Journal of Number Theory 131 (2011) 2219–2238

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On congruences related to central binomial coefficients ☆

Zhi-Wei Sun

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

ARTICLE INFO

Article history:
Received 16 September 2010
Revised 31 January 2011
Accepted 6 April 2011
Available online 18 July 2011
Communicated by David Goss

MSC:
primary 11B65
secondary 05A10, 11A07, 11B68, 11E25

Keywords:
Central binomial coefficients
Congruences modulo prime powers
Euler numbers
Binary quadratic forms

ABSTRACT

It is known that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)4^k}=\frac{\pi}{2}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3}.$$

In this paper we obtain their p-adic analogues such as

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)4^k}\equiv 3\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv pE_{p-3}\pmod{p^2},$$

where $p>3$ is a prime and $E_0,E_1,E_2,\ldots$ are Euler numbers. Besides these, we also deduce some other congruences related to central binomial coefficients. In addition, we pose some conjectures one of which states that for any odd prime p we have

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p \pmod{p^2}\\ \quad\text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2}\\ \quad\text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

☆ Supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

E-mail address: zwsun@nju.edu.cn.
URL: http://math.nju.edu.cn/~zwsun.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.04.004

1. Introduction

The following three series related to π are well known (cf. [Ma]):

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)4^k}=\frac{\pi}{2},\qquad \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3},$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}=\frac{\pi^2}{10}.$$

These three identities can be easily shown by using $1/(2k+1)=\int_0^1 x^{2k}\,dx$. In March 2010 the author [Su2] suggested that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7\pi^3}{216}$$

via a public message to Number Theory List, and then Olivier Gerard pointed out there is a computer proof via certain math. softwares like `Mathematica` (version 7). Our main goal in this paper is to investigate p-adic analogues of the above identities for powers of π.

For a prime p and an integer $a\not\equiv 0 \pmod p$, we let $q_p(a)$ denote the Fermat quotient $(a^{p-1}-1)/p$. For an odd prime p and an integer a, by $(\frac{a}{p})$ we mean the Legendre symbol. As usual, harmonic numbers refer to those $H_n=\sum_{0<k\leqslant n}1/k$ with $n\in\mathbb{N}=\{0,1,2,\dots\}$. Recall that Euler numbers $E_0,E_1,E_2,\dots$ are integers defined by $E_0=1$ and the recursion:

$$\sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\quad\text{for } n=1,2,3,\dots.$$

And Bernoulli numbers $B_0,B_1,B_2,\dots$ are rational numbers given by $B_0=1$ and

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad(n=1,2,3,\dots).$$

Now we state our first theorem which gives certain p-adic analogues of the first and the second identities mentioned at the beginning of this section.

Theorem 1.1. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)4^k}\equiv(-1)^{(p+1)/2}q_p(2)\pmod{p^2},\tag{1.1}$$

and

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)4^k}\equiv pE_{p-3}\pmod{p^2}\tag{1.2}$$

which is equivalent to the congruence

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{(2k-1)\binom{2k}{k}}\equiv E_{p-3}+(-1)^{(p-1)/2}-1\pmod{p}.\tag{1.3}$$

(ii) *Suppose $p>3$. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0\pmod{p^2},\tag{1.4}$$

and

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv\frac{p}{3}E_{p-3}\pmod{p^2}\tag{1.5}$$

which is equivalent to the congruence

$$\sum_{k=1}^{(p-1)/2}\frac{16^k}{k(2k-1)\binom{2k}{k}}\equiv\frac{8}{3}E_{p-3}\pmod{p}.\tag{1.6}$$

Remark 1.1. Motivated by the work of H. Pan and Z.W. Sun [PS], and Sun and R. Tauraso [ST1,ST2], the author [Su1] managed to determine $\sum_{k=0}^{p^a-1}\binom{2k}{k}/m^k$ modulo p^2, where p is a prime, a is a positive integer, and m is any integer not divisible by p. See also [SSZ,G-Z,Su3] for related results on p-adic valuations.

The congruences in Theorem 1.1 are somewhat sophisticated. Now we deduce some easier congruences via combinatorial identities. Using the software `Sigma`, we find the identities

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(2k+1)^2}=\frac{4^n}{(2n+1)\binom{2n}{n}}\sum_{k=0}^{n}\frac{1}{2k+1},$$
$$\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)\binom{n}{k}}=n+1-(n+1)\sum_{k=1}^{n}\frac{1-2(-1)^k}{(k+1)^2},$$

and

$$n\sum_{k=2}^{n}\frac{(-1)^k}{(k-1)^2\binom{n}{k}}=\sum_{k=2}^{n}\frac{1-2k+(-1)^k(1-k+2k^2)}{k(k-1)^2}=\frac{1+(-1)^n}{n}-\sum_{k=1}^{n-1}\frac{1+2(-1)^k}{k^2}.$$

If $p=2n+1$ is an odd prime, then

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod{p}\quad\text{for all } k=0,\ldots,p-1.$$

Thus, from the above three identities we deduce for any prime $p>3$ the congruences

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^2 4^k}\equiv(-1)^{(p+1)/2}\frac{q_p(2)^2}{2}\pmod{p},\tag{1.7}$$

$$\sum_{k=2}^{(p-1)/2}\frac{4^k}{(k-1)^2\binom{2k}{k}}\equiv 8E_{p-3}-4-12\left(\frac{-1}{p}\right)\pmod{p}\tag{1.8}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{4^k}{(k+1)\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)(4-2E_{p-3})-2\pmod{p}.\tag{1.9}$$

Note that the series $\sum_{k=0}^{\infty}4^k/((k+1)\binom{2k}{k})$ diverges while `Mathematica` (version 7) yields

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2 4^k}=\frac{\pi}{4}\log 2\quad\text{and}\quad\sum_{k=2}^{\infty}\frac{4^k}{(k-1)^2\binom{2k}{k}}=\pi^2-4$$

the latter of which appeared in [Sp].

