On a Fixed Point Index Method for the Analysis of the Asymptotic Behavior and Boundary Value Problems of Infinite Dimensional Dynamical Systems and Processes

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1. INTRODUCTION

The Ważewski principle [25] plays an important role in the study of ordinary differential equations. Its applicability is largely due to the fact that in a finite dimensional euclidean space the unit sphere is not a retract of the closed unit ball. Since this is no longer true in infinite dimensional Banach spaces the direct extension of Ważewski’s principle to processes or semidynamical systems of infinite dimensional Banach spaces has a very limited applicability.

Since in finite dimensional spaces the fact that the unit sphere is not a retract of the closed unit ball is equivalent to the fact that every continuous mapping of the unit closed convex ball has a fixed point, the main idea of this work is to develop a method based on fixed point index properties instead of retraction properties.

Our fixed point formulation Corollary 1 is essentially equivalent in finite dimension to the Ważewski theorem. Although in infinite dimension the Ważewski theorem is no longer applicable, Theorems 1 and 2 are applicable since fixed point index methods have proved to be very useful in the solution of problems related to differential equations either in finite or infinite dimensional spaces.

After the Ważewski paper several papers were written applying the Ważewski principle to the asymptotic behaviour of ordinary differential equations, Olech [19], Pliss [22], Mikolajska [16], Onuchic [20], Izé [12], and others. Kaplan, Lasota, and Yorke [13] applied the Ważewski method to the boundary value problem and Conley [4] also applied the Ważewski method to a boundary value problem for diffusion equations in biology. Since our approach uses Ważewski basic ideas in connection with fixed point index theory it should also give good results even in finite dimensions and it can also be applied to boundary value problems in Hilbert spaces.
Let $X$ be a topological space, $R^+ = [0, \infty)$, $A \subseteq X \times R^+$ a subset of $X \times R^+$ such that

$$\{0\} \times X \subseteq A,$$

and let $\pi$ be a mapping from $A$ into $X$. We put

$$I_x = \{t \in R^+ \mid (t, x) \in A\}, \quad w_x = \sup I_x;$$

$$w_x = \infty \text{ if } \sup I_x \text{ does not exist.}$$

**Definition 1.** We say that $(X, R^+, A, \pi)$ is a local semidynamical system if and only if

(a) the map $x \mapsto \omega_x$, $x \in X$, is lower semicontinuous in the sense that for every $x \in X$,

(i) if $\omega_x < \infty$ then for every $\eta > 0$ there exists a neighbourhood $V$ of $x$ such that

$$y \in V \Rightarrow \omega_y > \omega_x - \eta,$$

(ii) if $\omega_x = \infty$ then for every $C \in R^+$ there exists a neighbourhood $V$ of $x$ such that

$$y \in V \Rightarrow \omega_y > C,$$

(b) $\pi$ is continuous,

(c) $\pi(x, 0) = x$ for every $x \in X$,

(d) if $t \in I_x$ and $s \in I_{\pi(x,t)}$ then $s + t \in I_x$,

(e) $\pi(\pi(t, x), s) = \pi(x, s + t)$ for every $t \in I_x$, $s \in I_{\pi(x,t)}$.

Autonomous differential equations on Banach spaces and autonomous functional differential equations are examples of semidynamical systems. Dafermos [5] introduced a generalization of dynamical systems also to include nonautonomous differential equations in Banach spaces or nonautonomous functional differential equations.

**Definition 2.** [5]. Suppose $X$ is a Banach space $R^+ = [0, \infty)$, $u: R \times X \times R^+ \to X$ is a given mapping, and define $U(\sigma, t): X \to X$ for $\sigma \in R, t \in R^+$ by

$$U(\sigma, t) x = u(\sigma, x, t).$$
A process on $X$ is a mapping $u: R \times X \times R^+ \to X$ satisfying the following properties:

(i) $u$ is continuous,
(ii) $U(\sigma, 0) = I$ (identity),
(iii) $U(\sigma + s, t) U(\sigma, s) = U(\sigma, s + t)$.

A process is said to be an autonomous process or a semidynamical system if $U(\sigma, t)$ is independent of $\sigma$, that is, $T(t) = U(0, t), \ t \geq 0$. Then $T(t) x$ is continuous for $(t, x) \in R^+ \times X$.

