Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Algebra 313 (2007) 782–801
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generators, relations and symmetries in pairs of 3 × 3 unimodular matrices

Sean Lawton

Mathematics Department, Kansas State University, Manhattan, KS 66506, USA

Received 27 May 2006

Available online 10 January 2007

Communicated by Corrado de Concini

Abstract

Denote the free group on two letters by $\mathtt{F}_2$ and the $\mathrm{SL}(3,\mathbb{C})$-representation variety of $\mathtt{F}_2$ by $\mathfrak{R} = \mathrm{Hom}(\mathtt{F}_2, \mathrm{SL}(3,\mathbb{C}))$. There is a $\mathrm{SL}(3,\mathbb{C})$-action on the coordinate ring of $\mathfrak{R}$, and the geometric points of the subring of invariants is an affine variety $\mathfrak{X}$. We determine explicit minimal generators and defining relations for the subring of invariants and show $\mathfrak{X}$ is a degree 6 hyper-surface in $\mathbb{C}^9$ mapping onto $\mathbb{C}^8$. Our choice of generators exhibit $\mathrm{Out}(\mathtt{F}_2)$ symmetries which allow for a succinct expression of the defining relations.

Keywords: Character variety; Free group

1. Introduction

The purpose of this paper is to describe a minimal generating set and defining relations for the ring of invariants

$$\mathbb{C}\big[\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})\big]^{\mathrm{SL}(3,\mathbb{C})}.$$

This generating set exhibits symmetries which allow for an explicit and succinct expression of the invariant ring as a quotient.

Explicit minimal generators have been found by [22] and graphically by [19]; in an unpublished calculation [13] independently describe the defining relations. Our treatment provides the

E-mail address: slawton@math.ksu.edu.
URL: http://www.math.ksu.edu/~slawton.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.01.003

most succinct and transparent description by uncovering symmetries which provide a framework for generalization.

A related algebra, however different, is the ring of invariants of pairs of 3×3 matrices $M_3(\mathbb{C}) \times M_3(\mathbb{C})$ under simultaneous conjugation. The algebra of invariants $\mathbb{C}[M_3(\mathbb{C}) \times M_3(\mathbb{C})]^{\mathrm{GL}(3,\mathbb{C})}$ comes to bear on the algebra of invariants $\mathbb{C}[\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})]^{\mathrm{SL}(3,\mathbb{C})}$ by restriction. On the other hand, the former ring of invariants may be described, in part, by $\mathbb{C}[\mathfrak{sl}(3) \times \mathfrak{sl}(3)]^{\mathrm{SL}(3,\mathbb{C})}$; the infinitesimal invariants in the latter ring. In this more general context, similar questions about generators and relations have been addressed. In particular, explicit minimal generators were first found by [5] in 1935, and later by [10,20,21]. The much more general results of [1] additionally provide minimal generators. However, [11], and later [2] were the first to explicitly describe the defining relations. For the state-of-the-art, see [4].

We now describe the main results of this paper. Let $\mathfrak{X}$ be the variety whose coordinate ring is $\mathbb{C}[\mathfrak{X}] = \mathbb{C}[\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C})]^{\mathrm{SL}(3,\mathbb{C})}$. Theorem 8 asserts that $\mathfrak{X}$ is isomorphic to a degree 6 affine hyper-surface in $\mathbb{C}^9$ which generically maps 2-to-1 onto $\mathbb{C}^8$. Next, Theorem 9 explicitly describes the singular locus of $\mathfrak{X}$, and examples of non-singular representations in the branching locus are constructed. Lastly, Theorem 13 describes an 8-fold symmetry on $\mathbb{C}[\mathfrak{X}]$ which at once characterizes the algebraically independent generators and allows for a surprisingly simple description of the defining relations.

We hope that this paper will be of interest to algebraic-geometers, ring theorists, and geometers alike. In particular, results in this paper have recently been used in work concerning the hyperbolic geometry of spherical CR manifolds (see [17]). With this in mind, some of the exposition, for instance, may be "well-known" to a ring theorist but perhaps not to an algebraic-geometer or a geometer. The reader is encouraged to skip such exposition, as appropriate.

2. SL(3, ℂ) invariants

2.1. *Algebraic structure of* SL(3, ℂ)

The group $\mathrm{SL}(3,\mathbb{C})$ has the structure of an algebraic set since it is the zero set of the polynomial

$$D = \det\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} - 1$$

on $\mathbb{C}^9$. Here $x_{ij} \in \mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}]$, the polynomial ring over $\mathbb{C}$ in 9 indeterminates. As such denote $\mathrm{SL}(3,\mathbb{C})$ by $\mathfrak{G}$. The coordinate ring of $\mathfrak{G}$ is given by

$$\mathbb{C}[\mathfrak{G}] = \mathbb{C}[x_{ij} \mid 1 \leqslant i, j \leqslant 3]/(D).$$

Since D is irreducible, (D) is a prime ideal. So the algebraic set $\mathfrak{G}$ is in fact an affine variety.

2.2. *Representation and character varieties of a free group*

Let $\mathtt{F}_r$ be the free group of rank r generated by $\{\mathtt{x}_1, \ldots, \mathtt{x}_r\}$. The map

$$\mathrm{Hom}(\mathtt{F}_r, \mathfrak{G}) \to \mathfrak{G}^{\times r}$$

defined by sending

$$\rho \mapsto \big(\rho(\mathtt{x}_1), \rho(\mathtt{x}_2), \ldots, \rho(\mathtt{x}_r)\big)$$

is a bijection. Since $\mathfrak{G}^{\times r}$ is the r-fold product of irreducible algebraic sets, $\mathfrak{G}^{\times r} \cong \mathrm{Hom}(\mathtt{F}_r, \mathfrak{G})$ is an affine variety.

As such $\mathrm{Hom}(\mathtt{F}_r, \mathfrak{G})$ is denoted by $\mathfrak{R}$ and referred to as the $\mathrm{SL}(3,\mathbb{C})$-*representation variety of* $\mathtt{F}_r$.

Let $\mathbb{C}[\mathfrak{R}]$ be the coordinate ring of $\mathfrak{R}$. Our preceding remarks imply $\mathbb{C}[\mathfrak{R}] \cong \mathbb{C}[\mathfrak{G}]^{\otimes r}$. For $1 \leqslant k \leqslant r$, define a *generic matrix* of the complex polynomial ring in $9r$ indeterminates by

$$\mathbf{x}_k = \begin{pmatrix} x^k_{11} & x^k_{12} & x^k_{13} \\ x^k_{21} & x^k_{22} & x^k_{23} \\ x^k_{31} & x^k_{32} & x^k_{33} \end{pmatrix}.$$

Let Δ be the ideal $(\det(\mathbf{x}_k) - 1 \mid 1 \leqslant k \leqslant r)$ in $\mathbb{C}[\mathfrak{R}]$. Then

$$\mathbb{C}[\mathfrak{R}] = \mathbb{C}\big[x^k_{ij} \bigm| 1 \leqslant i, j \leqslant 3,\ 1 \leqslant k \leqslant r\big]/\Delta.$$

Let $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r)$ be an r-tuple of generic matrices. An element $f \in \mathbb{C}[\mathfrak{R}]$ is a function defined in terms of such r-tuples. There is a polynomial $\mathfrak{G}$-action on $\mathbb{C}[\mathfrak{R}]$ given by diagonal conjugation; that is, for $g \in \mathfrak{G}$

$$g \cdot f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = f\big(g^{-1}\mathbf{x}_1 g, \ldots, g^{-1}\mathbf{x}_r g\big).$$

The subring of invariants of this action $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is a finitely generated $\mathbb{C}$-algebra (see [3,14,15]). Consequently, the *character variety*

$$\mathfrak{X} = \mathrm{Spec}_{\max}\big(\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}\big)$$

is the irreducible algebraic set whose coordinate ring is the ring of invariants. For $r > 1$, the Krull dimension of $\mathfrak{X}$ is $8r - 8$ since generic elements have zero-dimensional isotropy (see [3, p. 98]). More generally, the dimension of the ring of invariants $\mathbb{C}[M_n(\mathbb{C})^{\times r}]^{\mathrm{GL}(n,\mathbb{C})}$ is $n^2(r-1)+1$ (see [4]). Consequently, the dimension of $\mathbb{C}[\mathfrak{sl}(n)^{\times r}]^{\mathrm{SL}(n,\mathbb{C})}$, which equals that of $\mathbb{C}[\mathfrak{X}]$, is $(n^2-1)(r-1)$.

There is a regular map $\mathfrak{R} \xrightarrow{\pi} \mathfrak{X}$ which factors through $\mathfrak{R}/\mathfrak{G}$: let $\mathfrak{m}$ be a maximal ideal corresponding to a point in $\mathfrak{R}$, then the composite isomorphism $\mathbb{C} \to \mathbb{C}[\mathfrak{R}] \to \mathbb{C}[\mathfrak{R}]/\mathfrak{m}$ implies that the composite map $\mathbb{C} \to \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}} \to \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}/(\mathfrak{m} \cap \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}})$ is an isomorphism as well. Hence the contraction $\mathfrak{m} \cap \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is maximal, and since for any $g \in \mathfrak{G}$, $(g\mathfrak{m}g^{-1}) \cap \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}} = \mathfrak{m} \cap \mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$, π factors through $\mathfrak{R}/\mathfrak{G}$ (see [6, p. 38]). Although $\mathfrak{R}/\mathfrak{G}$ is not generally an algebraic set, $\mathfrak{X}$ is the categorical quotient $\mathfrak{R}/\!\!/\mathfrak{G}$, and since $\mathfrak{G}$ is a (geometrically) reductive algebraic group π is surjective, maps closed $\mathfrak{G}$-invariant sets to closed sets, and separates distinct closed orbits (see [3]).

