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Permanence of a delayed *SIR* epidemic model with density dependent birth rate

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Abstract

In this paper, we consider the permanence of a modified delayed *SIR* epidemic model with density dependent birth rate which is proposed in [M. Song, W. Ma, Asymptotic properties of a revised *SIR* epidemic model with density dependent birth rate and time delay, Dynamic of Continuous, Discrete and Impulsive Systems, 13 (2006) 199–208]. It is shown that global dynamic property of the modified delayed *SIR* epidemic model is very similar as that of the model in [W. Ma, Y. Takeuchi, T. Hara, E. Beretta, Permanence of an *SIR* epidemic model with distributed time delays, Tohoku Math. J. 54 (2002) 581–591; W. Ma, M. Song, Y. Takeuchi, Global stability of an *SIR* epidemic model with time delay, Appl. Math. Lett. 17 (2004) 1141–1145].

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1. Introduction

Epidemic models with or without time delay are studied by many authors (see, for example, for the model with time delay [1,2,11–13,15], for one without time delay [7,9,10,14]). They consider the stability or permanence of the models by applying the theory on delay differential equations [3–6,8]. In this paper, we consider the permanence of the following modified delayed *SIR* epidemic model with density dependent birth rate which is proposed in [13],

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t-h) - \mu_1 S(t) + b\left(1 - \beta_1 \dfrac{N(t)}{1+N(t)}\right), \\ \dot{I}(t) = \beta S(t)I(t-h) - \mu_2 I(t) - \lambda I(t), \\ \dot{R}(t) = \lambda I(t) - \mu_3 R(t), \end{cases} \tag{1.1}$$

where $S(t)+I(t)+R(t) \equiv N(t)$ denotes the number of a population at time t; $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible members to the disease, of infective members and of members who have been removed from the possibility

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of infection through full immunity, respectively. It is assumed that all newborns are susceptible. The positive constants μ_1, μ_2 and μ_3 represent the death rates of susceptibles, infectives and recovered, respectively. It is natural biologically to assume that $\mu_1 \leqslant \min\{\mu_2, \mu_3\}$. The positive constants b and λ represent the birth rate of the population and the recovery rate of infectives, respectively. The constant β_1 $(0 \leqslant \beta_1 < 1)$ reflects the relation between the birth rate and the density of population. The nonnegative constant h is the time delay.

The initial condition of (1.1) is given as

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta) \quad (-h \leqslant \theta \leqslant 0), \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}} \in C$, such that $\varphi_i(\theta) \geqslant 0$ $(-h \leqslant \theta \leqslant 0,\ i = 1, 2, 3)$. C denotes the Banach space $C([-h, 0], \mathscr{R}^3)$ of continuous functions mapping the interval $[-h, 0]$ into $\mathscr{R}^3$. By a biological meaning, we further assume that $\varphi_i(0) > 0$ for $i = 1, 2, 3$. It is easily to show that the solution $(S(t), I(t), R(t))$ of (1.1) with the initial condition (1.2) exists for all $t \geqslant 0$ and is unique and positive for all $t \geqslant 0$.

With some simple computation, we see that (1.1) always has a disease free equilibrium (i.e., boundary equilibrium) $E_0 = (S_0, 0, 0)$, where

$$S_0 = \frac{1}{2\mu_1}\left[b(1-\beta_1) - \mu_1 + \sqrt{[b(1-\beta_1) - \mu_1]^2 + 4\mu_1 b}\right].$$

Furthermore, if

$$S_0 > S^* \equiv \frac{\mu_2 + \lambda}{\beta}, \tag{1.3}$$

then (1.1) also has an endemic equilibrium (i.e., interior equilibrium) $E_+ = (S^*, I^*, R^*)$, where

$$I^* = -P + \frac{\sqrt{P^2 - 4\beta S^* W Q}}{2\beta S^* W}, \quad R^* = \frac{\lambda I^*}{\mu_3},$$

$$W = 1 + \frac{\lambda}{\mu_3} > 0,$$

$$P = [\mu_1 S^* - b(1-\beta_1)]W + \beta S^*(1 + S^*),$$

$$Q = [\mu_1 S^* - b(1-\beta_1)](1 + S^*) - b\beta_1 < 0.$$

A detailed analysis on the local asymptotic stability of E_0 and E_+, and the global asymptotic stability of E_0 are given in [13]. The purpose of the paper is to consider the permanence of (1.1) with the initial condition (1.2).