Let $A \subset R \times X \times R^+$ and $u: A \to X$. We define

$$I_{(x,a)} = \{t > \sigma | (\sigma, x, t) \in A\}$$

$$\omega_{(x,a)} = \sup I_{(x,a)}$$

$\omega(x, a) = \infty$ if $\sup I_{(x,a)}$ does not exist.

Then if the map $(x, a) \to \omega_{(x,a)}$ is continuous in the sense of Definition 1, $u$ defines a local process. A local semidynamical system is an autonomous local process.

If $A$ is a bounded metric space we define the measure of noncompactness of $A$ to be $\inf\{d > 0 | A \text{ can be covered by a finite number of sets of diameter less than or equal to } d\}$. If $X$ is a Banach space and $A$ a bounded subset of $X$, $A$ inherits a metric from $X$ and we can give the same definition of the measure of noncompactness of $A$.

Let $X_1$ and $X_2$ be metric spaces and suppose $f: X_1 \to X_2$ is a continuous map. We say that $f$ is a $k$-set contraction if there exists a map $A \subset X_1$, $f(A)$ is bounded and $\gamma_i(f(A)) < k\gamma_i(A)$. Of course, $\gamma_i$ denotes the measure of noncompactness in $X_i$, $i = 1, 2$. We assume that $0 \leq k < 1$. If $f$ is a $k$-set contraction we define $\gamma(f) = \{k \geq 0 | f \text{ is a } k \text{-set contraction}\}$. We say that $f: X \to X$ is a local strict set contraction if for every $x \in X$ there is a neighbourhood $N(x)$ such that $f/N(x)$ is a $k_x$-set contraction.

Furi and Vignoli [8] and Sadovskii gave a slight generalization of $k$-set contraction. Given a continuous mapping $f: X_1 \to X_2$ we say that $f$ is a condensing map if for every bounded set $A \subset X_1$ such that $\gamma_1(A) \neq 0$, $\gamma_2(f(A)) \leq \gamma_1(A)$. We say that $f$ is a local condensing map if every $x \in X$ has a neighbourhood $N(x)$ such that $f/N(x)$ is a condensing map. If $f$ is a $k$-set contraction $f$ is condensing but the converse is not true, in general (see Nussbaum [18]). If $f$ is linear the two concepts are equivalent.

The Schauder fixed point theorem was generalized to $k$-set contraction by Darbo and to condensing maps by Sadovskii and Furi-Vignoli, that is, if $X$ is a Banach space, $C \subset X$ is a closed bounded convex set, and $f: C \to C$ is an $k$-set contraction then $f$ has a fixed point in $C$.

There are several examples of processes described by functional differential equations and partial differential equations of the evolution type that are compact or $\alpha$-set contractions.
EXAMPLE 1. Let $r > 0$, $C = C([-r, 0], \mathbb{R}^n)$ the space of continuous functions defined in $[-r, 0]$. If $f \in C([-r, 0] + A, \mathbb{R}^n)$, $A > 0$, $\sigma \in \mathbb{R}$ define $x(\theta) = x(t + \theta)$. Let $\Omega \subset \mathbb{R} \times C$, $\Omega$ open, and let $D$, $f: \Omega \to \mathbb{R}^n$ be a continuous function; $D$ is linear and $D(\phi) = \phi(0) - \int_0^t d\mu(t, \theta) \phi(\theta)$, where $\mu$ is a matrix function of bounded variation for $\theta \in [-r, 0]$. A functional differential equation of the neutral type is a relation of the form

$$\frac{d}{dt} D(t, x(t)) = f(t, x(t)), \quad x_0 = \phi. \quad (1.1)$$

We say that $D$ is an uniformly stable operator if there are constants $k > 1$, $\alpha > 0$ such that $|D(t, \phi)| \leq ke^{-\alpha(t-t_0)}$, $t \geq \sigma$. The solution of this equation describes a process $U(s, t) \phi = x(s, t, \phi)$. If $D$ is an uniformly stable operator and $t > r$, $U$ is a weak $a$-set contraction, that is, for every bounded set $A \subset \mathbb{R}^n \times C$, for which $U(A)$ is bounded $\gamma(U(A)) \leq k\gamma(A)$. When $D(t, \phi) = \phi(0)$ Eq. (1.1) is the equation $\dot{x} = f(t, x(t))$ and if $t > r$ the process $U$ is compact [10].