3. Trace identities for matrices

Let $\mathtt{F}_r^+$ be the free monoid generated by $\{\mathtt{x}_1, \dots, \mathtt{x}_r\}$, and let $\mathbf{M}_r^+$ be the monoid generated by $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$, as defined in Section 2.2, under matrix multiplication and with identity $\mathbb{I}$ the 3×3 identity matrix. There is a surjection $\mathtt{F}_r^+ \to \mathbf{M}_r^+$, defined by mapping $\mathtt{x}_i \mapsto \mathbf{x}_i$. Let $\mathbf{w} \in \mathbf{M}_r^+$ be the image of $\mathtt{w} \in \mathtt{F}_r^+$ under this map. Further, let $|\cdot|$ be the function that takes a reduced word in $\mathtt{F}_r$ to its word length. Then by [14,15], we know $\mathbb{C}[\mathfrak{X}]$ is not only finitely generated, but in fact generated by

$$\left\{\operatorname{tr}(\mathbf{w}) \;\middle|\; \mathtt{w} \in \mathtt{F}_r^+,\ |\mathtt{w}| \leqslant 7\right\}. \tag{1}$$

More generally, the length of the generators is bounded by the class of nilpotency of nil algebras of class n. With respect to matrix algebras, n is the size of the matrices under consideration. The best known upper bound is that of [15] and is n^2; the lower bound is $n(n+1)/2$ and is conjectured to be equality. For $n = 2, 3, 4$ this conjecture, known as Kuzmin's conjecture, has been verified (see [4]). In the proof of the Nagata–Higman theorem (see [4]), the bound is computed to be $2^n - 1$, which is how $|\mathtt{w}| \leqslant 7$ in (1) arises.

Let $\mathbf{x}_k^*$ be the transpose of the matrix of cofactors of $\mathbf{x}_k$. In other words, the (i, j)th entry of $\mathbf{x}_k^*$ is

$$(-1)^{i+j}\operatorname{Cof}_{ji}(\mathbf{x}_k);$$

that is, the determinant obtained by removing the jth row and ith column of $\mathbf{x}_k$. Let $\mathbf{M}_r^*$ be the monoid generated by $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ and $\{\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_r^*\}$. Observe that

$$(\mathbf{xy})^* = \mathbf{y}^*\mathbf{x}^*$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{M}_r^+$, and

$$\mathbf{xx}^* = \det(\mathbf{x})\mathbb{I}.$$

Now let $\mathbf{N}_r$ be the normal sub-monoid generated by

$$\left\{\det(\mathbf{x}_k)\mathbb{I} \;\middle|\; 1 \leqslant k \leqslant r\right\},$$

and subsequently define $\mathbf{M}_r = \mathbf{M}_r^*/\mathbf{N}_r$. Notice in $\mathbf{M}_r$, $\mathbf{x}^* = \mathbf{x}^{-1}$, and thus $\mathbf{M}_r$ is a group.

We will need the structure of an algebra, and to that end let $\mathbb{C}\mathbf{M}_r$ be the group algebra defined over $\mathbb{C}$ with respect to matrix addition and scalar multiplication in $\mathbf{M}_r$. Likewise, let $\mathbb{C}\mathbf{M}_r^*$ be the semi-group algebra of the monoid $\mathbf{M}_r^*$.

The following commutative diagram relates these objects:

$$\begin{array}{ccccccc}
\mathtt{F}_r^+ & \longrightarrow & \mathtt{F}_r & = & \mathtt{F}_r & & \\
\downarrow & & & & \downarrow & & \\
\mathbf{M}_r^+ & \longrightarrow & \mathbf{M}_r^* & \longrightarrow & \mathbf{M}_r & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{C}\mathbf{M}_r^+ & \longrightarrow & \mathbb{C}\mathbf{M}_r^* & \longrightarrow & \mathbb{C}\mathbf{M}_r & \xrightarrow{\operatorname{tr}} & \mathbb{C}[\mathfrak{X}].
\end{array}$$

3.1. Relations

The Cayley–Hamilton theorem applies to this context and so for any $\mathbf{x} \in \mathbb{C}\mathbf{M}_r$,

$$\mathbf{x}^3 - \operatorname{tr}(\mathbf{x})\mathbf{x}^2 + \operatorname{tr}(\mathbf{x}^*)\mathbf{x} - \det(\mathbf{x})\mathbb{I} = 0. \tag{2}$$

By direct calculation, or by Newton's trace formulas

$$\operatorname{tr}(\mathbf{x}^*) = \frac{1}{2}\left(\operatorname{tr}(\mathbf{x})^2 - \operatorname{tr}\left(\mathbf{x}^2\right)\right). \tag{3}$$

Together (2) and (3) imply

$$\det(\mathbf{x}) = \frac{1}{3}\operatorname{tr}\left(\mathbf{x}^3\right) + \frac{1}{6}\operatorname{tr}(\mathbf{x})^3 - \frac{1}{2}\operatorname{tr}(\mathbf{x})\operatorname{tr}\left(\mathbf{x}^2\right). \tag{4}$$

Remark 1. In general the coefficients of the characteristic polynomial of an $n \times n$ matrix are the elementary symmetric polynomials in the eigenvalues of the matrix. By Newton's formulas these are trace expressions in powers of the matrix. So one may use this and the general method of polarization, which we demonstrate below, to develop trace identities for larger size matrices.

Computations similar to those that follow may be found in [10,20]; the process is standard and is generally known as (partial) polarization, or multilinearization. For any $\mathbf{x}, \mathbf{y} \in \mathbb{C}\mathbf{M}_r$ and any $\lambda \in \mathbb{C}$, Eq. (2) implies

$$(\mathbf{x} + \lambda\mathbf{y})^3 - \operatorname{tr}(\mathbf{x} + \lambda\mathbf{y})(\mathbf{x} + \lambda\mathbf{y})^2 + \operatorname{tr}\left((\mathbf{x} + \lambda\mathbf{y})^*\right)(\mathbf{x} + \lambda\mathbf{y}) - \det(\mathbf{x} + \lambda\mathbf{y})\mathbb{I} = 0. \tag{5}$$

Using Eqs. (2), (3), and (4), Eq. (5) simplifies to

$$\begin{aligned} 0 = \lambda^2\Big(&\mathbf{xy}^2 + \mathbf{y}^2\mathbf{x} + \mathbf{yxy} - \operatorname{tr}(\mathbf{x})\mathbf{y}^2 - \operatorname{tr}(\mathbf{y})\mathbf{xy} - \operatorname{tr}(\mathbf{y})\mathbf{yx} + \frac{1}{2}\operatorname{tr}(\mathbf{y})^2\mathbf{x} - \frac{1}{2}\operatorname{tr}\left(\mathbf{y}^2\right)\mathbf{x} \\ &+ \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{y})\mathbf{y} - \operatorname{tr}(\mathbf{xy})\mathbf{y} - \operatorname{tr}\left(\mathbf{xy}^2\right)\mathbb{I} - \frac{1}{2}\operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{y})^2\mathbb{I} + \frac{1}{2}\operatorname{tr}(\mathbf{x})\operatorname{tr}\left(\mathbf{y}^2\right)\mathbb{I} + \operatorname{tr}(\mathbf{y})\operatorname{tr}(\mathbf{xy})\mathbb{I}\Big) \\ + \lambda\Big(&\mathbf{yx}^2 + \mathbf{x}^2\mathbf{y} + \mathbf{xyx} - \operatorname{tr}(\mathbf{y})\mathbf{x}^2 - \operatorname{tr}(\mathbf{x})\mathbf{yx} - \operatorname{tr}(\mathbf{x})\mathbf{xy} + \frac{1}{2}\operatorname{tr}(\mathbf{x})^2\mathbf{y} - \frac{1}{2}\operatorname{tr}\left(\mathbf{x}^2\right)\mathbf{y} \\ &+ \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{y})\mathbf{x} - \operatorname{tr}(\mathbf{xy})\mathbf{x} - \operatorname{tr}\left(\mathbf{yx}^2\right)\mathbb{I} - \frac{1}{2}\operatorname{tr}(\mathbf{y})\operatorname{tr}(\mathbf{x})^2\mathbb{I} + \frac{1}{2}\operatorname{tr}(\mathbf{y})\operatorname{tr}\left(\mathbf{x}^2\right)\mathbb{I} + \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{xy})\mathbb{I}\Big). \end{aligned}$$

Thus, by Vandermonde arguments (see [16]) we have the partial polarization of (2)

$$\begin{aligned} \mathbf{yx}^2 + \mathbf{x}^2\mathbf{y} + \mathbf{xyx} = {} & \operatorname{tr}(\mathbf{y})\mathbf{x}^2 + \operatorname{tr}(\mathbf{x})\mathbf{yx} + \operatorname{tr}(\mathbf{x})\mathbf{xy} - \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{y})\mathbf{x} + \operatorname{tr}(\mathbf{xy})\mathbf{x} + \operatorname{tr}\left(\mathbf{yx}^2\right)\mathbb{I} \\ & - \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{xy})\mathbb{I} - \frac{1}{2}\left(\operatorname{tr}(\mathbf{x})^2\mathbf{y} - \operatorname{tr}\left(\mathbf{x}^2\right)\mathbf{y} - \operatorname{tr}(\mathbf{y})\operatorname{tr}(\mathbf{x})^2\mathbb{I} + \operatorname{tr}(\mathbf{y})\operatorname{tr}\left(\mathbf{x}^2\right)\mathbb{I}\right). \end{aligned} \tag{6}$$

Define $\mathrm{pol}(\mathbf{x},\mathbf{y})$ to be the right-hand side of Eq. (6); that is,

$$\mathrm{pol}(\mathbf{x},\mathbf{y}) = \mathbf{y}\mathbf{x}^2 + \mathbf{x}^2\mathbf{y} + \mathbf{x}\mathbf{y}\mathbf{x}. \tag{7}$$

Then substituting $\mathbf{x}$ by the sum $\mathbf{x}+\mathbf{z}$ in Eq. (7), yields the full polarization of (2)

$$\mathbf{xzy} + \mathbf{zxy} + \mathbf{yxz} + \mathbf{yzx} + \mathbf{xyz} + \mathbf{zyx} = \mathrm{pol}(\mathbf{x}+\mathbf{z},\mathbf{y}) - \mathrm{pol}(\mathbf{x},\mathbf{y}) - \mathrm{pol}(\mathbf{z},\mathbf{y}). \tag{8}$$

If $\mathbf{x},\mathbf{y} \in \mathbf{M}_r$ then multiplying Eq. (2) on the right by $\mathbf{x}^{-1}\mathbf{y}$ yields,

$$\mathbf{x}^2\mathbf{y} - \mathrm{tr}(\mathbf{x})\mathbf{xy} + \mathrm{tr}(\mathbf{x}^{-1})\mathbf{y} - \mathbf{x}^{-1}\mathbf{y} = 0. \tag{9}$$