2. Permanence of (1.1)

In this section, we always assume that $S_0 > S^*$ which ensures the existence of the endemic equilibrium E_+ of (1.1). The following lemma is proved in [13].

Lemma 2.1. *For any solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2), *we have that*

$$\limsup_{t \to +\infty} N(t) \leqslant S_0. \tag{2.1}$$

We also have the following

Lemma 2.2. *For any solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2), *it has that*

$$\liminf_{t \to +\infty} S(t) \geqslant \frac{b(1-\beta_1)}{\beta S_0 + \mu_1} \equiv v_1. \tag{2.2}$$

Proof. For any sufficiently small $\varepsilon > 0$, from Lemma 2.1, there exists a large $t_1 > 0$ such that for $t \geqslant t_1$, $I(t) \leqslant S_0 + \varepsilon$. Hence, for $t \geqslant t_1 + h$,

$$\dot{S}(t) \geqslant -\beta S(t)(S_0+\varepsilon) - \mu_1 S(t) + b(1-\beta_1)$$
$$= -[\beta(S_0+\varepsilon)+\mu_1]S(t) + b(1-\beta_1),$$

which clearly implies that

$$\liminf_{t\to+\infty} S(t) \geqslant \frac{b(1-\beta_1)}{\beta(S_0+\varepsilon)+\mu_1} = v_1.$$

Note that ε may be arbitrarily small. It has that (2.2) holds. This completes the proof of Lemma 2.2. □

The following Lemma 2.3 plays an important role for the permanence of (1.1).

Lemma 2.3. *For any solution* $(S(t), I(t), R(t))$ *of* (1.1) *with* (1.2), *it has that*

$$\liminf_{t\to+\infty} I(t) \geqslant \rho I^* \mathrm{e}^{-(\mu_2+\lambda)(d+h)} \equiv v_2,$$

where $\rho > 0$ *and* $d > 0$ *satisfy*

$$q \equiv \frac{1}{\rho\beta I^* + \mu_1}\left(b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}\right) > S^*$$

and

$$S^{\Delta} \equiv q(1-\mathrm{e}^{-(\rho\beta I^*+\mu_1)d}) > S^*,$$

respectively. $\varepsilon_0 > 0$ *satisfies*

$$\frac{1}{\mu_1}\left(b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}\right) > S^*. \tag{2.3}$$

Proof. First, note that since S_0 satisfies

$$S_0 = \frac{1}{\mu_1}\left(b - \frac{b\beta_1 S_0}{1+S_0}\right) > S^*,$$

it is possible to choose $\varepsilon_0 > 0$ satisfying (2.3). Hence, there exist $\rho > 0$ and $d > 0$ such that $q > S^*$ and $S^{\Delta} > S^*$ hold.

Let us consider any solution $(S(t), I(t), R(t))$ of (1.1) with (1.2). For $t \geqslant 0$, we define a differentiable function $V(t)$ as follows:

$$V(t) = I(t) + \beta S^* \int_{t-h}^{t} I(u)\,\mathrm{d}u. \tag{2.4}$$

Then, the derivative of $V(t)$ along the solution of (1.1) with (1.2) satisfies

$$\begin{aligned}\dot{V}(t) &= \dot{I}(t) + \beta S^* I(t) - \beta S^* I(t-h)\\ &= \beta(S(t)-S^*)I(t-h) + [\beta S^* - (\mu_2+\lambda)]I(t)\\ &= \beta(S(t)-S^*)I(t-h).\end{aligned} \tag{2.5}$$

From Lemma 2.1, there is some $t_0 > 0$ such that for any $t \geqslant t_0$, it has that

$$N(t) \leqslant S_0 + \varepsilon_0.$$

Claim. *It is impossible that for all large* t, *it has that*

$$I(t) \leqslant \rho I^*.$$

In fact, if the claim is not true, there exists $t^* \geqslant t_0$ such that for any $t \geqslant t^*$, it has that

$$I(t) \leqslant \rho I^*.$$

Hence, it follows from the first equation of (1.1) that, for any $t \geqslant t^* + h$,

$$\begin{aligned}\dot{S}(t) &= -\beta S(t)I(t-h) - \mu_1 S(t) + b(1-\beta_1) + \frac{b\beta_1}{1+N(t)} \\ &\geqslant -(\beta\rho I^* + \mu_1)S(t) + b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}.\end{aligned}$$