Another general form of a neutral equation for which there is a reasonable existence and continuation of solutions theory [6] is the equation

$$\dot{x} = f(t, x(t), \dot{x}(t)),$$

where $x_0(\theta) = \psi(\theta) \in C([-r, 0], \mathbb{R}^n)$, $x_0(\theta) = \psi(\theta) \in L^p([-r, 0], \mathbb{R}^n)$, and $f$ satisfies a uniform Lipschitz condition with respect to $\psi$ in $L^p([-r, 0], \mathbb{R}^n)$, $1 \leq p \leq \infty$. The process described by these equations is also a $k$-set contraction.

The following example is given in [24].

EXAMPLE 2. Let $X$ be a Banach space and $A: D(A) \to X$ be a closed densely defined linear operator in $X$. $A$ is called sectorial if there are constants $\phi, M, a, 0 < \phi < \pi/2$, $M \geq 1$, $a \in \mathbb{R}$ such that the sector $S_{\phi,a} = \{ \lambda \in \mathbb{C} | \lambda \neq a, \phi < \arg |\lambda - a| \leq \phi \}$ is contained in $\rho(A)$ the resolvent set of $A$, and $\| (\lambda - a)^{-1} \| \leq M/(\lambda - a)$ for all $\lambda \in S_{\phi,a}$. If $A$ is sectorial then there is a $k \geq 0$ such that $\Re \sigma(A + kI) > 0$. Let $A_1 = A + kI$. For $0 < \alpha < 1$ define

$$A_1^{-\alpha} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda - A_1)^{-1} d\lambda,$$

$A_1^{-\alpha}$ is bounded and injective. Let $X^\alpha$ be the range of $A_1^{-\alpha}$, $X^0 = X$, $X' = D(A)$. Let $A_1^\alpha: X^\alpha \to X$ be the inverse of $A_1^{-\alpha}$, $A_0 = I_0$, $A' = A \cdot X^\alpha$ is dense in $X$. Define the norm $\| \cdot \|_\alpha$ on $X^\alpha$ by $\| u \|_\alpha = \| A_1^\alpha u \|$, where $\| \cdot \|$ is the norm for $X \cdot X^\alpha$ does not depend on the choice of $k$, and different choices of $k$ yield equivalent norms on $X^\alpha$. $X^\alpha$ is a Banach space under $\| \cdot \|_\alpha$.
Suppose $0 \leq \alpha < 1$, $V$ is open in $X^\alpha$, and $f: V \to X$ is a locally Lipschitz continuous mapping. Consider the equation

$$\frac{du}{dt} + Au = f(u). \quad (1.2)$$

Let $u_0 \in V$. By a solution of (1.2) on $(0, A)$ through $u_0$, we mean a continuous mapping $u: [0, A) \to V$ such that $u(0) = u_0$, $u$ is differentiable on $(0, A)$, $u(t) \in D(A)$ for $t \in (0, A)$, $t \to f(u(t))$ is locally Hölder continuous, $\int_0^a \| f(u(t)) \| \, dt < \infty$ for some $a > 0$, and (1.2) holds for $t \in (0, A)$. In this definition "$t \to g(t)$ is locally Hölder-continuous" means that for every $t_0$, there exists a neighbourhood $W$ of $t_0$ and $L, \theta > 0$ such that $\| g(t_1) - g(t_2) \| \leq L |t_1 - t_2|^{\theta}$ for $t_1, t_2 \in W$.

It follows from [11] that under the above assumptions, for every $u_0 \in V$ there exists a unique solution $u(u_0)$ of (1.2) through $u_0$, defined on a maximal interval $[0, W_{u_0})$. Defining $U(t) u_0 = u(u_0, t)$ for $t < W_{u_0}$ we obtain a local autonomous process or a local semidynamical system.