Let p be an odd prime. By a known result (see, e.g., [I]),

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv a(p)\pmod{p^2},$$

where the sequence $\{a(n)\}_{n\geqslant 1}$ is defined by

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}\left(1-q^{4n}\right)^6.$$

Clearly, $a(p)=0$ if $p\equiv 3\pmod 4$.

Recall that Catalan numbers are those integers

$$C_k=\frac{1}{k+1}\binom{2k}{k}=\binom{2k}{k}-\binom{2k}{k+1}\quad(k=0,1,2,\ldots).$$

They have many combinatorial interpretations (see, e.g., [St2, pp. 219–229]).

Now we present our second theorem.

Theorem 1.2. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{k^3\binom{2k}{k}^3}{64^k}\equiv\begin{cases}0\pmod{p} & \text{if } p\equiv 1\pmod 4,\\ -\frac{1}{640}\left(\frac{p+1}{4}!\right)^{-4}\pmod{p} & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{1.10}$$

If $p > 3$ *and* $p \equiv 3 \pmod 4$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2}. \tag{1.11}$$

(ii) *We have*

$$\sum_{k=0}^{p-1}\frac{C_k^2}{16^k} \equiv -3 \pmod p, \tag{1.12}$$

and

$$\sum_{k=0}^{p-1}\frac{C_k^3}{64^k} \equiv \begin{cases} 7 \pmod p & \text{if } p \equiv 1 \pmod 4, \\ 7 - \frac{3}{2}(\frac{p+1}{4}!)^{-4} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.13}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}C_k}{32^k} \equiv \begin{cases} p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ p + (4p + 2^p - 6)\binom{(p-3)/2}{(p-3)/4} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.14}$$

Remark 1.2. Let p be an odd prime. We conjecture that if $p \equiv 1 \pmod 4$ and $p > 5$ then

$$\sum_{k=0}^{p^a-1}\frac{k^3\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for all } a = 1, 2, 3, \ldots.$$

We also conjecture that $\sum_{k=0}^{(p-1)/2} kC_k^3/16^k \equiv 2p-2 \pmod{p^2}$ if $p \equiv 1 \pmod 3$, and $\sum_{k=0}^{(p-1)/2} C_k^3/64^k \equiv 8 \pmod{p^2}$ if $p \equiv 1 \pmod 4$.

In the next section we are going to provide several lemmas. Theorems 1.1 and 1.2 will be proved in Sections 3 and 4 respectively. Section 5 contains some open conjectures of the author for further research.

2. Some lemmas

For $n \in \mathbb{N}$ the Chebyshev polynomial $U_n(x)$ of the second kind is given by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

It is well known that

$$U_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-1)^k(2x)^{n-2k}.$$

Lemma 2.1. *For $n \in \mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4)^k}{2k+1}=\frac{(-1)^n}{2n+1} \tag{2.1}$$

and

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-1)^k}{2k+1}=\begin{cases}(-1)^n/(2n+1) & \text{if } 3\nmid 2n+1,\\ 2(-1)^{n-1}/(2n+1) & \text{if } 3\mid 2n+1.\end{cases} \tag{2.2}$$

Proof. Note that

$$U_{2n}(x)=\sum_{k=0}^{n}\binom{2n-k}{2n-2k}(-1)^k(2x)^{2n-2k}=\sum_{j=0}^{n}\binom{n+j}{2j}(-1)^{n-j}(2x)^{2j}.$$

Thus

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-4)^k}{2k+1}&=\int_0^1\sum_{k=0}^{n}\binom{n+k}{2k}(-4)^kx^{2k}\,dx=(-1)^n\int_0^1U_{2n}(x)\,dx\\&=(-1)^n\int_{\pi/2}^{0}U_{2n}(\cos\theta)(-\sin\theta)\,d\theta\\&=(-1)^n\int_0^{\pi/2}\sin\bigl((2n+1)\theta\bigr)\,d\theta\\&=\frac{-(-1)^n}{2n+1}\cos\bigl((2n+1)\theta\bigr)\Big|_0^{\pi/2}=\frac{(-1)^n}{2n+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-1)^k}{2k+1}&=\int_0^1\sum_{k=0}^{n}\binom{n+k}{2k}(-1)^kx^{2k}\,dx=(-1)^n\int_0^1U_{2n}\left(\frac{x}{2}\right)dx\\&=(-1)^n\int_{\pi/2}^{\pi/3}U_{2n}(\cos\theta)(-2\sin\theta)\,d\theta\\&=-2(-1)^n\int_{\pi/2}^{\pi/3}\sin\bigl((2n+1)\theta\bigr)\,d\theta\\&=\frac{2(-1)^n}{2n+1}\cos\bigl((2n+1)\theta\bigr)\Big|_{\pi/2}^{\pi/3}=\frac{2(-1)^n}{2n+1}\cos\left(\frac{2n+1}{3}\pi\right)\end{aligned}$$

$$= \begin{cases} (-1)^n/(2n+1) & \text{if } 3 \nmid 2n+1, \\ 2(-1)^{n-1}/(2n+1) & \text{if } 3 \mid 2n+1. \end{cases}$$