Suppose $\mathbf{x},\mathbf{y} \in \mathbf{M}_r$. Multiplying Eq. (6) on the left by $\mathbf{y}^{-1}\mathbf{x}^{-1}$ and on the right by $\mathbf{x}^{-1}$, followed by taking the trace, and using Eq. (9) appropriately, provides the commutator trace relation

$$\begin{aligned}\mathrm{tr}(\mathbf{xyx}^{-1}\mathbf{y}^{-1}) = {} & -\mathrm{tr}(\mathbf{yxy}^{-1}\mathbf{x}^{-1}) + \mathrm{tr}(\mathbf{x})\,\mathrm{tr}(\mathbf{x}^{-1})\,\mathrm{tr}(\mathbf{y})\,\mathrm{tr}(\mathbf{y}^{-1}) + \mathrm{tr}(\mathbf{x})\,\mathrm{tr}(\mathbf{x}^{-1}) + \mathrm{tr}(\mathbf{y})\,\mathrm{tr}(\mathbf{y}^{-1}) \\ & + \mathrm{tr}(\mathbf{xy})\,\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}) + \mathrm{tr}(\mathbf{xy}^{-1})\,\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}) - \mathrm{tr}(\mathbf{x}^{-1})\,\mathrm{tr}(\mathbf{y})\,\mathrm{tr}(\mathbf{xy}^{-1}) \\ & - \mathrm{tr}(\mathbf{x})\,\mathrm{tr}(\mathbf{y}^{-1})\,\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}) - \mathrm{tr}(\mathbf{x})\,\mathrm{tr}(\mathbf{y})\,\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}) \\ & - \mathrm{tr}(\mathbf{xy})\,\mathrm{tr}(\mathbf{x}^{-1})\,\mathrm{tr}(\mathbf{y}^{-1}) - 3.\end{aligned} \tag{10}$$

3.2. Generators

From (1), we need only consider words in $\mathtt{F}_r^+$ of length 7 or less. In [20] it is shown that this length may be taken to be 6. We give a similar argument here since the development of the result provides many useful relations, and a constructive algorithm for word reduction that is of computational significance.

The length of a reduced word is defined to be the number of letters, counting multiplicity, in the word. We now define the *weighted length*, denoted by $|\cdot|_w$, to be the number of letters of a reduced word having positive exponent plus twice the number of letters having negative exponent, again counting multiplicity.

For example, in $\mathtt{F}_2$, we have $|\mathtt{x}_1\mathtt{x}_2| = |\mathtt{x}_1\mathtt{x}_2|_w = 2$ but $|\mathtt{x}_1^3\mathtt{x}_2^{-2}| = 3 + 2 = 5$ while $|\mathtt{x}_1^3\mathtt{x}_2^{-2}|_w = 3 + 2\cdot 2 = 7$.

For a polynomial expression e in generic matrices with coefficients in $\mathbb{C}[\mathfrak{X}]$, we define the *degree of* e, denoted by $\|e\|$, to be the largest weighted length of monomial words in the expression of e that is minimal among all such expressions for e. Additionally, we define the *trace degree of* e, denoted by $\|e\|_{\mathrm{tr}}$, to be the maximal degree over all monomial words within a trace coefficient of e.

For example, when $\mathbf{x},\mathbf{y} \in \mathbf{M}_r$, $\|\mathrm{pol}(\mathbf{x},\mathbf{y})\| \leqslant \max\{2\|\mathbf{x}\|, \|\mathbf{x}\| + \|\mathbf{y}\|\}$, while $\|\mathrm{pol}(\mathbf{x},\mathbf{y})\|_{\mathrm{tr}} \leqslant 2\|\mathbf{x}\| + \|\mathbf{y}\|$.

We remark that given two such expressions e_1 and e_2,

$$\|e_1e_2\| \leqslant \|e_1\| + \|e_2\| \quad \text{and} \quad \|e_1e_2\|_{\mathrm{tr}} \leqslant \max\{\|e_1\|_{\mathrm{tr}}, \|e_2\|_{\mathrm{tr}}\}.$$

We are now prepared to characterize the generators of $\mathbb{C}[\mathfrak{X}]$.

Lemma 2. $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ *is generated by* $\operatorname{tr}(\mathbf{w})$ *such that* $\mathtt{w} \in \mathtt{F}_r$ *is cyclicly reduced,* $|\mathtt{w}|_w \leqslant 6$*, and all exponents of letters in* $\mathtt{w}$ *are* ± 1.

Proof. For $n \geqslant 2$, Eqs. (2) and (9) determine equation

$$\operatorname{tr}(\mathbf{u}\mathbf{x}^n\mathbf{v}) = \operatorname{tr}(\mathbf{x})\operatorname{tr}(\mathbf{u}\mathbf{x}^{n-1}\mathbf{v}) - \operatorname{tr}(\mathbf{x}^{-1})\operatorname{tr}(\mathbf{u}\mathbf{x}^{n-2}\mathbf{v}) + \operatorname{tr}(\mathbf{u}\mathbf{x}^{n-3}\mathbf{v}), \tag{11}$$

which recursively reduces $\operatorname{tr}(\mathbf{w})$ to a polynomial in traces of words having no letter with exponent other than ± 1. If however $n \leqslant -2$ then we first apply Eq. (9) and then use (11). Hence it follows that $\mathtt{w}$ may be taken to be cyclically reduced, having all letters with exponent ± 1.

It remains to show that the word length may be taken to be less than or equal to 6.

Substituting $\mathbf{x} \mapsto \mathbf{y}$ and $\mathbf{y} \mapsto \mathbf{x}\mathbf{z}$ in Eq. (7), and multiplying the resulting expression on the left by $\mathbf{x}$ gives

$$\mathbf{x}^2\mathbf{z}\mathbf{y}^2 = -(\mathbf{x}\mathbf{y}^2\mathbf{x})\mathbf{z} - (\mathbf{x}\mathbf{y}\mathbf{x})\mathbf{z}\mathbf{y} + \mathbf{x}\operatorname{pol}(\mathbf{y}, \mathbf{x}). \tag{12}$$

Replacing $\mathbf{y} \mapsto \mathbf{y}^2$ in Eq. (7) produces

$$\mathbf{y}^2\mathbf{x}^2 + \mathbf{x}^2\mathbf{y}^2 + \mathbf{x}\mathbf{y}^2\mathbf{x} = \operatorname{pol}(\mathbf{x}, \mathbf{y}^2),$$

which substituted into Eq. (12) yields equation

$$\mathbf{x}^2\mathbf{z}\mathbf{y}^2 = (\mathbf{y}^2\mathbf{x}^2 + \mathbf{x}^2\mathbf{y}^2 - \operatorname{pol}(\mathbf{x}, \mathbf{y}^2))\mathbf{z} + (\mathbf{y}\mathbf{x}^2 + \mathbf{x}^2\mathbf{y} - \operatorname{pol}(\mathbf{x}, \mathbf{y}))\mathbf{z}\mathbf{y} + \mathbf{x}\operatorname{pol}(\mathbf{y}, \mathbf{x}\mathbf{z}). \tag{13}$$

Now substituting

$$\operatorname{pol}(\mathbf{y}, \mathbf{x}^2\mathbf{z}) = \mathbf{x}^2\mathbf{z}\mathbf{y}^2 + \mathbf{y}^2\mathbf{x}^2\mathbf{z} + \mathbf{y}\mathbf{x}^2\mathbf{z}\mathbf{y},$$

and

$$\mathbf{x}^2\operatorname{pol}(\mathbf{y}, \mathbf{z}) = \mathbf{x}^2\mathbf{z}\mathbf{y}^2 + \mathbf{x}^2\mathbf{y}^2\mathbf{z} + \mathbf{x}^2\mathbf{y}\mathbf{z}\mathbf{y}$$

into Eq. (13) results in

$$3\mathbf{x}^2\mathbf{z}\mathbf{y}^2 = \operatorname{pol}(\mathbf{y}, \mathbf{x}^2\mathbf{z}) + \mathbf{x}\operatorname{pol}(\mathbf{y}, \mathbf{x}\mathbf{z}) - \operatorname{pol}(\mathbf{x}, \mathbf{y}^2)\mathbf{z} - \operatorname{pol}(\mathbf{x}, \mathbf{y})\mathbf{z}\mathbf{y} + \mathbf{x}^2\operatorname{pol}(\mathbf{y}, \mathbf{z}). \tag{14}$$

Thus,

$$\|\mathbf{x}^2\mathbf{z}\mathbf{y}^2\| < 2\|\mathbf{x}\| + \|\mathbf{z}\| + 2\|\mathbf{y}\| \quad \text{and} \quad \|\mathbf{x}^2\mathbf{z}\mathbf{y}^2\|_{\mathrm{tr}} \leqslant 2\|\mathbf{x}\| + \|\mathbf{z}\| + 2\|\mathbf{y}\|.$$

Remark 3. In the proof of the Nagata–Higman theorem, the two-sided ideal of polynomial trace relations, for $n = 3$, is shown to be generated as a vector space by $\operatorname{pol}(\mathbf{u}, \mathbf{v})$, $\mathbf{u}^3$, and equation (8) evaluated at monomial words $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Equation (14) shows $\mathbf{x}^2\mathbf{z}\mathbf{y}^2$ is in this ideal, and consequently its degree is less than its word length. However, one can conclude $\mathbf{x}^2\mathbf{z}\mathbf{y}^2$ is in this ideal from more general considerations and avoid the above calculation (see [4, p. 76]).