The most important example of a sectorial operator arises in the following way: let $\Omega$ be an open bounded set in $\mathbb{R}^n$ whose boundary is of class $C^m$ (an integer). Let $X = L^2(\Omega)$, $D(A) = H^{2m}(\Omega) \cap H^m(\Omega)$, $(Au)(x) = \sum a_\alpha(x) \cdot \nabla^\alpha u(x)$, where the $a_\alpha: \overline{\Omega} \to C^\infty$ are continuous mappings and $D^\alpha u$ is understood in the distributional sense. Suppose that $A$ is uniformly strongly elliptic on $\Omega$, i.e., there is a $C_0 > 0$ such that

$$(-1)^m \Re \left\{ \sum_{|\alpha| = m} a_\alpha(x) \cdot \xi^\alpha \right\} \geq C_0 |\xi|^{2m}$$

for all $\xi = (\xi_\alpha)_{|\alpha| \leq m}$, $\xi_\alpha \in \mathbb{R}$, and all $x \in \Omega$, then Eq. (1.2) is called a semilinear parabolic PDE. Results in [8] imply that $A$ is sectorial and $R(\lambda, A)$ is compact for every $\lambda \in \rho(A)$.

In the following we consider a process defined for all $t > \sigma$ but it becomes quite clear that the results are true for local processes or local semidynamical systems.

**Definition 3.** Suppose $u$ is a process on $X$. The trajectory $\tau^+(\sigma, x)$ through $(\sigma, x) \in R \times X$ is the set in $R \times X$ defined by

$$\tau^+(\sigma, x) = \{(\sigma + t, U(\sigma, t) x) \mid t \in R^+\}.$$ 

The orbit $\gamma^+(\sigma, x)$ through $(\sigma, x)$ is the set in $X$ defined by

$$\gamma^+(\sigma, x) = \{U(\sigma, t) x, t \in R^+\}.$$ 

**Definition 4.** If $u$ is a process on $X$ then an integral of the process on $R$ is a continuous function $y: R \to X$ such that for any $\sigma \in R$,
\[ \tau^+(\sigma, y(\sigma)) = \{(\sigma + t, y(\sigma + t)) | t \geq 0\}. \]

An integral \( y \) is an integral through \((\sigma, x) \in R \times X\) if \( y(\sigma) = x \). We assume in the following that the integral through each \((\sigma, x) \in R \times X\) is unique. We define \( \tau^{-1}(x) = \{(\sigma, y) \in R \times X | \exists t > 0 \text{ such that } U(\sigma, t)y = x\}. \) If \( P_0 = (\sigma, x) \in R \times X \) and \( z \in y^+(\sigma, x) \), we define

\[
\begin{align*}
t_z &= \inf\{t \geq 0 | U(\sigma, t)x = z\}, \\
Q_z &= (\sigma + t_z, U(\sigma, t_z)x),
\end{align*}
\]

\[
\begin{align*}
[P_0, Q_z] &= \{(\sigma + t, U(\sigma, t)x) | 0 \leq t \leq t_z\}, \\
[P_0, Q_z] &= \{(\sigma + t, U(\sigma, t)x) | 0 < t < t_z\}, \\
(P_0, Q_z) &= \{(\sigma + t, U(\sigma, t)x) | 0 < t < t_z\}.
\end{align*}
\]

3. Main Results

Let \( \Omega \) be an open set of \( R \times X \), \( \omega \) an open set of \( \Omega \), \( \omega \subset \Omega \), \( \omega \neq \emptyset \), and \( \partial \omega = \omega \cap (\bar{\Omega} - \omega) \) the boundary of \( \omega \) with respect to \( \Omega \). We put

\[
S^0 = \{P_0 = (\sigma, x) \in \partial \omega | \exists t > 0 \text{ and } z \in y^+(\sigma, x) \text{ with } (P_0, Q_z) \neq \emptyset \text{ and } (P_0, Q_z) \cap \bar{\omega} = \emptyset\},
\]

\[
S = \{Q \in \partial \omega | \exists P_0 = (\sigma, x) \in \omega, \text{ with } Q \in \tau^+(\sigma, x) \text{ and } (P_0, Q) \subset \omega\}
\]

\[
S^* = S^0 \cap S.
\]

The points of \( S \) are called egress points; the points of \( S^* \) are called strict egress points.