This concludes the proof. □

Lemma 2.2. *Let $p = 2n+1$ be an odd prime. For $k = 0, \ldots, n$ we have*

$$\binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}. \tag{2.3}$$

Proof. As observed by the author's brother Z.H. Sun,

$$\binom{n+k}{2k} = \frac{\prod_{0<j\leqslant k}(p^2-(2j-1)^2)}{4^k(2k)!} \equiv \frac{\prod_{0<j\leqslant k}(-(2j-1)^2)}{4^k(2k)!} = \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.$$

We are done. □

Remark 2.1. Using Lemma 2.2 and the identity

$$\sum_{k=0}^{n} \frac{\binom{n+k}{2k}(-2)^k}{2k+1} = \frac{(1+i)(-i)^n(1+(-1)^{n-1}i)}{2(2n+1)},$$

we can deduce for any prime $p > 3$ that

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \equiv -\left(\frac{-2}{p}\right)\frac{q_p(2)}{2} + \left(\frac{-2}{p}\right)\frac{p}{8}q_p^2(2) \pmod{p^2}.$$

Lemma 2.3. *Let p be any odd prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k\binom{2k}{k}} \equiv 2\big((-1)^{(p-1)/2}-1\big) \pmod{p}. \tag{2.4}$$

Proof. Clearly (2.4) holds for $p = 3$.

Now assume that $p > 3$. We can even show a stronger congruence

$$\frac{1}{2}\sum_{k=1}^{(p-1)/2} \frac{4^k}{k\binom{2k}{k}} \equiv (-1)^{(p-1)/2}\big(1 - p\,q_p(2) + p^2 q_p(2)^2\big) - 1 \pmod{p^3}.$$

Let us employ a known identity (cf. [G, (2.9)])

$$\sum_{k=1}^{n} \frac{2^{2k-1}}{k\binom{2k}{k}} = \frac{2^{2n}}{\binom{2n}{n}} - 1$$

which can be easily proved by induction. Taking $n = (p-1)/2$ and noting that

$$(-1)^n\binom{2n}{n} \equiv 4^{p-1} \pmod{p^3}$$

by Morley's congruence [Mo], we get

$$\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{4^k}{k\binom{2k}{k}} \equiv \frac{(-1)^{(p-1)/2}}{1+p\,q_p(2)}-1 \equiv (-1)^{(p-1)/2}\big(1-p\,q_p(2)+p^2q_p(2)^2\big)-1 \pmod{p^3}.$$

This ends the proof. □

Lemma 2.4. *For any $n\in\mathbb{N}$, we have the identity*

$$\sum_{k=-n}^{n}\frac{(-1)^k}{(2k+1)^2}\binom{2n}{n+k}=\frac{16^n}{(2n+1)^2\binom{2n}{n}}. \tag{2.5}$$

Proof. Let u_n and v_n denote the left-hand side and the right-hand side of (2.5) respectively. By the well-known Zeilberger algorithm (cf. [PWZ]),

$$(2n+3)(2n+5)^2u_{n+2}-16(n+2)(2n+3)^2u_{n+1}+64(n+1)(n+2)(2n+1)u_n=0$$

for all $n=0,1,2,\ldots$. It is easy to verify that $\{v_n\}_{n\geqslant 0}$ also satisfies this recurrence. Since $u_0=v_0=1$ and $u_1=v_1=8/9$, by the recursion we have $u_n=v_n$ for all $n\in\mathbb{N}$. □

Remark 2.2. (2.5) was discovered by the author during his study of Delannoy numbers (cf. [Su5]). The reader may consult [GZ] and [ZG] for some other combinatorial identities obtained via solving recurrence relations.

Lemma 2.5. *For any $n\in\mathbb{N}$ we have*

$$\sum_{k=0}^{n}\binom{2n-k}{k}(-1)^k=\left(\frac{1-n}{3}\right) \tag{2.6}$$

and

$$\sum_{k=0}^{n}\binom{2n-k}{k}\frac{1}{(-4)^k}=\frac{2n+1}{4^n}. \tag{2.7}$$

Remark 2.3. (2.6) and (2.7) are known identities, see (1.75) and (1.73) of [G].

Lemma 2.6. *Let $p>3$ be a prime. Then*

$$\sum_{0<k\leqslant\lfloor p/6\rfloor}\frac{(-1)^k}{k^2}\equiv(-1)^{(p-1)/2}10E_{p-3} \pmod{p}. \tag{2.8}$$

Proof. Recall that the Euler polynomial of degree n is defined by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(1-x)=(-1)^n E_n(x), \qquad E_n(x)+E_n(x+1)=2x^n,$$

and

$$E_n(x)=\frac{2}{n+1}\left(B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right),$$

where $B_m(x)$ denotes the Bernoulli polynomial of degree m.

Note that $E_{p-3}(0)=\frac{2}{p-2}(1-2^{p-2})B_{p-2}=0$ and $E_{p-3}(5/6)=E_{p-3}(1/6)$. Thus

$$\begin{aligned}
2\sum_{0<k\leqslant\lfloor p/6\rfloor}\frac{(-1)^k}{k^2} &\equiv \sum_{k=0}^{\lfloor p/6\rfloor}(-1)^k\big(2k^{p-3}\big)\\
&=\sum_{k=0}^{\lfloor p/6\rfloor}\big((-1)^k E_{p-3}(k)-(-1)^{k+1}E_{p-3}(k+1)\big)\\
&=E_{p-3}(0)-(-1)^{\lfloor p/6\rfloor+1}E_{p-3}\left(\left\lfloor\frac{p}{6}\right\rfloor+1\right)\\
&\equiv(-1)^{\lfloor p/6\rfloor}E_{p-3}\left(\frac{1}{6}\right) \pmod p.
\end{aligned}$$

