Lyapunov exponents of solutions to linear differential equations with periodic forcing functions

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Received 18 March 2007
Available online 14 December 2007
Submitted by R. Manásevich

Abstract

We give Lyapunov exponents of solutions to linear differential equations of the form \( x' = Ax + f(t) \), where \( A \) is a complex matrix and \( f(t) \) is a \( \tau \)-periodic continuous function. Notice that \( f(t) \) is not “small” as \( t \to \infty \). The proof is essentially based on a representation [J. Kato, T. Naito, J.S. Shin, A characterization of solutions in linear differential equations with periodic forcing functions, J. Difference Equ. Appl. 11 (2005) 1–19] of solutions to the above equation.

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Keywords: Linear differential equation with periodic forcing function; Representation of solution; Lyapunov exponent of solution

1. Introduction

The purpose of the present paper is to give Lyapunov exponents of solutions to linear differential equations with periodic forcing functions of the form

\[
\frac{d}{dt} x(t) = Ax + f(t),
\]

where \( A \) is a complex \( n \times n \) matrix and \( f : \mathbb{R} \to \mathbb{C}^n \) a nontrivial \( \tau \)-periodic continuous function.

It is well known that the Lyapunov exponents of solutions to linear differential equations

\[
\frac{d}{dt} x(t) = Ax
\]

are related to the real parts of the eigenvalues of \( A \). However, it seems that Lyapunov exponents of solutions to Eq. (1) is not yet calculated as far as we know. On the other hand, in [1, p. 97, Theorem 5] it is shown that the Lyapunov exponent of any solution to the perturbed equation of the form

\[
\frac{d}{dt} x(t) = Ax + f(t)
\]
\[
\frac{d}{dt} x(t) = Ax + f(t, x)
\]
(3)

coincides with the real part of some eigenvalue of \( A \) or \(-\infty\) under the condition \( \| f(t, x) \| \leqslant \gamma(t) \| x \| \) with

\[
\int_{t}^{t+1} \gamma(r) \, dr \to 0 \quad (t \to +\infty).
\]
(4)

The condition (4) means that the nonlinear term \( f(t, x) \) is “small” as \( t \to \infty \). Note that this condition on \( \gamma(t) \) holds if \( \gamma(t) \to 0 \) \((t \to +\infty)\) or \( \int_{0}^{\infty} \gamma(t) \, dt < \infty \).

For the case where \( f(t, x) = f(t) \) in Eq. (3) is a nontrivial \( \tau \)-periodic function, it seems that Lyapunov exponents of solutions cannot be calculated by traditional methods, since \( f(t) \) is not “small” as \( t \to \infty \).

Recently, a representation of solutions of Eq. (1) is given as the sum of exponential-like functions and \( \tau \)-periodic functions in the papers [2] and [4]. This representation of solutions naturally leads to the formula about the Lyapunov exponents of solutions in the main theorem in this paper.

2. Preliminaries and the main result

In this section we give some lemmas, and the main theorem in this paper.

**Definition 2.1.** Let \( \varphi(t) \) be a vector function defined for \( t \geqslant t_0 \). We define the Lyapunov exponent \( \chi(\varphi(t)) \) by

\[
\chi(\varphi(t)) = \limsup_{t \to \infty} \frac{\log \| \varphi(t) \|}{t}
\]

if the support of \( \varphi \) is not compact, and \( \chi(\varphi(t)) = -\infty \) if the support of \( \varphi \) is compact.

A fundamental property on Lyapunov exponents is given as follows: If \( \chi(\varphi) < \chi(\psi) \) for \( \varphi, \psi : [t_0, \infty) \to \mathbb{C}^n \), then \( \chi(\varphi + \psi) = \chi(\psi) \); if \( \chi(\varphi) = \chi(\psi) \), then \( \chi(\varphi + \psi) \leqslant \chi(\psi) \), see [5, p. 457]. It is easy to see that, if \( \varphi(t) = e^{\lambda t} p(t) \) with a polynomial \( p(t) \neq 0 \), then \( \chi(\varphi(t)) = \Re \lambda \).