Thus, it has that, for any $t \geqslant t^* + h + d$,

$$\begin{aligned}S(t) &\geqslant S(t^*+h)\mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-t^*-h)} + \left(b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}\right)\int_{t^*+h}^{t} \mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-\theta)}\,\mathrm{d}\theta \\ &> \frac{1}{\beta\rho I^*+\mu_1}\left(b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}\right)(1-\mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-t^*-h)}) \\ &\geqslant \frac{1}{\beta\rho I^*+\mu_1}\left(b(1-\beta_1) + \frac{b\beta_1}{1+S_0+\varepsilon_0}\right)(1-\mathrm{e}^{-(\beta\rho I^*+\mu_1)d}) \\ &= q(1-\mathrm{e}^{-(\rho\beta I^*+\mu_1)d}) \\ &= S^{\Delta} > S^*. \end{aligned} \tag{2.6}$$

Therefore, from (2.5) and (2.6), we have that, for any $t \geqslant t^* + d + h$,

$$\begin{aligned}\dot{V}(t) &= \beta(S(t) - S^*)I(t-h) \\ &> \beta(S^{\Delta} - S^*)I(t-h).\end{aligned}$$

Set

$$\underline{i} = \min_{\theta\in[-h,0]} I(t^* + d + 2h + \theta) > 0.$$

Now, we show that $I(t) \geqslant \underline{i}$ for all $t \geqslant t^* + d + h$.

In fact, if there is a $T \geqslant 0$ such that $I(t) \geqslant \underline{i}$ for $t^* + d + h \leqslant t \leqslant t^* + d + 2h + T$, $I(t^* + d + 2h + T) = \underline{i}$ and $\dot{I}(t^* + d + 2h + T) \leqslant 0$, it has from the second equation of (1.1) and (2.6) that, for $t = t^* + d + 2h + T$,

$$\begin{aligned}\dot{I}(t) &\geqslant [\beta S(t) - (\mu_2 + \lambda)]\underline{i} \\ &> \beta[S^{\Delta} - S^*]\underline{i} \\ &> 0.\end{aligned}$$

This is a contradiction to $\dot{I}(t^* + d + 2h + T) \leqslant 0$. Thus, $I(t) \geqslant \underline{i}$ for all $t \geqslant t^* + d + h$.

Therefore, for all $t \geqslant t^* + d + h$,

$$\dot{V}(t) > \beta(S^{\Delta} - S^*)\underline{i},$$

which implies that $V(t) \to +\infty$ as $t \to +\infty$. On the other hand, from Lemma 2.1 and (2.4), there exists a sufficiently large $\bar{T} > 0$ such that, for $t \geqslant \bar{T}$,

$$V(t) \leqslant S_0 + 1 + \beta S^* h(1 + S_0).$$

This is a contradiction to $V(t) \to +\infty$ as $t \to +\infty$. Hence, the claim is proved.

In the rest, we are left to consider two cases.

(i) $I(t)\geqslant \rho I^*$ for all large t.
(ii) $I(t)$ oscillates about ρI^* for all large t.

We show that $I(t)\geqslant \rho I^* \mathrm{e}^{-(\mu_2+\lambda)(d+h)}=v_2$ for all large t. Clearly, we only need to consider the case (ii). Let t_1 and t_2 be sufficiently large such that $t^*<t_1<t_2$,

$$I(t_1)=I(t_2)=\rho I^*,$$

$$I(t)<\rho I^* \quad (t_1<t<t_2).$$

If $t_2-t_1\leqslant d+h$, from the second equation of (1.1), we have that

$$\dot{I}(t)>-(\mu_2+\lambda)I(t),$$

which implies that, for $t\in(t_1,t_2)$,

$$I(t)>I(t_1)\mathrm{e}^{-(\mu_2+\lambda)(t-t_1)}.$$

It is obvious that, for $t_1<t<t_2$,

$$I(t)>\rho I^* \mathrm{e}^{-(\mu_2+\lambda)(d+h)}=v_2.$$

If $t_2-t_1>d+h$, we can easily obtain that $I(t)\geqslant v_2$ for $t\in[t_1,t_1+d+h]$. Then, proceeding exactly as the proof for the above claim, we can show that $I(t)\geqslant v_2$ for $t_1+d+h\leqslant t\leqslant t_2$.