Evidently $\lfloor p/6\rfloor\equiv(p-1)/2 \pmod 2$. As $E_n(1/6)=2^{-n-1}(1+3^{-n})E_n$ for all $n=0,2,4,\ldots$ (see, e.g., G.J. Fox [F]), we have

$$E_{p-3}\left(\frac{1}{6}\right)=2^{2-p}\big(1+3^{3-p}\big)E_{p-3}\equiv 2\big(1+3^2\big)E_{p-3}=20E_{p-3} \pmod p.$$

Therefore (2.8) follows from the above. □

3. Proof of Theorem 1.1

(a) Set $n=(p-1)/2$. By Lemmas 2.1 and 2.2,

$$\begin{aligned}
\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{(2k+1)4^k} &\equiv \sum_{k=0}^{n-1}\binom{n+k}{2k}\frac{(-4)^k}{2k+1}=\frac{(-1)^n-(-4)^n}{2n+1}\\
&=(-1)^n\frac{1-2^{p-1}}{p}=(-1)^{n+1}q_p(2) \pmod{p^2}.
\end{aligned}$$

This proves (1.1). When $p=2n+1>3$, again by Lemmas 2.1 and 2.2, we have

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv\sum_{k=0}^{n-1}\binom{n+k}{2k}\frac{(-1)^k}{2k+1}=0 \pmod{p^2}$$

and hence (1.4) holds.

(b) For $k \in \{1, \ldots, (p-1)/2\}$, it is clear that

$$\frac{1}{p}\binom{2(p-k)}{p-k} = \frac{1}{p} \times \frac{p!\prod_{s=1}^{p-2k}(p+s)}{((p-1)!/\prod_{0<t<k}(p-t))^2}$$
$$\equiv \frac{(k-1)!^2}{(p-1)!/(p-2k)!} \equiv -\frac{(k-1)!^2}{(2k-1)!} = -\frac{2}{k\binom{2k}{k}} \pmod{p}.$$

Therefore

$$\frac{1}{p}\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)4^k} = \sum_{k=1}^{(p-1)/2}\frac{\binom{2(p-k)}{p-k}/p}{(2(p-k)+1)4^{p-k}}$$
$$\equiv -2\sum_{k=1}^{(p-1)/2}\frac{4^{k-1}}{(1-2k)k\binom{2k}{k}} = \frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}} \pmod{p}.$$

Similarly,

$$\frac{1}{p}\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{1}{8}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k(2k-1)\binom{2k}{k}} \pmod{p}$$

and hence (1.5) and (1.6) are equivalent. Observe that

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}} = 2\sum_{k=1}^{(p-1)/2}\frac{4^k}{(2k-1)\binom{2k}{k}} - \sum_{k=1}^{(p-1)/2}\frac{4^k}{k\binom{2k}{k}}.$$

Thus, in view of (2.4), both (1.2) and (1.3) are equivalent to the congruence

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}} \equiv 2E_{p-3} \pmod{p} \tag{3.1}$$

which holds trivially when $p=3$.

(c) Now we prove (3.1) for $p>3$. It is easy to see that

$$(n+1)\big(2(n+1)-1\big)\binom{2(n+1)}{n+1} = 2(2n+1)^2\binom{2n}{n}$$

for any $n \in \mathbb{N}$. Thus

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k(2k-1)\binom{2k}{k}} = \sum_{k=0}^{(p-3)/2}\frac{4^{k+1}}{2(2k+1)^2\binom{2k}{k}}.$$

In view of Lemma 2.4,

$$\sum_{n=0}^{(p-3)/2}\frac{4^n}{(2n+1)^2\binom{2n}{n}}=\sum_{n=0}^{(p-3)/2}\frac{1}{4^n}\sum_{k=-n}^{n}\frac{(-1)^k}{(2k+1)^2}\binom{2n}{n-k}$$
$$=\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2 4^{|k|}}\sum_{n=|k|}^{(p-3)/2}\frac{\binom{2n}{n-|k|}}{4^{n-|k|}}.$$

For $k\in\{0,\dots,(p-3)/2\}$, with the help of Lemma 2.5 we have

$$\sum_{n=k}^{(p-3)/2}\frac{\binom{2n}{n-k}}{4^{n-k}}=\sum_{r=0}^{(p-3)/2-k}\frac{\binom{2k+2r}{r}}{4^r}=\sum_{r=0}^{(p-3)/2-k}\frac{\binom{-2k-r-1}{r}}{(-4)^r}$$
$$\equiv\sum_{r=0}^{(p-1)/2-k}\frac{\binom{p-1-2k-r}{r}}{(-4)^r}-\frac{1}{(-4)^{(p-1)/2-k}}$$
$$=\frac{p-2k-(-1)^{(p-1)/2-k}}{4^{(p-1)/2-k}}\equiv\frac{(-1)^{(p+1)/2-k}-2k}{4^{-k}}\pmod p.$$

Therefore

$$\sum_{n=0}^{(p-3)/2}\frac{4^n}{(2n+1)^2\binom{2n}{n}}\equiv\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left((-1)^{(p+1)/2-k}-2|k|\right)$$
$$=\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left((-1)^{(p+1)/2-k}-2k\right)$$
$$+\sum_{k=1}^{(p-3)/2}\frac{(-1)^{-k}}{(-2k+1)^2}\left((-1)^{(p+1)/2+k}-2k\right)$$
$$\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{(2k-1)^2}\left((-1)^{(p+1)/2-k+1}-2(k-1)\right)$$
$$+\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{(2k-1)^2}\left((-1)^{(p+1)/2+k}-2k\right)\pmod p$$

and hence

$$\sum_{n=0}^{(p-3)/2}\frac{4^n}{(2n+1)^2\binom{2n}{n}}\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{(2k-1)^2}\left(2(-1)^{(p+1)/2-k}+2(k-1)-2k\right)$$
$$=4(-1)^{(p+1)/2}\sum_{\substack{1\leqslant k\leqslant(p-1)/2\\ k\equiv(p-1)/2\ (\mathrm{mod}\ 2)}}\frac{1}{(2k-1)^2}\pmod p.$$