For \( \lambda \in \mathbb{C} \) we denote by \( \Re \lambda \) its real part. Let \( I \) be the \( n \times n \) identity matrix. For the \( n \times n \) complex matrix \( A \) in Eq. (1) we denote by \( \sigma(A) \) the set of all eigenvalues of \( A \), and by \( h(\lambda) := h_A(\lambda) \) the geometric multiplicity of \( \lambda \in \sigma(A) \). Let \( G_A(\lambda) := N((A - \lambda I)^{h(\lambda)}) \) be the generalized eigenspace corresponding to \( \lambda \in \sigma(A) \) and \( P_\lambda : \mathbb{C}^n \to G_A(\lambda) \) the projection corresponding to the direct sum decomposition \( \mathbb{C}^n = \bigoplus_{\lambda \in \sigma(A)} G_A(\lambda) \). The family \( \{P_\lambda\}_{\lambda \in \sigma(A)} \) of projections is uniquely defined, and \( I = \sum_{\lambda \in \sigma(A)} P_\lambda \) holds. If such a family \( \{P_\lambda\}_{\lambda \in \sigma(A)} \) is given, then for any continuous function \( \varphi : [t_0, \infty) \to \mathbb{C}^n \) the following relation holds:

\[
\chi(\varphi(t)) = \max \left\{ \chi(P_\lambda \varphi(t)) \mid \lambda \in \sigma(A) \right\}.
\]

Indeed, since \( \varphi(t) = \sum_{\lambda \in \sigma(A)} P_\lambda \varphi(t) \), by the fundamental property on Lyapunov exponents we have \( \chi(\varphi(t)) \leqslant \max \{ \chi(P_\lambda \varphi(t)) \mid \lambda \in \sigma(A) \} \). On the other hand, since \( \| P_\lambda \varphi(t) \| \leqslant \| P_\lambda \| \| \varphi(t) \| \), it follows that \( \chi(P_\lambda \varphi(t)) \leqslant \chi(\varphi(t)) \).

As a special case, the following result holds true.

**Lemma 2.2.** If \( x(t) \) is any solution of Eq. (1), then

\[
\chi(x(t)) = \max \left\{ \chi(P_\lambda x(t)) \mid \lambda \in \sigma(A) \right\}.
\]

It is well known that \( e^{tA} \) is expressed as

\[
e^{tA} = \sum_{\lambda \in \sigma(A)} e^{\lambda t} \sum_{k=0}^{h(\lambda)-1} \frac{t^k}{k!} (A - \lambda I)^k P_\lambda.
\]
(5)

For the Lyapunov exponents of solutions to Eq. (2) the following result holds, which is a special case of [1, p. 97, Theorem 5]. Notice that its proof is easily given by using Lemma 2.2 and the representation (5).
Lemma 2.3.

1) Let \( \lambda \in \sigma(A) \). If \( x(t) \) is a solution of Eq. (2) and \( P_\lambda x(t) \neq 0 \), then
\[
\chi(P_\lambda x(t)) = \Re \lambda.
\]

2) If \( x(t) \) is a nontrivial solution of Eq. (2), then
\[
\chi(x(t)) = \max \{ \Re \lambda \mid \lambda \in \sigma(A), P_\lambda x(t) \neq 0 \}.
\]

Put
\[
b_f = \int_{0}^{\tau} e^{(t-s)A} f(s) \, ds.
\]

For \( \lambda \in \sigma(A) \) and \( w \in \mathbb{C}^n \), two vectors \( \alpha_\lambda(w, b_f) \) and \( \beta_\lambda(w, b_f) \) are defined as follows:

If \( e^{\lambda \tau} \neq 1 \), then
\[
\alpha_\lambda(w, b_f) = P_\lambda w + \sum_{k=0}^{h(\lambda)-1} \frac{d^k}{dz^k} \left( e^z - 1 \right)^{-1} \bigg|_{z=\lambda \tau} \frac{\tau^k}{k!} (A - \lambda I)^k P_\lambda b_f.
\]

If \( e^{\lambda \tau} = 1 \), then
\[
\beta_\lambda(w, b_f) = \tau (A - \lambda I) P_\lambda w + \sum_{k=0}^{h(\lambda)-1} B_k \frac{\tau^k}{k!} (A - \lambda I)^k P_\lambda b_f,
\]

where \( B_i, i \in \{0, 1, 2, \ldots\} \), stand for Bernoulli’s numbers (cf. [3]).

