1\nu}}{3}(\mu_+ + \omega - \epsilon) \\ -\frac{m_{1\nu}}{3}(\mu_+ + \omega^2 - \epsilon) & -\frac{m_{1\nu}}{3}(1 - \mu_- - \omega^2 + 2\epsilon) & m_{0\nu} + \frac{m_{1\nu}}{3}(1 + \epsilon) \\ -\frac{m_{1\nu}}{3}(\mu_+ + \omega - \epsilon) & m_{0\nu} + \frac{m_{1\nu}}{3}(1 + \epsilon) & -\frac{m_{1\nu}}{3}(1 - \mu_- - \omega + 2\epsilon) \end{pmatrix} \quad (37)$$

The interesting features of this matrix are:

- Elements of $M_{\nu f}$ satisfy

$$\sum_i (M_{\nu f})_{ei} = \sum_i (M_{\nu f})_{\mu i} = \sum_i (M_{\nu f})_{\tau i}.$$

This condition implies that one of the column vectors of U_{PMNS} has a tri-maximal form as is the case with the zeroth order mixing matrix, Eq. (33). Thus one gets the prediction $\sin^2 \theta_{12} \cos^2 \theta_{13} = \frac{1}{3}$ if perturbation makes the state with an eigenvector corresponding to the first column in Eq. (33) heavier compared to the second degenerate state. Perturbation in this case does not change the zeroth order solar angle but it stabilizes it to that value by splitting the degenerate states.

- If parameters $m_{0\nu}$, $m_{1\nu}$ and ϵ are real then the $M_{\nu f}$ satisfies $(M_{\nu f})_{12} = (M_{\nu f})_{13}^*$ and $(M_{\nu f})_{22} = (M_{\nu f})_{33}^*$. Thus $M_{\nu f}$ simultaneously enjoys the $Z_2 \times Z_2$ symmetries corresponding to a tri-maximal solar angle and generalized μ - τ as envisaged and studied in [37]. In particular, one gets the maximal atmospheric angle and the maximal CP violating phase as predictions even after perturbation.

As an example, we give a set of specific values of ϵ , $m_{0\nu}$, $m_{1\nu}$ determined numerically which fit the experimental values:

$$m_{0\nu} = 0.025 \text{ eV}, \quad m_{1\nu} = 0.016 \text{ eV}, \quad \epsilon = 0.228$$

This gives the following mixing matrix

$$|U_{\text{PMNS}}|^2 = \begin{pmatrix} 0.6421 & 0.3333 & 0.0246 \\ 0.179 & 0.3333 & 0.4877 \\ 0.179 & 0.3333 & 0.4877 \end{pmatrix}$$

corresponding to

$$\sin^2 \theta_{12} \cos^2 \theta_{13} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \sin^2 \theta_{13} = 0.0246.$$

The δ_{CP} gets stabilized to the value $-\pi/2$. The neutrino masses giving correct Δm_{sol}^2 and Δm_{atm}^2 are determined by the above values of parameters as

$$m_{\nu_1} = 0.0097 \text{ eV}, \quad m_{\nu_2} = 0.0131 \text{ eV}, \quad m_{\nu_3} = 0.0522 \text{ eV}.$$

The maximality of θ_{23} can be changed by introducing small imaginary parts in parameters but the tri-maximal value of θ_{12} remains unchanged. Small deviations can be introduced by perturbing other component of the VEVs or by perturbing the charge lepton mass matrix. Since general perturbations are already studied in [25], we shall not pursue them further.

5. Summary

The generalized μ - τ symmetry of the leptonic mixing matrix is known to predict maximal atmospheric mixing angle and maximal Dirac CP violation in case of nonzero θ_{13} . Both these predictions are consistent with the current experimental observations within 3σ and their future precision measurements will reveal whether such a symmetry is indeed realized in nature in its exact form. It is therefore interesting to explore the symmetries of the leptons which lead to generalized μ - τ symmetry in the lepton mixing predicting such special values of θ_{23} and δ_{CP} .

Assuming the Majorana neutrinos, we have shown in this paper that generalized μ - τ symmetry naturally follows if the symmetry group G_f of leptons, is a discrete subgroup of $O(3)$. It is required that the G_f is broken into Z_m with $m \geq 3$ as the residual symmetry of the charged lepton mass matrix. The residual symmetry of the Majorana neutrino mass matrix can be either (a) $Z_2 \times Z_2 \in G_f$ or (b) $Z_n \in G_f$ with $n \geq 3$. The possibility (a) leads to three non-degenerate neutrinos while one obtains two of the three neutrinos degenerate in the case (b). The possible candidates of G_f are only A_4 , S_4 and A_5 which can predict all the three mixing angles at the leading order. Among these, only A_5 predicts θ_{13} very close to its experimentally observed value in the case of two degenerate neutrinos which are identified with the solar pair. The corrections to the leading order neutrino mass matrix are needed to generate viable θ_{13} , θ_{12} and the solar mass difference. We have discussed in detail the group A_5 in the context of generalized μ - τ symmetry and provided an explicit model in which the leading order predictions are realized. We have also discussed the perturbations which lead to the realistic neutrino masses and mixing angles while maintaining the predictions $\theta_{23} = \pi/4$ and $\delta_{\text{CP}} = \pm\pi/2$.