Since $p>3$ and $\sum_{k=1}^{p-1}1/(2k)^2\equiv\sum_{k=1}^{p-1}1/k^2 \pmod p$, we have

$$2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod p$$

and hence

$$\sum_{\substack{1\leqslant k\leqslant (p-1)/2\\ 2k+1\equiv p \ (\mathrm{mod}\ 4)}}\frac{1}{(2k-1)^2}\equiv\sum_{\substack{1\leqslant k\leqslant (p-1)/2\\ p+1-2k\equiv 2 \ (\mathrm{mod}\ 4)}}\frac{1}{(p+1-2k)^2}=\sum_{\substack{k=1\\ k\equiv 2 \ (\mathrm{mod}\ 4)}}^{p-1}\frac{1}{k^2}$$
$$\equiv-\sum_{\substack{k=1\\ 4\mid k}}^{p-1}\frac{1}{k^2}=-\frac{1}{16}\sum_{k=1}^{\lfloor p/4\rfloor}\frac{1}{k^2} \pmod p.$$

As $\sum_{k=1}^{\lfloor p/4\rfloor}1/k^2\equiv(-1)^{(p-1)/2}4E_{p-3} \pmod p$ by Lehmer [L, (20)], from the above we obtain that

$$\sum_{n=0}^{(p-3)/2}\frac{4^n}{(2n+1)^2\binom{2n}{n}}\equiv 4(-1)^{(p+1)/2}\frac{(-1)^{(p-1)/2}4E_{p-3}}{-16}=E_{p-3} \pmod p$$

and hence (3.1) holds.

(d) Finally we show (1.6) for $p>3$. In view of Lemmas 2.4 and 2.5, arguing as in (c) we get

$$\frac{1}{8}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k(2k-1)\binom{2k}{k}}=\sum_{n=0}^{(p-3)/2}\frac{16^n}{(2n+1)^2\binom{2n}{n}}$$
$$\equiv\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\sum_{r=0}^{(p-3)/2-|k|}\binom{p-1-2|k|-r}{r}(-1)^r$$
$$=\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left(\left(\frac{|k|-(p-3)/2}{3}\right)-(-1)^{(p-1)/2-|k|}\right)$$
$$=\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left(\left(\frac{p-2|k|}{3}\right)+(-1)^{(p+1)/2-|k|}\right) \pmod p.$$

Observe that

$$\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{1}{(2k+1)^2}=\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}+\sum_{k=1}^{(p-3)/2}\frac{1}{(-2k+1)^2}$$
$$=2\sum_{k=1}^{(p-1)/2}\frac{1}{(2k-1)^2}-\frac{1}{(p-2)^2}$$
$$\equiv 2\left(\sum_{k=1}^{p-1}\frac{1}{k^2}-\sum_{k=1}^{(p-1)/2}\frac{1}{(2k)^2}\right)-\frac{1}{4}\equiv-\frac{1}{4} \pmod p$$

and

$$\begin{aligned}
&\sum_{k=-(p-3)/2}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left(\frac{p-2|k|}{3}\right)\\
&=\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{(2k+1)^2}\left(\frac{p+k}{3}\right)+\sum_{k=1}^{(p-3)/2}\frac{(-1)^k}{(-2k+1)^2}\left(\frac{p+k}{3}\right)\\
&=\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{(2k-1)^2}\left(\left(\frac{p+k}{3}\right)-\left(\frac{p+k-1}{3}\right)\right)-\frac{(-1)^{(p-1)/2}}{(p-2)^2}\left(\frac{p+(p-1)/2}{3}\right)\\
&\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{(2k-1)^2}-3\sum_{\substack{k=1\\ 3\mid p+k+1}}^{(p-1)/2}\frac{(-1)^k}{(2k-1)^2}+\frac{(-1)^{(p+1)/2}}{4}\pmod p.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{8}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k(2k-1)\binom{2k}{k}}&\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{(p-(2k-1))^2}-3\sum_{\substack{k=1\\ 3\mid 2k-1-p}}^{(p-1)/2}\frac{(-1)^k}{(p-(2k-1))^2}\\
&=\sum_{k=1}^{(p-1)/2}\frac{(-1)^{(p+1)/2-k}}{(2k)^2}-3\sum_{0<k\leqslant\lfloor p/6\rfloor}\frac{(-1)^{(p+1)/2-3k}}{(6k)^2}\pmod p.
\end{aligned}$$

Since

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k+1}{k^2}=\frac{1}{2}\sum_{k=1}^{\lfloor p/4\rfloor}\frac{1}{k^2}\equiv 2(-1)^{(p-1)/2}E_{p-3}\pmod p,$$

with the help of Lemma 2.6 we finally get

$$\frac{1}{8}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k(2k-1)\binom{2k}{k}}\equiv-\frac{E_{p-3}}{2}+\frac{10}{12}E_{p-3}=\frac{E_{p-3}}{3}\pmod p$$

which proves (1.6).