Denote by \( x(t; w, f) \) the solution of Eq. (1) satisfying the initial condition \( x(0) = w \).

Lemma 2.4. (See [2,4].) Let \( \lambda \in \sigma(A) \) and \( x(t) := x(t; w, f) \) be the solution of Eq. (1).

1) If \( e^{\lambda \tau} \neq 1 \), then
\[
P_\lambda x(t) = e^{\lambda t} \sum_{k=0}^{h(\lambda)-1} \frac{\tau^k}{k!} (A - \lambda I)^k \alpha_\lambda(w, b_f) + u_\lambda(t, f)
\]
where \( u_\lambda(t, f) \) is a \( \tau \)-periodic solution of Eq. (1) in \( G_\lambda(\lambda) \).

2) If \( e^{\lambda \tau} = 1 \), then
\[
P_\lambda x(t) = \frac{e^{\lambda t}}{\tau} \sum_{k=0}^{h(\lambda)-1} \frac{\tau^{k+1}}{(k+1)!} (A - \lambda I)^k \beta_\lambda(w, b_f) + v_\lambda(t, w, f),
\]
where \( v_\lambda(t, w, f) \) is a \( \tau \)-periodic function.

The following result is a main theorem in the present paper.

Theorem 2.5. Let \( x(t; w, f) \) be any solution of Eq. (1). If there exists \( \lambda \in \sigma(A) \) such that \( \Re \lambda > 0 \) and that \( \alpha_\lambda(w, b_f) \neq 0 \), then
\[
\chi(x(t; w, f)) = \max \{ \Re \lambda \mid \lambda \in \sigma(A), \Re \lambda > 0, \alpha_\lambda(w, b_f) \neq 0 \},
\]
otherwise \( \chi(x(t; w, f)) = 0 \).
3. Proof of Theorem 2.5

Using Lemma 2.2, we give a proof of Theorem 2.5. To do so, we will calculate the Lyapunov exponent of the component $P_{\lambda}x(t)$ of solution $x(t)$ to Eq. (1). The component $P_{\lambda}x(t)$ satisfies the equation $\frac{d}{dt}P_{\lambda}x(t) = AP_{\lambda}x(t) + P_{\lambda}f(t)$. Thus it is obvious that $P_{\lambda}x(t) \equiv 0$ if and only if $P_{\lambda}x(0) = 0$ and $P_{\lambda}f(t) = 0$. From this fact it is possible that $P_{\lambda}x(t) \equiv 0$ for any solution of Eq. (1).

**Proposition 3.1.** Let $\lambda \in \sigma(A)$ and $x(t) := x(t; w, f)$ be a solution of Eq. (1).

1) Assume that $P_{\lambda}f(t) \equiv 0$. Then $\chi(P_{\lambda}x(t)) = \Re \lambda$ if $P_{\lambda}w \neq 0$, and $\chi(P_{\lambda}x(t)) = -\infty$ if $P_{\lambda}w = 0$.

2) Assume that $P_{\lambda}f(t) \neq 0$.

(1) If $e^{\lambda t} \neq 1$, then

$$\chi(P_{\lambda}x(t)) = \max\{\Re \lambda, 0\} \quad (\alpha_{\lambda}(w, b_f) \neq 0) \quad \text{and} \quad \chi(P_{\lambda}x(t)) = 0 \quad (\alpha_{\lambda}(w, b_f) = 0).$$

(2) If $e^{\lambda t} = 1$, then $\chi(P_{\lambda}x(t)) = 0$.

**Proof.** Set $x_{\lambda}(t) = P_{\lambda}x(t; w, f)$, $u_{\lambda}(t) = u_{\lambda}(t, f)$, $v_{\lambda}(t) = v_{\lambda}(t, w, f)$ in Lemma 2.4.