Given a point \( P_0 = (\sigma, x) \in \omega \), if the trajectory \( \tau^+(\sigma, x) \) of the process is contained in \( \omega \) for every \( t > 0 \), we say that the trajectory is asymptotic with respect to \( \omega \); if the trajectory is not asymptotic with respect to \( \omega \) then there is a \( t > 0 \) such that \( (\sigma + t, U(\sigma, t)x) \in \partial \omega \). Taking

\[
\begin{align*}
t_{P_0} &= \min\{t > 0 | (\sigma + t, U(\sigma, t)x) \in \partial \omega\}, \\
Q &= (\sigma + t_{P_0}, U(\sigma, t_{P_0})x) = C(P_0),
\end{align*}
\]

we have

\[
(P_0, Q) \subset \omega.
\]
The point \( C(P_o) \) is called the consequent of \( P_o \). Define \( G \) to be the set of all \( P_o = (\sigma, x) \in \omega \) such that there are \( C(P_o) \) and \( C(P_o) \in S^* \).

Consider the mapping
\[
\tilde{U}: S^* \cup G \to S^*, \quad \tilde{U}(P_o) = C(P_o),
\]
if \( P_o \in \omega \) and \( \tilde{U}(P_o) = P_o \), if \( P_o \in S^* \).

The proof of the following is standard, see, for example, [25, 21].

**Lemma 1.** The mapping \( \tilde{U}: S^* \cup G \to S^* \) is continuous.

To prove Theorem 1 we will need to know the basic properties of the fixed point index theory as well as the extensions made by Nussbaum [18] for \( k \)-set contraction and condensing maps. We shall say that a topological space \( X \) is an absolute neighbourhood retract (ANR) if given any metric space \( M \), a closed subspace \( A \subset M \), and a continuous map \( f: A \to X \) there exists an open neighbourhood \( U \) of \( A \) and a continuous map \( F: U \to X \) such that \( F(a) = f(a) \) for \( a \in A \), \( X \) is called an absolute retract (AR) if, as above, can be defined on all of \( M \). A theorem of Dugundji [6] asserts that any convex subset of a locally convex topological space is an AR. Let \( \mathcal{A} \) be the category of compact metric absolute neighbourhood retract (ANR). Let \( A \in \mathcal{A}, G \) be an open subset of \( A \), and \( f: G \to A \) be a continuous function which has no fixed points on \( \partial G \). Then there is a unique integer valued function \( i_A(f, G) \) which satisfies the following four properties [2]:

1. If \( f: G \to A \) has no fixed point on \( \partial G \) and the fixed points of \( f \) lie in \( G_1 \cup G_2 \), where \( G_1 \) and \( G_2 \) are two disjoint open sets included in \( G \), then \( i_A(f, G) = i_A(f, G_1) + i_A(f, G_2) \). In particular, if \( f \) has no fixed points in \( G \), this means that \( i_A(f, G) = 0 \) (the additive property).

2. Let \( I \) denote the closed unit interval \([0, 1]\). If \( F: G \times I \to A \) \((A \in \mathcal{A}, \text{ of course})\) is a continuous map, and \( F_t(x) = F(x, t) \) has no fixed points on \( \partial G \) for \( 0 \leq t \leq 1 \) then \( i_A(F_0, G) = i_A(F_1, G) \) (the homotopy property).

3. If \( G = A \) then \( i_A(f, G) = A(f) \), the Lefschetz number of \( f \), equals \( \sum (-1)^k \text{trace}(f_{**k}) \), where \( f_{**k}: H_k(A) \to H_k(A) \) is the vector space homomorphism of \( H_k(A) \) to \( H_k(A) \) and \( H_k(A) \) is the Cech homology of \( A \) with rational coefficients (the normalization property).

4. Let \( A \) and \( B \) be two spaces which belong to \( \mathcal{A} \). Let \( f: A \to B \) be a continuous map. Let \( V \) be an open subset of \( B \) and \( g: \bar{V} \to A \) a continuous map. Assume \( fg \) has no fixed points on \( \partial V \). Let \( U = f^{-1}(V) \). Then \( gf \) has no fixed points on \( \partial U \) and \( i_B(fg, V) = i_A(gf, U) \) (the commutative property).

Let \( G \) be an open subset of a Banach space \( X \) and \( g: G \to X \) a continuous map such that \( g(x) \neq x \) for \( x \in \partial G \). Assume that \( g \) is compact, that is, \( g(G) \)
INFINITE DIMENSIONAL DYNAMICAL SYSTEMS

has compact closure. Leray and Schauder defined a fixed point index for \( g \) and, consequently, a degree for \( I - g, I \) the identity function. We shall denote this degree by \( \text{deg}(I - g, G, 0) \). It turns out that the Leray–Schauder degree satisfies all four properties of the fixed point index listed above. So we can define the fixed point index by

\[
i_x(g/X \cap G, X \cap G) = \text{deg}(I - g, G, 0),
\]

where \( X = \overline{g}(G) \).