Combining (a)–(d) we have completed the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Lemma 4.1. *For any $n\in\mathbb{N}$ we have*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\frac{(3n)!}{(n!)^3}. \tag{4.1}$$

Proof. By Dixon's identity (cf. [St1, p. 45]) we have

$$\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^3=\frac{(3n)!}{(n!)^3},$$

which is equivalent to the desired identity. □

Lemma 4.2. *(See [DPSW, (2)].) For any positive odd integer n we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}^3(-1)^kH_k=\frac{(-1)^{(n+1)/2}}{3}\cdot\frac{(3n)!!}{(n!!)^3}, \tag{4.2}$$

where $(2m+1)!!$ *refers to* $\prod_{k=0}^{m}(2k+1)$.

Lemma 4.3. *For each* $n=1,2,3,\ldots,$ *we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{C_k}{(-2)^k}=\begin{cases}(-1)^{(n-1)/2}C_{(n-1)/2}/2^n & \text{if } 2\nmid n,\\ 0 & \text{if } 2\mid n.\end{cases} \tag{4.3}$$

Proof. The desired identity can be easily proved by the WZ method (cf. [PWZ]); in fact, if we denote by $S(n)$ the sum of the left-hand side or the right-hand side of (4.3), then we have the recursion $S(n+2)=-nS(n)/(n+3)$ $(n=1,2,3,\ldots)$. □

Proof of Theorem 1.2. Let us recall that

$$\binom{(p-1)/2}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\quad\text{for } k=0,1,\ldots,p-1.$$

Note also that for any positive odd integer n we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k\binom{n}{k}^3+(-1)^{n-k}\binom{n}{n-k}^3\right)=0.$$

These two basic facts will be frequently used in the proof.

(i) Clearly,

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{k^3\binom{2k}{k}^3}{64^k}&\equiv\sum_{k=0}^{(p-1)/2}(-1)^kk^3\binom{(p-1)/2}{k}^3\\&=-\left(\frac{p-1}{2}\right)^3\sum_{k=1}^{(p-1)/2}(-1)^{k-1}\binom{(p-3)/2}{k-1}^3\\&\equiv\frac{1}{8}\sum_{k=0}^{(p-3)/2}(-1)^k\binom{(p-3)/2}{k}^3\pmod p.\end{aligned}$$

So, if $p\equiv 1\pmod 4$ then $(p-3)/2$ is odd and hence $\sum_{k=0}^{p-1}k^3\binom{2k}{k}^3/64^k\equiv 0\pmod p$. When $p=4n+3$ with $n\in\mathbb{N}$, applying Lemma 4.1 we get

$$8\sum_{k=0}^{p-1}\frac{k^3\binom{2k}{k}^3}{64^k}\equiv(-1)^n\frac{(3n)!}{(n!)^3}=\frac{(-1)^n((p+1)/4)^3}{((p+1)/4)!^3}\times\frac{(p-1)!}{\prod_{0<k<p-3n}(p-k)}$$
$$\equiv\frac{(-1)^{n+1}}{64((p+1)/4)!^3(-1)^{p-1-3n}(p-1-3n)!}$$
$$\equiv-\frac{1}{64((p+1)/4)!^4(p+5)/4}\pmod p.$$

So (1.10) holds.

For $k=0,1,\ldots,p-1$, clearly

$$\binom{p-1}{k}(-1)^k=\prod_{0<j\leqslant k}\left(1-\frac{p}{j}\right)\equiv 1-pH_k\pmod{p^2}.$$

When $p>3$ and $p\equiv 3\pmod 4$, $\sum_{k=0}^{p-1}\binom{2k}{k}^3/64^k\equiv 0\pmod{p^2}$ as mentioned in the first section, hence with the help of Lemma 4.2 we get

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\sum_{k=0}^{p-1}(1-pH_k)\frac{\binom{2k}{k}^3}{64^k}$$
$$\equiv -p\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}^3(-1)^kH_k$$
$$\equiv -p\frac{(-1)^{(p+1)/4}}{3}\times\frac{(3(p-1)/2)!!}{((p-1)/2)!!^3}\equiv 0\pmod{p^2}.$$

This proves (1.11) for $p\equiv 3\pmod 4$ with $p\neq 3$.

(ii) Below we set $n=(p-1)/2$ and want to show (1.12)–(1.14). Note that $C_k\equiv 0\pmod p$ when $n<k<p-1$. Also,

$$C_{p-1}=\frac{1}{p}\binom{2p-2}{p-1}=\frac{1}{2p-1}\binom{2p-1}{p}\equiv-\prod_{k=1}^{p-1}\frac{p+k}{k}\equiv -1\pmod p.$$

Thus,

$$\sum_{k=0}^{p-1}\frac{C_k^2}{16^k}\equiv\sum_{k=0}^{n}\frac{C_k^2}{16^k}+1\equiv\sum_{k=0}^{n}\frac{1}{(k+1)^2}\binom{n}{k}^2+1\pmod p$$

and

$$\sum_{k=0}^{p-1}\frac{C_k^3}{64^k}\equiv\sum_{k=0}^{n}\frac{C_k^3}{64^k}-1\equiv\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^3}\binom{n}{k}^3-1\pmod p.$$

Clearly,

$$\begin{aligned}
&(n+1)^2\sum_{k=0}^{n}\frac{1}{(k+1)^2}\binom{n}{k}^2\\
&=\sum_{k=0}^{n}\binom{n+1}{k+1}^2=\sum_{k=0}^{n+1}\binom{n+1}{k}^2-1=\sum_{k=0}^{n+1}\binom{n+1}{k}\binom{n+1}{n+1-k}-1\\
&=\binom{2n+2}{n+1}-1\quad\big(\text{by the Chu–Vandermonde identity }\big(\text{cf. [GKP, p. 169]}\big)\big)\\
&=\binom{p+1}{(p+1)/2}-1=\frac{2p}{(p-1)/2}\binom{p-1}{(p-3)/2}-1\equiv-1\pmod p
\end{aligned}$$

and

$$-(n+1)^3\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^3}\binom{n}{k}^3=\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}^3=\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}^3-1.$$