For the remainder of the argument assume $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are of length 1. Replacing $\mathbf{y} \mapsto \mathbf{u} + \mathbf{v}$ in Eq. (14) we deduce $\|\mathbf{x}^2\mathbf{z}(\mathbf{u}^2 + \mathbf{uv} + \mathbf{vu} + \mathbf{v}^2)\| \leqslant 4$. This in turn implies $\|\mathbf{x}^2\mathbf{z}(\mathbf{uv} + \mathbf{vu})\| \leqslant 4$ and so $\|\mathbf{x}^2\mathbf{zw}(\mathbf{uv} + \mathbf{vu})\| \leqslant 5$. In a like manner, we have that both $\|\mathbf{x}^2\mathbf{z}(\mathbf{wuv} + \mathbf{vwu})\| \leqslant 5$ and $\|\mathbf{x}^2\mathbf{z}(\mathbf{wv} + \mathbf{vw})\mathbf{u}\| \leqslant 5$. Hence we conclude that

$$\left\|2\mathbf{x}^2\mathbf{zwuv}\right\| = \left\|\mathbf{x}^2\mathbf{zw}(\mathbf{uv} + \mathbf{vu}) + \mathbf{x}^2\mathbf{z}(\mathbf{wuv} + \mathbf{vwu}) - \mathbf{x}^2\mathbf{z}(\mathbf{wv} + \mathbf{vw})\mathbf{u}\right\| \leqslant 5,$$

and

$$\left\|2\mathbf{x}^2\mathbf{zwuv}\right\|_{\mathrm{tr}} \leqslant 6.$$

Replacing $\mathbf{x} \mapsto \mathbf{x} + \mathbf{y}$ in $\mathbf{x}^2\mathbf{zwuv}$ we come to the conclusion that $\|\mathbf{xyzwuv} + \mathbf{yxzwuv}\| \leqslant 5$. In other words, permuting $\mathbf{x}$ and $\mathbf{y}$ introduces a factor of -1 and a polynomial term of lesser degree. Slight variation in our analysis concludes the same result for any transposition of two letters in the word $\mathbf{xyzwuv}$.

Therefore, if σ is a permutation of the letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ then

$$\left\|\mathbf{xyzuvw} + \operatorname{sgn}(\sigma)\sigma(\mathbf{xyzuvw})\right\| \leqslant 5 \quad \text{while } \left\|\mathbf{xyzuvw} + \operatorname{sgn}(\sigma)\sigma(\mathbf{xyzuvw})\right\|_{\mathrm{tr}} \leqslant 6.$$

Lastly, making the substitutions $\mathbf{x} \mapsto \mathbf{xy}$, $\mathbf{y} \mapsto \mathbf{zu}$, and $\mathbf{z} \mapsto \mathbf{vw}$ in the fundamental expression (8), we derive

$$\begin{aligned}\mathbf{xyvwzu} + \mathbf{vwxyzu} + \mathbf{zuxyvw} + \mathbf{zuvwxy} + \mathbf{xyzuvw} + \mathbf{vwzuxy}\\ = \mathrm{pol}(\mathbf{xy} + \mathbf{vw}, \mathbf{zu}) - \mathrm{pol}(\mathbf{xy}, \mathbf{zu}) - \mathrm{pol}(\mathbf{vw}, \mathbf{zu}).\end{aligned} \tag{15}$$

However, each word on the left-hand side of Eq. (15) is an even permutation of the first, so

$$\|6\mathbf{xyvwzu}\| \leqslant 5 \quad \text{and} \quad \|6\mathbf{xyvwzu}\|_{\mathrm{tr}} \leqslant 6.$$

Hence, if $\mathbf{w}$ is a word of length 7 or more, then $\|\operatorname{tr}(\mathbf{w})\|_{\mathrm{tr}} \leqslant 6$. Moreover, this process gives an iterative algorithm for reducing such an expression. □

As an immediate consequence we have the following description of sufficient generators of $\mathbb{C}[\mathfrak{X}]$.

Corollary 4. $\mathbb{C}[\mathfrak{X}]$ *is generated by traces of the form*

$$\begin{gathered}\operatorname{tr}(\mathbf{x}_i), \operatorname{tr}(\mathbf{x}_i^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j^{-1}), \operatorname{tr}(\mathbf{x}_i^{-1}\mathbf{x}_j^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k^{-1}),\\ \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l\mathbf{x}_m), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_j^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j^{-1}\mathbf{x}_k^{-1}),\\ \operatorname{tr}(\mathbf{x}_i^{-1}\mathbf{x}_j^{-1}\mathbf{x}_k^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k^{-1}\mathbf{x}_l^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k^{-1}\mathbf{x}_j^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_i^{-1}\mathbf{x}_j^{-1}),\\ \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l\mathbf{x}_m^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l\mathbf{x}_k^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l\mathbf{x}_j^{-1}), \operatorname{tr}(\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k\mathbf{x}_l\mathbf{x}_m\mathbf{x}_n),\end{gathered}$$

where $1 \leqslant i \neq j \neq k \neq l \neq m \neq n \leqslant r$.

Proof. First, consider generators of type $\mathrm{tr}(\mathbf{u}^{-1}\mathbf{w}\mathbf{x}^{-1}\mathbf{z})$. It can be shown that $\mathrm{tr}(\mathbf{uvwxyz}) + \mathrm{tr}(\mathbf{uvwyxz}) + \mathrm{tr}(\mathbf{vuwxyz}) + \mathrm{tr}(\mathbf{vuwyxz})$ has trace degree 5. Setting $\mathbf{u} = \mathbf{v}$ and $\mathbf{x} = \mathbf{y}$ and subsequently interchanging words with squares to those with inverses, we find generators of the form $\mathrm{tr}(\mathbf{u}^{-1}\mathbf{w}\mathbf{x}^{-1}\mathbf{z})$ can be freely eliminated; that is, inverses can be assumed to be adjacent. It remains to show that letters may be taken to be distinct. Equation (7) implies that for any letter $\mathbf{x}$ and any monomial words $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$,

$$\mathrm{tr}\big(\mathbf{w}_1\mathbf{x}^{\pm 1}\mathbf{w}_2\mathbf{x}^{\pm 1}\mathbf{w}_3\big) = -\mathrm{tr}\big(\mathbf{w}_1\mathbf{x}^{\pm 2}\mathbf{w}_2\mathbf{w}_3\big) - \mathrm{tr}\big(\mathbf{w}_1\mathbf{w}_2\mathbf{x}^{\pm 2}\mathbf{w}_3\big) + \mathrm{tr}\big(\mathbf{w}_1\mathrm{pol}\big(\mathbf{x}^{\pm 1}, \mathbf{w}_2\big)\mathbf{w}_3\big).$$

However, by subsequently reducing the words having letters with exponent not ± 1, we conclude that expressions of the form $\mathrm{tr}(\mathbf{w}_1\mathbf{x}^{\pm 1}\mathbf{w}_2\mathbf{x}^{\pm 1}\mathbf{w}_3)$ are unnecessary. □

This result can be refined using the work of [1], where explicit *minimal* generators are formulated in a more general context. In an upcoming paper, we will address the issue of minimality for our generators, as well as provide a maximal subset that is algebraically independent. This subset will allow for a generalization of the symmetry described in Section 5.

4. Structure of $\mathbb{C}[\mathfrak{G} \times \mathfrak{G}]^{\mathfrak{G}}$

4.1. Minimal generators

As a consequence of Corollary 4, we have

Lemma 5. $\mathbb{C}[\mathfrak{G} \times \mathfrak{G}]^{\mathfrak{G}}$ *is generated by*

$$\begin{aligned}&\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2), \mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2^{-1}\big), \mathrm{tr}\big(\mathbf{x}_1^{-1}\big),\\ &\mathrm{tr}\big(\mathbf{x}_2^{-1}\big), \mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big), \mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2\big), \mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big).\end{aligned}$$

Proof. The words of weighted length 1, 2, 3, 4 with exponents ± 1 are unambiguously cyclically equivalent to one of

$$\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_1\mathtt{x}_2, \mathtt{x}_1\mathtt{x}_2^{-1}, \mathtt{x}_2\mathtt{x}_1^{-1}, \mathtt{x}_1^{-1}\mathtt{x}_2^{-1}, (\mathtt{x}_1\mathtt{x}_2)^2.$$

But Eq. (9) reduces the latter most of these in terms of the others. All words in two letters of length 5 are cyclically equivalent to a word with an exponent whose magnitude is greater than 1, except $\mathtt{x}_1\mathtt{x}_2^{-1}\mathtt{x}_1\mathtt{x}_2$, and $\mathtt{x}_2\mathtt{x}_1^{-1}\mathtt{x}_2\mathtt{x}_1$. Both are cyclically equivalent to $(\mathtt{x}_i\mathtt{x}_j)^2\mathtt{x}_j^{-2}$ which in turn, by Eq. (11) reduces to expressions in the other variables. The only words of weighted length 6 and with exponents only ± 1 are $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_1^{-1}\mathtt{x}_2^{-1}$, its inverse, and $(\mathtt{x}_1\mathtt{x}_2)^3$. But the latter most of these is reduced by Eq. (2). Lastly, letting $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{y} = \mathbf{x}_2$ in Eq. (10), we have

$$\begin{aligned}\mathrm{tr}\big(\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{x}_1^{-1}\big) = &-\mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big) + \mathrm{tr}(\mathbf{x}_1)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\big)\,\mathrm{tr}(\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_2^{-1}\big)\\ &+ \mathrm{tr}(\mathbf{x}_1)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\big) + \mathrm{tr}(\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_2^{-1}\big) + \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big)\\ &+ \mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2^{-1}\big)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2\big) - \mathrm{tr}\big(\mathbf{x}_1^{-1}\big)\,\mathrm{tr}(\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2^{-1}\big)\\ &- \mathrm{tr}(\mathbf{x}_1)\,\mathrm{tr}\big(\mathbf{x}_2^{-1}\big)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2\big) - \mathrm{tr}(\mathbf{x}_1)\,\mathrm{tr}(\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big)\\ &- \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)\,\mathrm{tr}\big(\mathbf{x}_1^{-1}\big)\,\mathrm{tr}\big(\mathbf{x}_2^{-1}\big) - 3,\end{aligned} \tag{16}$$

which expresses the trace of the inverse of the commutator in terms of the other expressions. □

4.2. $\mathbb{Z}_3^{\times 2}$-grading

The center of $\mathfrak{G}$ is $\zeta(\mathfrak{G}) = \{\omega\mathbb{I} \mid \omega^3 = 1\} \cong \mathbb{Z}_3$. There is an action of $\zeta(\mathfrak{G})^{\times 2}$ on $\mathbb{C}[\mathfrak{X}]$ given by

$$(\omega_1\mathbb{I}, \omega_2\mathbb{I}) \cdot \mathrm{tr}\big(\mathbf{w}(\mathbf{x}_1, \mathbf{x}_2)\big) = \mathrm{tr}\big(\mathbf{w}(\omega_1\mathbf{x}_1, \omega_2\mathbf{x}_2)\big) = \omega_1^{|\mathbf{w}(\mathbf{x}_1,\mathbb{I})|_w}\omega_2^{|\mathbf{w}(\mathbb{I},\mathbf{x}_2)|_w}\,\mathrm{tr}\big(\mathbf{w}(\mathbf{x}_1, \mathbf{x}_2)\big).$$

Applying this action to the generators and recording the orbit by a 9-tuple, all generators are distinguished. Consequently, we have

Proposition 6.

$$\mathbb{C}[\mathfrak{X}] = \sum_{(\omega_1,\omega_2)\in\mathbb{Z}_3\times\mathbb{Z}_3} \mathbb{C}[\mathfrak{X}]_{(\omega_1,\omega_2)}$$

is a $\mathbb{Z}_3 \times \mathbb{Z}_3$-graded ring. The summand $\mathbb{C}[\mathfrak{X}]_{(\omega_1,\omega_2)}$ is the linear span over $\mathbb{C}$ of all monomials whose orbit under $\mathbb{Z}_3 \times \mathbb{Z}_3$ equals one of the orbits of the nine orbit types corresponding to the minimal generators.