Let \( D \) be a closed subset of a Banach space \( X \). We shall say that \( D \in F \) or that \( D \) is admissible if \( D \) has a locally finite covering \( \{C_j : j \in J\} \) by closed convex sets \( C_j \in D \). If \( D \in F, G \) is an open subset of \( D \), and \( f : G \to D \) is a local condensing map such that \( S = \{x \in G : f(x) = x\} \) is compact (possibly empty) then there is defined an integer \( i_\varnothing(f, G) \) called the generalized fixed point index of \( f \) satisfying conditions (1–4). We refer to [18] for details.

**Theorem 1.** Assume that there exist sets, \( \omega \) open,

\[
S_1 \subset S \subset \partial \omega, \text{ and } Z \subset \omega \cup S_1, Z \neq \emptyset,
\]

satisfying the conditions:

(i) \( S = S^* \),

(ii) \( Z \) is a compact ANR,

(iii) there is a retraction \( r : S_1 \to Z \cap S_1 \),

(iv) there is a continuous map \( \Phi : Z \cap S_1 \to Z \cap S_1 \) such that \( \Phi(P) \neq P \) for every \( P \in Z \cap S_1 \),

(v) \( i_x(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) \neq 0 \).

Then there exists at least one point \( P_0 = (\sigma, x) \in Z \cap \omega \) such that either \( C(P_0) \in S - S_1 \) or \( C(P_0) \) does not exist, that is, \( \tau^+(\sigma, x) \subset \omega \).

**Proof:** Assume that the theorem is not true. Then for every \( P_0 \in Z \cap \omega, C(P_0) \in S_1 \), and then \( Z \cap \omega \subset G \). Then \( Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G \). From (i) \( S = S^* \), from Lemma 1 the map \( \tilde{U} \) is continuous, and the restriction of \( \tilde{U} \) to \( Z \cup S_1 \) is also continuous. From condition (iii) there is a retraction \( r : S_1 \to Z \cap S_1 \).

Then the map \( R = r \cdot \tilde{U} : Z \cup S_1 \to Z \cap S_1 \) is continuous and takes \( P_0 \) into \( C(P_0) \subset Z \cap S_1 \).

From condition (iv) the map \( \Phi \) takes \( C(P_0) \) into \( \Phi(C(P_0)) = C'(P_0) \neq C(P_0) \) and then the composite map \( \Phi \cdot r \cdot \tilde{U} : Z \to Z \) is continuous and \( \Phi \cdot r \cdot \tilde{U}(P) \neq P \) for every \( P \in Z \cap \omega \). From property 2 of the fixed point index \( i_x(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) = 0 \), which is a contradiction with (v). Then there exists at least one point \( P_0 \in Z \cap \omega \) such that the trajectory
through $P_0$ is asymptotic with respect to $\omega$, that is, $\tau^+(\sigma, x)\subset \omega$ or $C(P_0)\in S - S_1$.

**Remark.** When $S = S_1$ the only final conclusion is that the trajectory $\tau^+(\sigma, x)$ is asymptotic with respect to $\omega$. The following corollary of Theorem 1 is essentially equivalent to the Ważewski theorem and can be used in the applications either in finite or infinite dimensions.

**Corollary 1.** Assume that there exist sets, $\omega$ open, $S \subset \partial \omega$ and $Z \subset \omega \cup S$, $Z \neq \emptyset$ satisfying the conditions

(a) $S = S^*$,
(b) $Z$ is compact and convex,
(c) $Z \cap S$ is a retract of $S$,
(d) there is a continuous map $\Phi: Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$.

Then there exists at least one point $P_0 \in Z \cap \omega$ such that the trajectory $\tau^+(\sigma, x)$ is contained in $\omega$.