If $p\equiv 1\pmod 4$, then $n+1$ is odd and hence

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}^3=0.$$

When $p=4m-1$ with $m\in\mathbb{Z}$, by Lemma 4.1

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}^3=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3=(-1)^m\frac{(3m)!}{(m!)^3},$$

and in the case $m>1$ we have

$$(-1)^m(3m)!=(-1)^m\frac{(p-1)!}{\prod_{0<k<m-1}(p-k)}\equiv-\frac{1}{(m-2)!}=-\frac{m(m-1)}{m!}\equiv\frac{3}{16(m!)}\pmod p.$$

Therefore, if $p\equiv 3\pmod 4$ then

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}^3\equiv\frac{3}{16}\left(\frac{p+1}{4}!\right)^{-4}\pmod p.$$

By the above,

$$\sum_{k=0}^{p-1}\frac{C_k^2}{16^k}\equiv 1-\frac{1}{(n+1)^2}=1-\frac{4}{(p+1)^2}\equiv-3\pmod p.$$

If $p\equiv 1\pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{C_k^3}{64^k}\equiv\frac{1}{(n+1)^3}-1=\frac{8}{(p+1)^3}-1\equiv 7\pmod p.$$

If $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{C_k^3}{64^k} \equiv \frac{\frac{3}{16}(\frac{p+1}{4}!)^{-4} - 1}{-(n+1)^3} - 1 \equiv 7 - \frac{3}{2}\left(\frac{p+1}{4}!\right)^{-4} \pmod p.$$

This proves (1.12) and (1.13).

With the help of Lemma 2.2, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}C_k}{32^k} = \frac{pC_{p-1}^2}{32^{p-1}} + \sum_{k=0}^{p-2} \frac{\binom{2k}{k}C_k}{32^k} \equiv p + \sum_{k=0}^{n} \binom{n+k}{2k} \frac{C_k}{(-2)^k} \pmod{p^2}.$$

If $p \equiv 1 \pmod 4$, then $n = (p-1)/2$ is even and hence

$$\sum_{k=0}^{n} \binom{n+k}{2k} \frac{C_k}{(-2)^k} = 0$$

by Lemma 4.3.

Now assume that $p \equiv 3 \pmod 4$. In view of Lemma 4.3,

$$\begin{aligned}\sum_{k=0}^{n} \binom{n+k}{2k} \frac{C_k}{(-2)^k} &= (-1)^{(n-1)/2} \frac{C_{(n-1)/2}}{2^n} \\ &= \frac{(-1)^{(p-3)/4}}{2^{(p-1)/2}((p-3)/4+1)} \binom{(p-3)/2}{(p-3)/4} \\ &\equiv \frac{4(p-1)}{1+(\frac{2}{p})(2^{(p-1)/2} - (\frac{2}{p}))} \binom{(p-3)/2}{(p-3)/4} \pmod{p^2}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{4(p-1)}{1+(\frac{2}{p})(2^{(p-1)/2} - (\frac{2}{p}))} &\equiv 4(p-1)\left(1 - \left(\frac{2}{p}\right)\left(2^{(p-1)/2} - \left(\frac{2}{p}\right)\right)\right) \\ &\equiv (4p-4)\left(1 - \frac{2^{p-1}-1}{2}\right) \equiv 4p - 4 + 2\big(2^{p-1}-1\big) \pmod{p^2}.\end{aligned}$$

By the above, the congruence (1.14) also holds. We are done. □

5. Some open conjectures

In this section we pose some conjectures for further research.

Motivated by the identities $\sum_{k=0}^{\infty}\binom{2k}{k}/((2k+1)16^k) = \pi/3$,

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} = \frac{\pi^2}{10} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} = \frac{7\pi^3}{216},$$

we formulate the following conjecture based on our computation via `Mathematica`.

Conjecture 5.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv (-1)^{(p-1)/2}\left(\frac{H_{p-1}}{12} + \frac{3p^4}{160}B_{p-5}\right) \pmod{p^5}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} \equiv (-1)^{(p-1)/2}\left(\frac{H_{p-1}}{4p^2} + \frac{p^2}{36}B_{p-5}\right) \pmod{p^3}.$$

We also have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv \frac{H_{p-1}}{5p} \pmod{p^3},$$

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\frac{p}{4}B_{p-3} \pmod{p^2}.$$

Remark 5.1. It is known that $H_{p-1} \equiv -p^2 B_{p-3}/3 \pmod{p^3}$ for any prime $p > 3$ (see, e.g., [S]). Thus the first congruence in the conjecture is a refinement of (1.4).

Motivated by the known identities

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2\binom{2k}{k}} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{3^k}{k^2\binom{2k}{k}} = \frac{2}{9}\pi^2$$

(cf. [Ma]), we raise the following related conjecture.

Conjecture 5.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k2^k} \equiv -\frac{H_{(p-1)/2}}{2} + \frac{7}{16}p^2 B_{p-3} \pmod{p^3}.$$

When $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{(-2)^k}{k^2}\binom{2k}{k} \equiv -2q_p(2)^2 \pmod{p},$$

$$p\sum_{k=1}^{p-1} \frac{2^k}{k^2\binom{2k}{k}} \equiv -q_p(2) + \frac{p^2}{16}B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k3^k} \equiv -2 \sum_{\substack{k=1\\ k\not\equiv p \pmod 3}}^{p-1} \frac{1}{k} \pmod{p^3},$$

and

$$p\sum_{k=1}^{p-1}\frac{3^k}{k^2\binom{2k}{k}}\equiv-\frac{3}{2}q_p(3)+\frac{4}{9}p^2B_{p-3}\pmod{p^3}.$$

Now we propose three more conjectures.