Table 1

List of all the non-trivial elements of A_5 . The last column gives the list of diagonalizing matrices for the corresponding elements which are used as the residual symmetries of neutrino and/or charged lepton mass matrices. The T^p collectively denotes a list of four elements $T^p = (T, T^2, T^3, T^4)$ while $a = 1, 2, 3$.

(Set, order)	Set of elements	Diagonalizing matrix
$(O_2, 2)$	f_a	I
	H	U_H
	$f_a H f_a$	$f_a U_H$
	$E H E^{-1}$	$E U_H$
	$E^{-1} H E$	$E^{-1} U_H$
	$E f_a H f_a E^{-1}$	$E f_a U_H$
	$E^{-1} f_a H f_a E$	$E^{-1} f_a U_H$
	$(O_3, 3)$	$E, E^2 = E^{-1}$
$f_a E f_a, f_a E^{-1} f_a$		$f_a U_\omega$
$A, A^2 = A^{-1}$		U_A
$E A E^{-1}, E A^2 E^{-1}$		$E U_A$
$E^{-1} A E, E^{-1} A^2 E$		$E^{-1} U_A$
$A E A^{-1}, A E^{-1} A^{-1}$		$A U_\omega$
$A f_{2,3} E f_{2,3} A^{-1}, A f_{2,3} E^{-1} f_{2,3} A^{-1}$		$A f_{2,3} U_\omega$
$(O_5, 5)$		T^p
	$f_2 T^p f_2$	$f_2 U_T$
	$E T^p E^{-1}$	$E U_T$
	$E^{-1} T^p E$	$E^{-1} U_T$
	$E f_2 T^p f_2 E^{-1}$	$E f_2 U_T$
	$E^{-1} f_2 T^p f_2 E$	$E^{-1} f_2 U_T$

Some example ansatz and symmetries of neutrino mass matrix leading to the generalized μ - τ symmetry have already been discussed in the literature. Our findings of an emergence of generalized μ - τ symmetry from the discrete subgroups of $O(3)$ are more general and they accommodate some of the symmetries and models proposed in literature to obtain $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$. In particular, we have shown that the generalized μ - τ symmetry in the lepton mixing can follow without imposing μ - τ symmetry and/or CP on the neutrino mass matrix. The μ - τ symmetry with CP conjugation is realized in our approach only accidentally when an additional assumption is made on the free parameters.

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Appendix A

We list all the sixty elements belonging to A_5 in terms of their presentation matrices E, F and H defined in Eq. (18). For brevity, we have defined the following matrices which are used to characterize various elements.

$$f_1 = F, \quad f_2 = E f_1 E^{-1}, \quad f_3 = E f_2 E^{-1},$$

$$T = f_1 E H, \quad A = H f_1.$$

The elements are listed in Table 1. Here U_ω diagonalizes E, E^2 and is defined in Eq. (16). The unitary matrices U_A, U_T and U_H respectively diagonalize $(A, A^2), T^p$ and H and are given by

$$U_A = \begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{\mu_-}{\sqrt{6}} & \frac{\mu_-}{\sqrt{6}} & -\frac{\mu_+}{\sqrt{3}} \\ \frac{\mu_+}{\sqrt{6}} & \frac{\mu_+}{\sqrt{6}} & \frac{\mu_-}{\sqrt{3}} \end{pmatrix},$$

$$U_T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ x\mu_- & x^*\mu_- & -\frac{\sqrt{2}\mu_-}{(1+\mu_-^2)^{1/2}} \\ -x(\mu_- - 1) & -x^*(\mu_- - 1) & -\frac{\sqrt{2}}{(1+\mu_-^2)^{1/2}} \end{pmatrix}$$

$$U_H = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{\mu_+}{\sqrt{3}} & \frac{\mu_-}{2\sqrt{3}} & \frac{\mu_-}{2} \\ -\frac{\mu_-}{\sqrt{3}} & \frac{\mu_+}{2\sqrt{3}} & \frac{\mu_+}{2} \end{pmatrix} U_{12}. \quad (38)$$

Here U_{12} denotes an arbitrary unitary rotation in the 12 plane arising due to degeneracy in two of the eigenvalues of H and $x = \frac{\lambda+1}{\lambda-1}$ with $\lambda = \frac{1}{2}(\mu_- + i\sqrt{4 - \mu_-^2})$. All the non-trivial elements of A_5 given in the table are expressed in the form $Q P Q^{-1}$ with $P = E, E^2, A, T^p, H$ and $Q = I, E, E^2, f_a, E f_a, E^2 f_a, A f_a$. This simplifies diagonalization of all the elements since $U_{Q P Q^{-1}} = Q U_P$. This allows one in principal to calculate all possible U_{PMNS} in A_5 analytically in terms the diagonalizing matrices of E, A, T, H . In particular, matrices diagonalizing A, T, E and therefore all elements in O_5, O_3 are seen to have μ - τ symmetric form given in Eq. (10).

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