The proof follows easily since a compact convex set is an ANR and then (b) implies (ii). Since $\Phi \cdot r \cdot \tilde{U}: Z \to Z$ is continuous, $\Phi \cdot r \cdot \tilde{U}$ has a fixed point in $Z$, and then $i_\omega(\Phi \cdot r \cdot \tilde{U}, Z \cap \omega) \neq 0$ which implies (v).

**Theorem 2.** Assume that there exist sets, $\omega$ open in $\Omega$,

$S_1 \subset S \subset \partial \omega$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$

satisfying the conditions:

(i) $S = S^*$,
(ii) $Z \cup S_1$ is a compact ANR,
(iii) there exists a continuous map $\Phi: S_1 \to S_1$ such that $\Phi(P) \neq P$ for every $P \in S_1$,
(iv) $i_{Z \cup S_1}(\Phi \tilde{U}, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that either $C(P_0) \in S - S_1$ or $C(P_0)$ does not exist, that is, the trajectory $\tau^+(\sigma, x)$ is asymptotic with respect to $\omega$.

The proof follows as in Theorem 1. If the theorem is not true $C(P_0) \in S_1$ for every $P_0 \in Z \cap \omega$. Then, $Z = (Z \cap S_1 \cup (Z \cap \omega) \subset S \cup G$. From Lemma 1 the map $\tilde{U}$ is continuous and the restriction of $\tilde{U}$ to $Z \cup S_1$ is also
continuous. The map $\Phi \cdot \tilde{U} : Z \cup S_1 \rightarrow Z \cup S_1$ is continuous and $\Phi \cdot K(P) \neq P$ for every $P \in Z \cup S_1$. Then $i_{Z \cup S_1}(\Phi \cdot \tilde{U}, Z \cap \omega) = 0$, a contradiction and the theorem is proved.

Theorems 1 and 2 or Corollary 1 can be used in finite dimension, but if the space is infinite dimensional it is not easy, in general, to find a $Z$ or $Z \cup S_1$ compact unless we consider the solutions of the equations as elements of a finite dimensional space. However, as done in proofs of existence theorems using the Schauder fixed point theorem, it is much easier to use the fact that the transformation $\Phi \tilde{U}$ or $\Phi \cdot r \cdot \tilde{U}$ is compact or a condensing map. A second formulation of Theorems 1 and 2 is given in Theorems 3 and 5.

**Theorem 3.** Assume that there exist sets, $\omega$ open in $\Omega$, 

$$S_1 \subset S \subset \partial \omega, \text{ and } Z \subset \omega \cup S_1, \ Z \neq \emptyset,$$

satisfying the conditions:

(i) $S = S^*$,

(ii) $Z \cup S_1$ is admissible,

(iii) there exists a continuous map $\Phi : S_1 \rightarrow S_1$ such that $\Phi(P) \neq P$ for every $P \in S_1$,

(iv) $\Phi \cdot \tilde{U}$ is a condensing map,

(v) $i_{Z \cup S_1}(\Phi \cdot \tilde{U}, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that either, $C(P_0) \in S - S_1$ or $C(P_0)$ does not exist, that is, the trajectory $\tau^+(\sigma, P_0)$ is asymptotic with respect to $\omega$.

**Proof.** Assume that the theorem is not true. Then $C(P_0) \in S_1$ for every $P_0 \in Z \cap \omega$ and then $Z \cap \omega \subset G$. Then $Z = (Z \cap S_1) \cup (Z \cap \omega) \subset S \cup G$. From Lemma 1 the map $\tilde{U}$ is continuous and the restriction of $\tilde{U}$ to $Z \cup S_1$ is also continuous. The transformation $\Phi \tilde{U}$ is condensing and $\Phi \tilde{U}(P) \neq P$ for every $P \in Z \cup S_1$, hence, $i(\Phi \tilde{U}, Z \cap \omega) = 0$, which is a contradiction. Then there exists at least one point $P_0 \in Z \cap \omega$ such that either $C(P_0) \subset S - S_1$ or the trajectory through $P_0$ is asymptotic with respect to $\omega$.

**Theorem 4.** Assume that there exist sets, $\omega$ open in $\Omega$,

$$S \subset \partial \omega, \quad Z \subset \omega \cup S, \quad Z \neq \emptyset,$$

satisfying the conditions:

(a) $S = S^*$,

(b) $Z \cup S$ is admissible,
Then the trajectory $t^+(\sigma, x)$ through $(\sigma, x)$ is contained in $\omega$.