In fact the situation is general. For a rank r free group, $\mathbb{Z}_3^{\times r}$ acts on the generators of $\mathbb{C}[\mathfrak{X}]$ and gives a filtration. However, since the relations are polarizations of the Cayley–Hamilton polynomial, which itself has a zero grading, no relation can compromise summands. So the filtration is a grading.

4.3. Hypersurface in $\mathbb{C}^9$

Let

$$\overline{R} = \mathbb{C}[t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}, t_{(4)}, t_{(-4)}, t_{(5)}, t_{(-5)}]$$

be the complex polynomial ring freely generated by $\{t_{(\pm i)},\ 1 \leqslant i \leqslant 5\}$, and let

$$R = \mathbb{C}[t_{(1)}, t_{(-1)}, t_{(2)}, t_{(-2)}, t_{(3)}, t_{(-3)}, t_{(4)}, t_{(-4)}]$$

be its subring generated by $\{t_{(\pm i)},\ 1 \leqslant i \leqslant 4\}$, so $\overline{R} = R[t_{(5)}, t_{(-5)}]$. Define the following ring homomorphism,

$$R[t_{(5)}, t_{(-5)}] \xrightarrow{\Pi} \mathbb{C}[\mathfrak{G} \times \mathfrak{G}]^{\mathfrak{G}}$$

by

$$\begin{aligned}
t_{(1)} &\mapsto \mathrm{tr}(\mathbf{x}_1), & t_{(-1)} &\mapsto \mathrm{tr}\big(\mathbf{x}_1^{-1}\big),\\
t_{(2)} &\mapsto \mathrm{tr}(\mathbf{x}_2), & t_{(-2)} &\mapsto \mathrm{tr}\big(\mathbf{x}_2^{-1}\big),\\
t_{(3)} &\mapsto \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2), & t_{(-3)} &\mapsto \mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big),\\
t_{(4)} &\mapsto \mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2^{-1}\big), & t_{(-4)} &\mapsto \mathrm{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2\big),\\
t_{(5)} &\mapsto \mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big), & t_{(-5)} &\mapsto \mathrm{tr}\big(\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{x}_1^{-1}\big).
\end{aligned}$$

It follows from Lemma 5 that

$$\mathbb{C}[\mathfrak{X}] \cong R[t_{(5)}, t_{(-5)}]/\ker(\Pi).$$

In other words, Π is a surjective algebra morphism.

We define

$$\begin{aligned} P = {} & t_{(1)}t_{(-1)}t_{(2)}t_{(-2)} - t_{(1)}t_{(2)}t_{(-3)} - t_{(-1)}t_{(-2)}t_{(3)} - t_{(1)}t_{(-2)}t_{(-4)} - t_{(-1)}t_{(2)}t_{(4)} \\ & + t_{(1)}t_{(-1)} + t_{(2)}t_{(-2)} + t_{(3)}t_{(-3)} + t_{(4)}t_{(-4)} - 3, \end{aligned}$$

and so $P \in R$. Moreover, by Eq. (16),

$$P - (t_{(5)} + t_{(-5)}) \in \ker(\Pi).$$

Hence it follows that the composite map

$$R[t_{(5)}] \hookrightarrow R[t_{(5)}, t_{(-5)}] \twoheadrightarrow R[t_{(5)}, t_{(-5)}]/\ker(\Pi),$$

is an epimorphism. Let I be the kernel of this composite map, and suppose there exists $Q \in R$ so $Q - t_{(5)}t_{(-5)} \in \ker(\Pi)$ as well.

Then under this assumption, we prove

Lemma 7. *I is principally generated by the polynomial*

$$t_{(5)}^2 - P t_{(5)} + Q. \tag{17}$$

Proof. The following argument is an adaptation of one found in [11].

Certainly, $t_{(5)}^2 - P t_{(5)} + Q \in I$ for it maps into $R[t_{(5)}, t_{(-5)}]/\ker(\Pi)$ to the coset representative $t_{(5)}^2 - (t_{(5)} + t_{(-5)})t_{(5)} + t_{(5)}t_{(-5)} = 0$.

On the other hand, observe

$$R[t_{(5)}]/I \cong R[t_{(5)}, t_{(-5)}]/\ker(\Pi) \cong \mathbb{C}[\mathfrak{X}],$$

the dimension of $\mathfrak{X}$ is 8, and $R[t_{(5)}]/I$ has at most 9 generators. Then it must be the case that I is principally generated since $R[t_{(5)}]$ is a U.F.D., and thus a co-dimension 1 irreducible subvariety of $\mathbb{C}^9$ must be given by one equation (see [18, p. 69]). Moreover, I is non-zero since otherwise the resulting dimension would necessarily be too large.

Seeking a contradiction, suppose there exists a polynomial identity comprised of only elements of R. Then Krull's dimension theorem (see [18, p. 68]) implies $t_{(5)}$ is free. In other words, given any restriction of the generators of R, $t_{(5)}$ is not determined. Consider $\mathrm{SL}_3(\mathrm{SL}(2,\mathbb{C}))^{\times 2} \subset \mathfrak{G}^{\times 2}$; that is, matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $ad - bc = 1$. Then by restricting to pairs of such matrices, we deduce that

$$\operatorname{tr}\bigl(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\bigr) = \operatorname{tr}\bigl(\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{x}_1^{-1}\bigr),$$

since for all $\mathbf{x} \in \mathrm{SL}(2, \mathbb{C})$, $\operatorname{tr}(\mathbf{x}) = \operatorname{tr}(\mathbf{x}^{-1})$. Then Eq. (16) becomes

$$t_{(5)} = P/2,$$

which is decidedly not free of the generators of R. Thus, the generators of R are algebraically independent in $R[t_{(5)}]/I$.

Since I is principal and contains a monic quadratic over R, its generator is expression (17), or a factor thereof. We have just showed that there are no degree zero relations, with respect to $t_{(5)}$. However, if I is generated by a linear polynomial over R then $t_{(5)}$ is determined by the generators of R alone. However this in turn would imply that all representations who agree by evaluation in R also agree by evaluation under $t_{(5)}$.

Consider the representations

$$\mathrm{F}_2 \xrightarrow{\rho_1} \mathfrak{G}, \qquad\qquad \mathrm{F}_2 \xrightarrow{\rho_2} \mathfrak{G},$$

$$\mathrm{x}_1 \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1/ab \end{pmatrix} \quad \text{and} \quad \mathrm{x}_1 \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1/ab \end{pmatrix},$$

$$\mathrm{x}_2 \mapsto \frac{1}{4^{1/3}} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathrm{x}_2 \mapsto \frac{1}{4^{1/3}} \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

It is a direct calculation to verify that they agree upon evaluation in R but disagree under $t_{(5)}$. □

Lemmas 5 and 7 together imply the following theorem whose result, in part, was given by [22], and later by [19], and may also be inferred by the work of [21].

Theorem 8. *$\mathfrak{G}^{\times 2}/\!\!/\mathfrak{G}$ is isomorphic to a degree 6 affine hyper-surface in $\mathbb{C}^9$, which maps onto $\mathbb{C}^8$.*

Proof. The degree of Q will be apparent when we explicitly write it down. It remains to show that $\mathfrak{X} \to \mathbb{C}^8$ is a surjection. To this end, let $(z_1 - \zeta_1, \ldots, z_8 - \zeta_8)$ be a maximal ideal in the coordinate ring of $\mathbb{C}^8$. Moreover, let ζ_9 be defined to be a solution to $t^2 - P(\zeta_1, \ldots, \zeta_8)t + Q(\zeta_1, \ldots, \zeta_8) = 0$. Then $(t_{(1)} - \zeta_1, t_{(-1)} - \zeta_2, \ldots, t_{(-4)} - \zeta_8, t_{(5)} - \zeta_9) + I$ is a maximal ideal in $\mathbb{C}[\mathfrak{X}]$, and so all maximal ideals of $\mathbb{C}[\mathbb{C}^8]$ are images of such in $\mathbb{C}[\mathfrak{X}]$. □

4.4. Singular locus of $\mathfrak{X}$

The surjection $\mathfrak{X} \to \mathbb{C}^8$ is generically 2-to-1; that is, there are exactly two solutions to

$$t^2 - P(\zeta_1, \ldots, \zeta_8)t + Q(\zeta_1, \ldots, \zeta_8) = 0$$

for every point in $\mathbb{C}^8$ except where $P^2 - 4Q = 0$. In this case,

$$0 = (t_{(5)} + t_{(-5)})^2 - 4t_{(5)}t_{(-5)} = (t_{(5)} - t_{(-5)})^2$$

which implies $t_{(5)} = t_{(-5)} = P/2$. On the other hand, at points in $\mathfrak{X}$ if $t_{(5)} = P/2$, then $P^2 - 4Q = 0$. Let $\mathfrak{L}$ denote the locus of solutions to $P^2 - 4Q = 0$ in $\mathfrak{X}$, which is a closed subset of $\mathfrak{X}$.