Conjecture 5.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Remark 5.2. Let p be an odd prime with $(\frac{p}{7})=1$. As $(\frac{-7}{p})=1$, and the quadratic field $\mathbb{Q}(\sqrt{-7})$ has class number one, p can be written uniquely in the form

$$\frac{a+b\sqrt{-7}}{2}\times\frac{a-b\sqrt{-7}}{2}=\frac{a^2+7b^2}{4}$$

with $a,b\in\mathbb{Z}$ and $a\equiv b \pmod 2$. Obviously a and b must be even (otherwise $a^2+7b^2\equiv 0 \pmod 8$), and $p=x^2+7y^2$ with $x=a/2$ and $y=b/2$.

Conjecture 5.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}$$

Remark 5.3. It is well known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11})=1$ we can write $4p=x^2+11y^2$ with $x,y\in\mathbb{Z}$. Concerning the parameters in the representation $4p=x^2+11y^2$, Jacobi obtained the following result (see, e.g., [BEW] and [HW]): If $p=11f+1$ is a prime and $4p=x^2+11y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 2 \pmod{11}$, then $x\equiv\binom{6f}{3f}\binom{3f}{f}/\binom{4f}{2f} \pmod p$.

Conjecture 5.5. *Let p be any odd prime. If $p\equiv 1 \pmod 4$ and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}.$$

If $p\equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^2}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv 0\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\pmod{p^3}.$$

Remark 5.4. The author could prove all the congruences in Conjecture 5.5 modulo p.

For more conjectures of the author on congruences related to central binomial coefficients, the reader may consult [Su4].

Acknowledgment

The author is grateful to the referee for helpful comments.

References

[BEW] B.C. Berndt, R.J. Evans, K.S. Williams, Gauss and Jacobi Sums, John Wiley & Sons, 1998.
[DPSW] K. Driver, H. Prodinger, C. Schneider, J. Weideman, Padé approximations to the logarithm III: Alternative methods and additional results, Ramanujan J. 12 (2006) 299–314.
[F] G.J. Fox, Congruences relating rational values of Bernoulli and Euler polynomials, Fibonacci Quart. 39 (2001) 50–57.
[G] H.W. Gould, Combinatorial Identities, Morgantown Printing and Binding Co., 1972.
[GKP] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, 2nd ed., Addison–Wesley, New York, 1994.
[GZ] G. Grossman, A. Zeleke, On linear recurrence relations, J. Concr. Appl. Math. 1 (2003) 229–245.
[G-Z] V.J.W. Guo, J. Zeng, Some congruences involving central q-binomial coefficients, Adv. in Appl. Math. 45 (2010) 303–316.
[HW] R.H. Hudson, K.S. Williams, Binomial coefficients and Jacobi sums, Trans. Amer. Math. Soc. 281 (1984) 431–505.
[I] T. Ishikawa, On Beukers' congruence, Kobe J. Math. 6 (1989) 49–52.
[L] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, Ann. of Math. 39 (1938) 350–360.
[Ma] R. Matsumoto, A collection of formulae for π, on-line version is available from the website http://www.pluto.ai.kyutech.ac.jp/plt/matumoto/pi_small.
[Mo] F. Morley, Note on the congruence $2^{4n}\equiv(-1)^n(2n)!/(n!)^2$, where $2n+1$ is a prime, Ann. of Math. 9 (1895) 168–170.
[PS] H. Pan, Z.W. Sun, A combinatorial identity with application to Catalan numbers, Discrete Math. 306 (2006) 1921–1940.
[PWZ] M. Petkovšek, H.S. Wilf, D. Zeilberger, $A=B$, A.K. Peters, Wellesley, 1996.
[Sp] R. Sprugnoli, Sums of reciprocals of the central binomial coefficients, Integers 6 (2006), #A27, 18 pp. (electronic).
[St1] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, Cambridge, 1999.
[St2] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
[SSZ] N. Strauss, J. Shallit, D. Zagier, Some strange 3-adic identities, Amer. Math. Monthly 99 (1992) 66–69.
[S] Z.H. Sun, Congruences concerning Bernoulli numbers and Bernoulli polynomials, Discrete Appl. Math. 105 (2000) 193–223.
[Su1] Z.W. Sun, Binomial coefficients, Catalan numbers and Lucas quotients, Sci. China Math. 53 (2010) 2473–2488.
[Su2] Z.W. Sun, Conjecture on a new series for π^3, A Message to Number Theory List (sent on March 31, 2010). Available from the website http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind1003&L=nmbrthry&T=0&P=1956.
[Su3] Z.W. Sun, p-adic valuations of some sums of multinomial coefficients, Acta Arith. 148 (2011) 63–76.
[Su4] Z.W. Sun, Super congruences and Euler numbers, Sci. China Math., in press, http://arxiv.org/abs/1001.4453.
[Su5] Z.W. Sun, On Delannoy numbers and Schröder numbers, preprint, http://arxiv.org/abs/1009.2486.
[ST1] Z.W. Sun, R. Tauraso, New congruences for central binomial coefficients, Adv. in Appl. Math. 45 (2010) 125–148.
[ST2] Z.W. Sun, R. Tauraso, On some new congruences for binomial coefficients, Int. J. Number Theory 7 (2011) 645–662.
[ZG] X. Zhu, G. Grossman, On zeros of polynomial sequence, J. Comput. Anal. Appl. 11 (2009) 140–158.