**Theorem 5.** Assume that there exist sets, $\omega$ open in $\Omega$, $S_1 \subset S \subset \partial \omega$, and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, $Z$ admissible satisfying the conditions:

(a) $S = S^*$,

(b) $Z \cap S_1$ is a retract of $S_1$, that is, there exists a retraction $r: S_1 \to Z \cap S_1$.

(c) there exists a continuous map $\Phi: Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$,

(d) $\Phi \circ r \circ 0$ is condensing,

(e) $i_2(\Phi r \hat{U}, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_0 \in Z \cap \omega$ such that either $C(P_0) \in S - S_1$ or the trajectory $t^+(\sigma, x)$ through $(\sigma, x)$ is contained in $\omega$.

The proof follows as in Theorem 3.

**Theorem 6.** Assume that there exist sets, $\omega$ open,

$$S \subset \partial \omega \quad \text{and} \quad Z \subset \omega \cup S, \quad Z \neq \emptyset$$

such that

(a) $S = S^*$,

(b) $Z$ is admissible,

(c) $Z \cap S$ is a retract of $S$, that is, there is a retraction $r: S \to Z \cap S$,

(d) there is a continuous map $\Phi: Z \cap S \to Z \cap S$ such that $\Phi(P) \neq P$ for every $P \in Z \cap S$, $\Phi \circ r \circ \hat{U}$ is condensing and $\Phi \circ r \circ \hat{U}$ has a fixed point in $Z$.

Then there exists at least one point $P_0 = (t, x) \in Z \cap \omega$ such that $C(P_0)$ does not exist, that is, $t^+(\sigma, x) \subset \omega$.

Several authors applied the Ważewski method to delay differential equations and partial differential equations [20, 21, 23], considering the solutions of these equations as elements of a finite dimensional euclidean space and defining properly the concepts of egress and ingress points. The reason they considered the solutions of these equations as elements of a euclidean space is that they used the Ważewski retract theorem in its original
form which is not applicable in a natural way for infinite dimensions. Rybakowski [23], following an idea already used by Razumikhin [10] to extend Lyapunov stability theorems to delay equations, considered the space of continuous functions $\phi \in C = C([-r, 0], R^n)$. In addition Rybakowski considered the Ważewski theorem in its original form for what is called a system of curves in $R^n$. The set $\omega$ is a subset of $R^n$ and the condition $S = S^*$ is verified by computing the derivatives along the solution of the equation $\dot{x} = F(x, t)$ in $\partial \omega$ in a way similar to Razumikhin's original paper. The conditions imposed on the boundary $\partial \omega$ of $\omega$ are weaker than the one imposed by Onuchic [20] and the results can be applied to a larger class of delay equations.

Remark. The definition of egress points given above follows closely the definitions given by Ważewski and it is assumed that $\omega$ is open. However, if the space is infinite dimensional $\omega$ does not generally have any interior. For instance, if $\omega$ is a cone in the sequence space $l^\infty$ it has a nonempty interior and if $\omega$ is a cone in $l^p$, $p \geq L$, $\omega$ does not have an interior. In the following definition of Ważewski sets given by Conley [3] it is not assumed that $\omega$ has a nonempty interior.

DEFINITION. Given $W \subset \Omega$, let $W^* = \{ (\sigma, x) \in W \}$ for some $t \in R^+$, $(\sigma + t, U(\sigma, t) x) \notin W$ and let $W^- = \{ (\sigma, x) \in W \}$ if $\varepsilon > 0$ then $(\sigma + t, U(\sigma, [0, \varepsilon]) x) \notin W$. A set $W$ is called a Ważewski set if

(a) $(\sigma, x) \in W$ and $U(\sigma, [0, t]) \subset W^*$ imply $U(\sigma, [0, t]) x \subset W$,

(b) $W^*$ is closed relative to $W^*$.

Then it follows that the following proposition is true.

PROPOSITION. If $W$ is a Ważewski set, then $W^-$ is a retract of $W^*$ and $W^*$ is open relative to $W$.

$W$ is a Ważewski set in the above definition and it means that $W^- = S^*$ and $W^* = G \cup W^-$. By using this proposition instead of Lemma 1 all results of this paper carry over to this more general case. A paper applying these results is in preparation.

REFERENCES