It is readily observed that the partial derivative with respect to $t_{(5)}$ of $t_{(5)}^2 - Pt_{(5)} + Q$ is zero if and only if $t_{(5)} = P/2$. The singular set in $\mathfrak{X}$, denoted by $\mathfrak{J}$, is the closed subset cut out by the Jacobian ideal; that is, the ideal generated by the formal partial derivatives of $t_{(5)}^2 - Pt_{(5)} + Q$. Thus $\mathfrak{J} \subset \mathfrak{L}$. If $\mathfrak{H} \hookrightarrow \mathfrak{G}$ is a sub-algebraic group, then we define $\mathfrak{H}^{\times r}/\!\!/\mathfrak{G}$ to be the image of $\mathfrak{H}^{\times r} \hookrightarrow \mathfrak{R}$ to $\mathfrak{X}$. In the proof of Lemma 7, we observed $\mathrm{SL}_3(\mathrm{SL}(2,\mathbb{C}))^{\times 2}/\!\!/\mathfrak{G} \subset \mathfrak{L}$. Additionally, since matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1/ab \end{pmatrix}$$

commute, restricting to pairs of such matrices enforces the relation

$$\mathrm{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big) = 3 = \mathrm{tr}\big(\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{x}_1^{-1}\big).$$

Let $\mathrm{SL}_3(\mathbb{C}^* \times \mathbb{C}^*)$ denote the subset of such matrices in $\mathfrak{G}$. Consequently, $\mathrm{SL}_3(\mathbb{C}^* \times \mathbb{C}^*)^{\times 2}/\!\!/\mathfrak{G} \subset \mathfrak{L}$ as well. We claim both sets satisfy all the generators of the Jacobian ideal, and so are singular. The Jacobian ideal is generated by the polynomials $-t_{(5)}\frac{\partial P}{\partial t_{(i)}} + \frac{\partial Q}{\partial t_{(i)}}$ for $1 \leqslant |i| \leqslant 4$, and $2t_{(5)} - P$. Using the formulas for P and Q (see Section 4.5), we explicitly write out the generators of the Jacobian ideal (see [8] for details). Then evaluating these polynomials at pairs of generic matrices in either $\mathrm{SL}_3(\mathrm{SL}(2,\mathbb{C}))$ or $\mathrm{SL}_3(\mathbb{C}^* \times \mathbb{C}^*)$ we verify that all partials vanish using *Mathematica* [23]. It turns out these examples are prototypical.

Let $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))$ be the subset of $\mathfrak{G}$ consisting of elements of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \frac{1}{ad-bc} \end{pmatrix}.$$

Notice that $\mathrm{SL}_3(\mathbb{C}^* \times \mathbb{C}^*)^{\times 2}/\!\!/\mathfrak{G}$ and $\mathrm{SL}_3(\mathrm{SL}(2,\mathbb{C}))^{\times 2}/\!\!/\mathfrak{G}$ are contained in $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))^{\times 2}/\!\!/\mathfrak{G}$. Again, using *Mathematica* we evaluate all generating polynomials of the Jacobian ideal on pairs of generic matrices in $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))$. Since all partials vanish, $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))^{\times 2}/\!\!/\mathfrak{G}$ is singular in $\mathfrak{X}$ as well.

In general, if $[\rho] \in \mathfrak{G}^{\times r}/\!\!/\mathfrak{G}$ is singular, then its orbit has positive-dimensional isotropy. Any *completely reducible* representation (these parameterize $\mathfrak{G}^{\times r}/\!\!/\mathfrak{G}$ as an orbit space), that is not *irreducible* is conjugate to an element in $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))^{\times r}$. This follows since there must be a shared eigenvector with respect to its generic matrices, if the representation reduces at all. Irreducible representations are known to be non-singular, and their isotropy is zero-dimensional. Consequently, it follows that in general the singular set of $\mathfrak{G}^{\times r}/\!\!/\mathfrak{G}$ is contained in $\mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))^{\times r}/\!\!/\mathfrak{G}$.

In the case of a free group of rank 1, there are no singular points in the quotient and so the identity, which has maximal isotropy, remains non-singular. Hence the converse inclusion does not generally hold. In the case of a free group of rank 2, the situation is much better. In fact, we have already established

Theorem 9. *A completely reducible representation in $\mathfrak{G}^{\times 2}/\!\!/\mathfrak{G}$ is singular if and only if its orbit has positive-dimensional isotropy*; *that is,* $\mathfrak{J} = \mathrm{SL}_3(\mathrm{GL}(2,\mathbb{C}))^{\times 2}/\!\!/\mathfrak{G}$.

As a final note, we give an example of a non-singular representation in the branching locus (actually we give a 2-dimensional family in $\mathfrak{L} - \mathfrak{J}$):

$$\mathtt{F}_2 \stackrel{\rho}{\to} \mathfrak{G},$$

$$\mathtt{x}_1 \mapsto \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a^2 \end{pmatrix}, \qquad \mathtt{x}_2 \mapsto \frac{c^{1/3}}{4^{1/3}} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1/c & -1/c & -1/c \end{pmatrix},$$

so long as $a^3 \neq 1$ and $c \neq 0$. Calculating the Jacobian relations we determine that all partial derivatives are 0 except for

$$-t_{(5)} \frac{\partial P}{\partial t_{(1)}} + \frac{\partial Q}{\partial t_{(1)}} = -\frac{(-1+a^3)^3}{4a^4} \quad \text{and} \quad -t_{(5)} \frac{\partial P}{\partial t_{(-1)}} + \frac{\partial Q}{\partial t_{(-1)}} = \frac{(-1+a^3)^3}{4a^5},$$

which are clearly not always 0.

4.5. Determining Q

For the proofs of Lemma 7 and subsequently Theorem 8 to be complete, it only remains to establish that there exists $Q \in R$ so $Q - t_{(5)}t_{(-5)} \in \ker(\Pi)$.

Before doing so, we state and prove the following technical fact, which may be found in [11].

Fact 10. Define a bilinear form on the vector space of $n \times n$ matrices over $\mathbb{C}$ by

$$\mathbb{B}(A, B) = n\,\mathrm{tr}(AB) - \mathrm{tr}(A)\,\mathrm{tr}(B).$$

Then given vectors $A_1, \ldots, A_{n^2}, B_1, \ldots, B_{n^2}$, the $n^2 \times n^2$ matrix $\mathbb{A} = (\mathbb{B}(A_i, B_j))$ is singular.

Proof. Consider the co-vector

$$v(\,) = \begin{bmatrix} \mathbb{B}(A_1,\ ) \\ \mathbb{B}(A_2,\ ) \\ \vdots \\ \mathbb{B}(A_{n^2},\ ) \end{bmatrix}.$$

If $B_1, \ldots, B_{n^2}$ are linearly dependent then so are $v(B_1), v(B_2), \ldots, v(B_{n^2})$, which implies the columns of $\mathbb{A}$ are linearly dependent. Otherwise there exists coefficients, not all zero, so

$$c_1 B_1 + c_2 B_2 + \cdots + c_{n^2} B_{n^2} = \mathbb{I},$$

which implies

$$c_1 v(B_1) + c_2 v(B_2) + \cdots + c_{n^2} v(B_{n^2}) = 0$$

since the identity $\mathbb{I}$ is in the kernel of $\mathbb{B}(A,\ )$. So again the columns of $\mathbb{A}$ are linearly dependent. Either way, $\mathbb{A}$ is singular. □

Lemma 11. *There exists a polynomial $Q \in R$ so $Q - t_{(5)}t_{(-5)} \in \ker(\Pi)$, and in particular*

$$\begin{aligned}
Q = 9 &- 6t_{(1)}t_{(-1)} - 6t_{(2)}t_{(-2)} - 6t_{(3)}t_{(-3)} - 6t_{(4)}t_{(-4)} + t_{(1)}^3 + t_{(2)}^3 + t_{(3)}^3 \\
&+ t_{(4)}^3 + t_{(-1)}^3 + t_{(-2)}^3 + t_{(-3)}^3 + t_{(-4)}^3 - 3t_{(-4)}t_{(-3)}t_{(-1)} - 3t_{(4)}t_{(3)}t_{(1)} \\
&- 3t_{(-4)}t_{(2)}t_{(3)} - 3t_{(4)}t_{(-2)}t_{(-3)} + 3t_{(-4)}t_{(-2)}t_{(1)} + 3t_{(4)}t_{(2)}t_{(-1)} \\
&+ 3t_{(1)}t_{(2)}t_{(-3)} + 3t_{(-1)}t_{(-2)}t_{(3)} + t_{(-2)}t_{(-1)}t_{(2)}t_{(1)} + t_{(-3)}t_{(-2)}t_{(3)}t_{(2)} \\
&+ t_{(-4)}t_{(-1)}t_{(4)}t_{(1)} + t_{(-4)}t_{(-2)}t_{(4)}t_{(2)} + t_{(-3)}t_{(-1)}t_{(3)}t_{(1)} + t_{(-3)}t_{(-4)}t_{(3)}t_{(4)} \\
&+ t_{(-4)}^2t_{(-3)}t_{(-2)} + t_{(4)}^2t_{(3)}t_{(2)} + t_{(-1)}^2t_{(-2)}t_{(-4)} + t_{(1)}^2t_{(2)}t_{(4)} + t_{(1)}t_{(-2)}^2t_{(-3)} \\
&+ t_{(-1)}t_{(2)}^2t_{(3)} + t_{(-4)}t_{(-3)}t_{(1)}^2 + t_{(4)}t_{(3)}t_{(-1)}^2 + t_{(-4)}t_{(2)}t_{(-3)}^2 + t_{(4)}t_{(-2)}t_{(3)}^2 \\
&+ t_{(-1)}^2t_{(-3)}t_{(2)} + t_{(1)}^2t_{(3)}t_{(-2)} + t_{(-4)}t_{(1)}t_{(2)}^2 + t_{(4)}t_{(-1)}t_{(-2)}^2 + t_{(-4)}t_{(3)}t_{(-2)}^2 \\
&+ t_{(4)}t_{(-3)}t_{(2)}^2 + t_{(1)}t_{(3)}t_{(-4)}^2 + t_{(-1)}t_{(-3)}t_{(4)}^2 + t_{(-1)}t_{(-4)}t_{(3)}^2 \\
&+ t_{(1)}t_{(4)}t_{(-3)}^2 - 2t_{(-3)}^2t_{(-2)}t_{(-1)} - 2t_{(3)}^2t_{(2)}t_{(1)} - 2t_{(-4)}^2t_{(-1)}t_{(2)} \\
&- 2t_{(4)}^2t_{(1)}t_{(-2)} + t_{(-1)}^2t_{(-2)}^2t_{(-3)} + t_{(1)}^2t_{(2)}^2t_{(3)} + t_{(-4)}t_{(-1)}^2t_{(2)}^2 \\
&+ t_{(4)}t_{(1)}^2t_{(-2)}^2 - t_{(-4)}t_{(-2)}^2t_{(2)}t_{(1)} - t_{(4)}t_{(2)}^2t_{(-2)}t_{(-1)} - t_{(-3)}t_{(1)}^2t_{(-1)}t_{(2)} \\
&- t_{(3)}t_{(-1)}^2t_{(1)}t_{(-2)} - t_{(-3)}t_{(2)}^2t_{(-2)}t_{(1)} - t_{(3)}t_{(-2)}^2t_{(2)}t_{(-1)} - t_{(-4)}t_{(-2)}t_{(-1)}t_{(1)}^2 \\
&- t_{(4)}t_{(2)}t_{(1)}t_{(-1)}^2 - t_{(-1)}t_{(-2)}^3t_{(1)} - t_{(-1)}t_{(2)}^3t_{(1)} - t_{(-1)}^3t_{(-2)}t_{(2)} - t_{(1)}^3t_{(-2)}t_{(2)} \\
&- t_{(-4)}t_{(-3)}t_{(-2)}t_{(-1)}t_{(2)} - t_{(4)}t_{(3)}t_{(2)}t_{(1)}t_{(-2)} - t_{(-1)}t_{(1)}t_{(2)}t_{(-4)}t_{(3)} \\
&- t_{(-1)}t_{(1)}t_{(-2)}t_{(4)}t_{(-3)} + t_{(-2)}t_{(-1)}^2t_{(1)}^2t_{(2)} + t_{(-1)}t_{(-2)}^2t_{(2)}^2t_{(1)}. \qquad (18)
\end{aligned}$$