In fact, if not, there exists a $T^*\geqslant 0$ such that $I(t)\geqslant v_2$ for $t_1\leqslant t\leqslant t_1+d+h+T^*\leqslant t_2$, $I(t_1+d+h+T^*)=v_2$ and $\dot{I}(t_1+d+h+T^*)\leqslant 0$. On the other hand, it has that, for $t_1\leqslant t\leqslant t_1+d+h+T^*\leqslant t_2$,

$$I(t)\leqslant \rho I^*.$$

Then, it has from the first equation of (1.1) that, for $t_1+h\leqslant t\leqslant t_1+d+h+T^*\leqslant t_2$,

$$\begin{aligned}\dot{S}(t)&=-\beta S(t)I(t-h)-\mu_1 S(t)+b(1-\beta_1)+\frac{b\beta_1}{1+N(t)}\\&\geqslant -(\beta\rho I^*+\mu_1)S(t)+b(1-\beta_1)+\frac{b\beta_1}{1+S_0+\varepsilon_0},\end{aligned}$$

which implies that, for $t_1+d+h\leqslant t\leqslant t_1+d+h+T^*\leqslant t_2$,

$$\begin{aligned}S(t)&\geqslant S(t_1+h)\mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-t_1-h)}+\left(b(1-\beta_1)+\frac{b\beta_1}{1+S_0+\varepsilon_0}\right)\int_{t_1+h}^{t}\mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-\theta)}\,\mathrm{d}\theta\\&>\frac{1}{\beta\rho I^*+\mu_1}\left(b(1-\beta_1)+\frac{b\beta_1}{1+S_0+\varepsilon_0}\right)(1-\mathrm{e}^{-(\beta\rho I^*+\mu_1)(t-t_1-h)})\\&\geqslant\frac{1}{\beta\rho I^*+\mu_1}\left(b(1-\beta_1)+\frac{b\beta_1}{1+S_0+\varepsilon_0}\right)(1-\mathrm{e}^{-(\beta\rho I^*+\mu_1)d})\\&=q(1-\mathrm{e}^{-(\beta\rho I^*+\mu_1)d})\\&=S^{\Delta}>S^*.\end{aligned}\tag{2.7}$$

Thus, it has from the second equation of (1.1) and (2.7) that, for $t = t_1 + d + h + T^*$,

$$\begin{aligned}\dot{I}(t) &= \beta S(t)I(t-h) - \mu_2 I(t) - \lambda I(t)\\ &\geqslant [\beta S(t) - (\mu_2 + \lambda)]v_2\\ &> \beta[S^{\Delta} - S^*]v_2\\ &> 0.\end{aligned}$$

This is a contradiction to $\dot{I}(t_1 + d + h + T^*) \leqslant 0$. Therefore, we have that $I(t) \geqslant v_2$ for $t \in [t_1, t_2]$. Since this kind of interval $[t_1, t_2]$ is chosen in an arbitrary way, we conclude that $I(t) \geqslant v_2$ for all large t in the case (ii). Hence, it has that

$$\liminf_{t \to +\infty} I(t) \geqslant v_2.$$

The proof of Lemma 2.3 is completed. □

Theorem 2.4. *If* $S_0 > S^*$, *then*, (1.1) *is permanent for any time delay* h.

Proof. Lemma 2.1 shows that the solution $(S(t), I(t), R(t))$ of (1.1) is uniformly ultimately bounded. Lemmas 2.2–2.3 show that $S(t)$ and $I(t)$ are ultimately strictly positive with some positive constants. Furthermore, it follows from the third equation of (1.1) and Lemma 2.3 that $\liminf_{t \to +\infty} R(t) \geqslant (\lambda v_2)/\mu_3$. Hence, (1.1) is permanent. This proves Theorem 2.4. □

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