Proof. The following argument is an adaptation of an existence argument given in [11], which we use not only to show existence of Q, but to derive the explicit formulation of Q as well. Indeed, with respect to Fact 10, let

$$\begin{aligned}
&A_1 = B_1 = \mathbf{x}_1, && A_4 = B_4 = \mathbf{x}_2^{-1}, && A_7 = B_7 = \mathbf{x}_1\mathbf{x}_2^{-1}, \\
&A_2 = B_2 = \mathbf{x}_2, && A_5 = B_5 = \mathbf{x}_1\mathbf{x}_2, && A_8 = B_8 = \mathbf{x}_2^{-1}\mathbf{x}_1, \\
&A_3 = B_3 = \mathbf{x}_1^{-1}, && A_6 = B_6 = \mathbf{x}_2\mathbf{x}_1, && A_9 = B_9 = \mathbf{x}_2\mathbf{x}_1^{-1}.
\end{aligned}$$

Then we see that $\mathbb{A}$ has exactly two entries with $\operatorname{tr}(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1})$. After rewriting all matrix entries in terms of our generators of $\mathbb{C}[\mathfrak{X}]$, we have

$$0 = \det(\mathbb{A}) = P_1 \cdot \operatorname{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big)^2 + P_2 \cdot \operatorname{tr}\big(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big) + P_3,$$

where P_1, P_2, P_3 are polynomials in terms of

$$\tilde{R} = \big\{\operatorname{tr}(\mathbf{x}_1), \operatorname{tr}\big(\mathbf{x}_1^{-1}\big), \operatorname{tr}(\mathbf{x}_2), \operatorname{tr}\big(\mathbf{x}_2^{-1}\big), \operatorname{tr}(\mathbf{x}_1\mathbf{x}_2), \operatorname{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}\big), \operatorname{tr}\big(\mathbf{x}_1\mathbf{x}_2^{-1}\big), \operatorname{tr}\big(\mathbf{x}_1^{-1}\mathbf{x}_2\big)\big\}.$$

If $P_1 = 0$ then we have a non-trivial relation among the elements of $\tilde{R}$, which we have already seen cannot exist. Alternatively, one can evaluate the elements of $\tilde{R}$ with the aid of a computer algebra system to verify that $P_1 \neq 0$. Then by direct calculation, using *Mathematica*, we verify that $P_2 = -P \cdot P_1$. Hence it follows that

$$-P_3 = P_1\big(t_{(5)}^2 - Pt_{(5)}\big) = P_1\big(t_{(5)}^2 - (t_{(5)} + t_{(-5)})t_{(5)}\big) = -P_1 t_{(5)} t_{(-5)},$$

and so we have shown the existence of

$$Q = t_{(5)} t_{(-5)}.$$

Lastly, we simplify P_3/P_1, with the aid of *Mathematica*, which turns out to be Eq. (18). □

5. Outer automorphisms

Given any $\alpha \in \mathrm{Aut}(\mathtt{F}_2)$, we define $a_\alpha \in \mathrm{End}(\mathbb{C}[\mathfrak{X}])$ by extending the following mapping

$$a_\alpha\big(\mathrm{tr}(\mathbf{w})\big) = \mathrm{tr}\big(\alpha(\mathbf{w})\big).$$

If $\alpha \in \mathrm{Inn}(\mathtt{F}_2)$, then there exists $\mathtt{u} \in \mathtt{F}_2$ so for all $\mathtt{w} \in \mathtt{F}_2$,

$$\alpha(\mathtt{w}) = \mathtt{uwu}^{-1},$$

which implies

$$a_\alpha\big(\mathrm{tr}(\mathbf{w})\big) = \mathrm{tr}\big(\mathbf{uwu}^{-1}\big) = \mathrm{tr}(\mathbf{w}).$$

Thus $\mathrm{Out}(\mathtt{F}_2)$ acts on $\mathbb{C}[\mathfrak{X}]$. By results of Nielsen (see [9,12]), $\mathrm{Out}(\mathtt{F}_2)$ is generated by the following mappings

$$\tau = \begin{cases} \mathtt{x}_1 \mapsto \mathtt{x}_2, \\ \mathtt{x}_2 \mapsto \mathtt{x}_1, \end{cases} \tag{19}$$

$$\iota = \begin{cases} \mathtt{x}_1 \mapsto \mathtt{x}_1^{-1}, \\ \mathtt{x}_2 \mapsto \mathtt{x}_2, \end{cases} \tag{20}$$

$$\eta = \begin{cases} \mathtt{x}_1 \mapsto \mathtt{x}_1\mathtt{x}_2, \\ \mathtt{x}_2 \mapsto \mathtt{x}_2. \end{cases} \tag{21}$$

Let $\mathfrak{D}$ be the subgroup generated by τ and ι, and let $\mathbb{C}\mathfrak{D}$ be the corresponding group ring. Then $\mathbb{C}[\mathfrak{X}]$ is a $\mathbb{C}\mathfrak{D}$-module.

Lemma 12. *The action of $\mathbb{C}\mathfrak{D}$ preserves R, and $\mathfrak{D}$ fixes P and Q.*

Proof. First we note that it suffices to check

$$\{\iota, \tau\}$$

on

$$\{t_{(\pm i)},\ 1 \leqslant i \leqslant 4\},$$

since the former generates $\mathbb{C}\mathfrak{D}$ and the latter generates R. Secondly we observe that both ι and τ are idempotent.

Indeed, ι maps the generators of R as follows:

$$t_{(1)} \mapsto t_{(-1)} \mapsto t_{(1)},$$
$$t_{(3)} \mapsto t_{(-4)} \mapsto t_{(3)},$$
$$t_{(2)} \mapsto t_{(2)},$$
$$t_{(-2)} \mapsto t_{(-2)},$$
$$t_{(4)} \mapsto t_{(-3)} \mapsto t_{(4)}.$$

Likewise, τ maps the generators of R by:

$$t_{(1)} \mapsto t_{(2)} \mapsto t_{(1)},$$
$$t_{(-1)} \mapsto t_{(-2)} \mapsto t_{(-1)},$$
$$t_{(3)} \mapsto t_{(3)},$$
$$t_{(-3)} \mapsto t_{(-3)},$$
$$t_{(4)} \mapsto t_{(-4)} \mapsto t_{(4)}.$$

Hence both map into R.

For the second part of the lemma, it suffices to observe $\iota(t_{(\pm 5)}) = t_{(\mp 5)} = \tau(t_{(\pm 5)})$, because in $\mathbb{C}[\mathfrak{X}]$,

$$P = t_{(-5)} + t_{(5)} \quad \text{and} \quad Q = t_{(5)} t_{(-5)}. \qquad \square$$

Observing $\iota(t_{(5)}) = \tau(t_{(5)}) = t_{(-5)} = P - t_{(5)}$, it is apparent that $\mathfrak{D}$ does not act as a permutation group on the entire coordinate ring of $\mathfrak{X}$. However, when restricted to R there is

Theorem 13. *$\mathfrak{D}$ restricted to R is group isomorphic to the dihedral group, D_4, of order* 8. *Moreover, the algebraically independent generators are characterized as those which $\mathfrak{D}$ acts on as a permutation group.*

Proof. Let $S = \mathrm{Sym}(\pm 1, \pm 2, \pm 3, \pm 4)$ be the symmetric group of all permutations on the eight letters $\pm i$ for $1 \leqslant i \leqslant 4$. Then we have worked out, in the proof of Lemma 12, that τ acts on the subscripts of $t_{(\pm i)}$ as the permutation

$$(1, 2)(-1, -2)(4, -4)$$

and likewise, ι acts as the permutation

$$(1, -1)(3, -4)(-3, 4).$$

Since $\mathfrak{D}$ is generated by these elements, we certainly have a well defined injection $\mathfrak{D} \to S$. The Cayley table for $\mathfrak{D}$ is:

	id	ι	τ	$\iota\tau$	$\tau\iota$	$\tau\iota\tau$	$\iota\tau\iota$	$\tau\iota\tau\iota$
id	id	ι	τ	$\iota\tau$	$\tau\iota$	$\tau\iota\tau$	$\iota\tau\iota$	$\tau\iota\tau\iota$
ι	ι	id	$\iota\tau$	τ	$\iota\tau\iota$	$\tau\iota\tau\iota$	$\tau\iota$	$\tau\iota\tau$
τ	τ	$\tau\iota$	id	$\tau\iota\tau$	ι	$\iota\tau$	$\tau\iota\tau\iota$	$\iota\tau\iota$
$\iota\tau$	$\iota\tau$	$\iota\tau\iota$	ι	$\tau\iota\tau\iota$	id	τ	$\tau\iota\tau$	$\tau\iota$
$\tau\iota$	$\tau\iota$	τ	$\tau\iota\tau$	id	$\tau\iota\tau\iota$	$\iota\tau\iota$	ι	$\iota\tau$
$\tau\iota\tau$	$\tau\iota\tau$	$\tau\iota\tau\iota$	$\tau\iota$	$\iota\tau\iota$	τ	id	$\iota\tau$	ι
$\iota\tau\iota$	$\iota\tau\iota$	$\iota\tau$	$\tau\iota\tau\iota$	ι	$\tau\iota\tau$	$\tau\iota$	id	τ
$\tau\iota\tau\iota$	$\tau\iota\tau\iota$	$\tau\iota\tau$	$\iota\tau\iota$	$\tau\iota$	$\iota\tau$	ι	τ	id

where

$$\begin{aligned} &id \mapsto (1), && \iota \mapsto (1,-1)(3,-4)(-3,4),\\ &\tau \mapsto (1,2)(-1,-2)(4,-4), && \iota\tau \mapsto (1,2,-1,-2)(3,-4,-3,4),\\ &\tau\iota \mapsto (1,-2,-1,2)(3,4,-3,-4), && \tau\iota\tau \mapsto (2,-2)(3,4)(-3,-4),\\ &\iota\tau\iota \mapsto (1,-2)(2,-1)(3,-3), && \tau\iota\tau\iota \mapsto (1,-1)(2,-2)(3,-3)(4,-4). \end{aligned}$$

It is an elementary exercise in group theory (see [7]) to show any group presentable as

$$\{a, b \mid |a| = n \geqslant 3,\ |b| = 2,\ ba = a^{-1}b\}$$

is isomorphic to the dihedral group D_n of order $2n$. However, letting $a = \tau\iota$ and $b = \iota$ we see $|a| = 4$, $|b| = 2$, $\mathfrak{D}$ is generated by a and b, and

$$ba = \iota\tau\iota = (\tau\iota)^{-1}\iota = a^{-1}b.$$

The last statement in the theorem follows from the fact that $\{t_{(\pm i)} \mid 1 \leqslant i \leqslant 4\}$ are algebraically independent and $\mathfrak{D}$ does not act as a permutation group if $t_{(5)}$ were included. □

Remark 14. The action of $\mathfrak{D}$ on $\mathbb{C}[\mathfrak{X}]$ determines an action on $\mathfrak{X}$. Since $\mathfrak{D}$ acts as a permutation group on R the surjection from Theorem 8, $\mathfrak{X} \to \mathbb{C}^8$, is $\mathfrak{D}$-equivariant. In this way $\mathfrak{X}$ exhibits 8-fold symmetry.

As already noted, the group ring $\mathbb{C}\mathfrak{D}$ acts on $\mathbb{C}[\mathfrak{X}]$. By brute force computation, one can establish the following succinct expressions for the polynomial relations P and Q.

Corollary 15. *In $\mathbb{C}\mathfrak{D}$ define $\mathbb{S}_{\mathfrak{D}}$ to be the group "symmetrizer"*

$$\sum_{\sigma\in\mathfrak{D}} \sigma.$$

Then $P = \mathbb{S}_{\mathfrak{D}}(p) - 3$ and $Q = \mathbb{S}_{\mathfrak{D}}(q) + 9$ where p and q are given by:

$$p = \frac{1}{8}\left(t_{(1)}t_{(-1)}t_{(2)}t_{(-2)} - 4t_{(1)}t_{(-2)}t_{(-4)} + 2t_{(1)}t_{(-1)} + 2t_{(3)}t_{(-3)}\right),$$

$$\begin{aligned} q = \frac{1}{8}\big(&2t_{(-2)}t_{(-1)}^2t_{(1)}^2t_{(2)} + 4t_{(1)}^2t_{(2)}^2t_{(3)} - 4t_{(1)}^3t_{(-2)}t_{(2)} - 8t_{(-4)}t_{(-2)}t_{(-1)}t_{(1)}^2 - 4t_{(4)}t_{(3)}t_{(2)}t_{(1)}t_{(-2)}\\ &+ 8t_{(1)}t_{(3)}t_{(-4)}^2 + 8t_{(-4)}t_{(1)}t_{(2)}^2 - 8t_{(3)}^2t_{(2)}t_{(1)} + 4t_{(4)}t_{(-3)}t_{(2)}^2 + t_{(-2)}t_{(-1)}t_{(2)}t_{(1)} \end{aligned}$$

$$+ t_{(-3)}t_{(-4)}t_{(3)}t_{(4)} + 4t_{(-3)}t_{(-1)}t_{(3)}t_{(1)} + 4t_{(1)}^3 + 4t_{(3)}^3 + 12t_{(-4)}t_{(-2)}t_{(1)}$$
$$- 12t_{(-4)}t_{(2)}t_{(3)} - 12t_{(1)}t_{(-1)} - 12t_{(3)}t_{(-3)}\big).$$

Proof. We work out P only since the computation for Q is established in the same way but longer. Indeed,

$$\begin{aligned}
\mathbb{S}_{\mathfrak{D}}(p) &= \frac{1}{8}\big(\mathbb{S}_{\mathfrak{D}}(t_{(1)}t_{(-1)}t_{(2)}t_{(-2)}) - 4\mathbb{S}_{\mathfrak{D}}(t_{(1)}t_{(-2)}t_{(-4)}) + 2\mathbb{S}_{\mathfrak{D}}(t_{(1)}t_{(-1)}) + 2\mathbb{S}_{\mathfrak{D}}(t_{(3)}t_{(-3)})\big) \\
&= \frac{1}{8}\big(8t_{(1)}t_{(-1)}t_{(2)}t_{(-2)} - 4(2t_{(1)}t_{(2)}t_{(-3)} + 2t_{(-1)}t_{(-2)}t_{(3)} \\
&\quad + 2t_{(1)}t_{(-2)}t_{(-4)} + 2t_{(-1)}t_{(2)}t_{(4)}) + 2(4t_{(1)}t_{(-1)} + 4t_{(2)}t_{(-2)} + 4t_{(3)}t_{(-3)} + 4t_{(4)}t_{(-4)})\big) \\
&= P + 3.
\end{aligned}$$

With the help of *Mathematica* or a tedious hand calculation, the formula for Q is equally verified. □

In [1] an algorithm is deduced that can be adapted to write *minimal* generators for $\mathbb{C}[\mathfrak{X}]$ when $\mathtt{F}_r$ is free of arbitrary rank, which we do is an upcoming paper. It is the hope of the author that exploiting symmetries as above will simplify the calculations involved in describing the ideals for free groups of rank 3 or more. Consequently, this would allow for subsequent advances in determining the defining relations of $\mathfrak{X}$ in general.

Acknowledgments

The author thanks William Goldman for overall guidance, substantial suggestions, and edits that always improved the quality of earlier drafts of this paper. He thanks Richard Schwartz, John Millson, and the University of Maryland's VIGRE-NSF program for generously supporting this work. In particular, he thanks John Millson for suggesting an analysis of the singular locus, which has greatly improved the quality of this paper. The author has benefited from fruitful conversations with Ben Howard, Elisha Peterson, Adam Sikora, and Joseph Previte, and thanks them for their time and insight. He especially thanks Joseph Previte and Eugene Xia for generously sharing their calculations with him. Lastly, the author thanks the reviewer, whose suggestions and corrections have helped make this paper more readable.

References

[1] A. Abeasis, M. Pittaluga, On a minimal set of generators for the invariants of 3 × 3 matrices, Comm. Algebra 17 (2) (1989) 487–499.
[2] H. Aslaksen, V. Drensky, L. Sadikova, Defining relations of invariants of two 3 × 3 matrices, J. Algebra 298 (2006) 41–57.
[3] I. Dolgachev, Lectures on Invariant Theory, London Math. Soc. Lecture Note Ser., vol. 296, Cambridge University Press, 2003.
[4] V. Drensky, E. Formanek, Polynomial Identity Rings, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2004.
[5] J. Dubnov, Sur une généralisation de l'équation de Hamilton–Cayley et sur les invariants simultanés de plusieurs affineurs, in: Proc. Seminar on Vector and Tensor Analysis, Mechanics Research Inst., Moscow State Univ. 2/3, 1935, pp. 351–367.

[6] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Grad. Texts in Math., vol. 150, Springer-Verlag, New York, 1995.
[7] T. Hungerford, Algebra, Grad. Texts in Math., vol. 73, Springer-Verlag, New York, 1974.
[8] S. Lawton, SL(3, $\mathbb{C}$)-character varieties and $\mathbb{RP}^2$-structures on a trinion, PhD thesis, University of Maryland, 2006.
[9] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, Pure Appl. Math., vol. 13, Interscience Publishers, 1966.
[10] A. Marincuk, K. Sibirskii, Minimal polynomial bases of affine invariants of square matrices of order three, Mat. Issled. 6 (1971) 100–113.
[11] K. Nakamoto, The structure of the invariant ring of two matrices of degree 3, J. Pure Appl. Algebra 166 (2002) 125–148.
[12] J. Nielsen, Die Isomorphismengruppe der freien Gruppen, Math. Ann. 91 (1924) 169–209.
[13] J. Previte, E. Xia, Various letters to the author.
[14] C. Procesi, Invariant theory of $n \times n$ matrices, Adv. Math. 19 (1976) 306–381.
[15] Y. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974) 723–756 (in Russian).
[16] L.H. Rowen, Polynomial Identities in Ring Theory, Academic Press, New York, 1980.
[17] R. Schwartz, Spherical CR Geometry and Dehn Surgery, Ann. of Math. Stud., vol. 165, 2007, in press.
[18] I. Shafarevich, Basic Algebraic Geometry 1, second ed., Springer-Verlag, Berlin, 1994.
[19] A. Sikora, SL_n-character varieties as space of graphs, Trans. Amer. Math. Soc. 353 (7) (2001) 2773–2804.
[20] A. Spencer, R. Rivlin, Further results in the theory of matrix polynomials, Arch. Ration. Mech. Anal. 4 (1960) 214–230.
[21] Y. Teranishi, The ring of invariants of matrices, Nagoya Math. J. 104 (1986) 149–161.
[22] Z.X. Wen, Relations polynomiales entre les traces de produits de matrices, C. R. Acad. Sci. Paris 318 (2) (1994) 99–104.
[23] S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, Wolfram Press, 2000.