1) If $P_{\lambda}f(t) = 0$, then $x_{\lambda}(t)$ is a solution of Eq. (2) satisfying $x_{\lambda}(0) = P_{\lambda}w$; hence, $x_{\lambda}(t) = e^{tA}P_{\lambda}w$. By Lemma 2.3 we have that $\chi(x_{\lambda}(t)) = \Re \lambda$ if $P_{\lambda}w \neq 0$, and $\chi(x_{\lambda}(t)) = -\infty$ if $P_{\lambda}w = 0$.

2) If $P_{\lambda}f(t) \neq 0$, then any solution of Eq. (1) with $f(t) = P_{\lambda}f(t)$ has a noncompact support. Since $u_{\lambda}(t)$ is a nontrivial $\tau$-periodic solution, we have $\chi(u_{\lambda}(t)) = 0$.

Consider the case that $e^{\lambda t} \neq 1$. If $\alpha_{\lambda}(w, b_f) = 0$, then $x_{\lambda}(t) = u_{\lambda}(t)$; hence, $\chi(x_{\lambda}(t)) = 0$. If $\alpha_{\lambda}(w, b_f) \neq 0$, then $e^{tA}\alpha_{\lambda}(w, b_f)$ is a nontrivial solution of Eq. (2). Using Lemma 2.3, we have $\chi(e^{tA}\alpha_{\lambda}(w, b_f)) = \Re \lambda$. If $\Re \lambda \neq 0$, then

$$\chi(x_{\lambda}(t)) = \chi(e^{tA}\alpha_{\lambda}(w, b_f)) = \max\{\Re \lambda, 0\}.$$ 

In the case that $\Re \lambda = 0$, we directly compute $\chi(x_{\lambda}(t))$ as follows. There is $h (0 \leq h \leq h(\lambda) - 1)$ such that $(A - \lambda I)^{h}\alpha_{\lambda}(w, b_f) \neq 0$ and $(A - \lambda I)^{h+1}\alpha_{\lambda}(w, b_f) = 0$. Put $c_{k} = \frac{1}{h!}(A - \lambda I)^{k}\alpha_{\lambda}(w, b_f)$, $k = 0, 1, \ldots, h$. Then $c_{h} \neq 0$ and we have

$$\|x_{\lambda}(t)\| = e^{\lambda t} \left\| \frac{1}{h!} \sum_{k=0}^{h} t^{k} (A - \lambda I)^{k}\alpha_{\lambda}(w, b_f) + u_{\lambda}(t) \right\| = e^{\lambda t} \left\| c_{h} + \sum_{k=0}^{h-1} \frac{1}{(h-k)!} c_{k} + \frac{1}{h!} e^{-\lambda t} u_{\lambda}(t) \right\|.$$ 

By definition of the Lyapunov exponent we have that $\chi(x_{\lambda}(t)) = 0$. Therefore, in any case, if $\alpha_{\lambda}(w, b_f) \neq 0$, then we have

$$\chi(P_{\lambda}x(t)) = \max\{\Re \lambda, 0\}.$$ 

Consider the case that $e^{\lambda t} = 1$. If $\beta_{\lambda}(w, b_f) = 0$, then $x_{\lambda}(t) = v_{\lambda}(t)$, and it is a nontrivial $\tau$-periodic solution; hence, $\chi(x_{\lambda}(t)) = 0$. If $\beta_{\lambda}(w, b_f) \neq 0$, as in the above case it is easily proved that $\chi(P_{\lambda}x(t)) = 0$. Summarizing these results, we complete the proof of the proposition. □

**Proof of Theorem 2.5.** The proof is based on Proposition 3.1 and Lemma 2.2. Since $f(t)$ in Eq. (1) is a nontrivial $\tau$-periodic function, there is at least one $\lambda_{0} \in \sigma(A)$ such that $P_{\lambda_{0}}f(t) \neq 0$ and $P_{\lambda_{0}}x(t; w, f) \neq 0$. Hence $\chi(P_{\lambda_{0}}x(t; w, f)) \geq 0$, which implies that $\chi(x(t; w, f)) \geq 0$.

Notice that $\alpha_{\lambda}(w, b_f) = P_{\lambda}w$ if $P_{\lambda}f(t) = 0$. Therefore, if there exists $\lambda$ such that $\Re \lambda \geq 0$ and that $\alpha_{\lambda}(w, b_f) \neq 0$, then

$$\chi(x(t; w, f)) = \max\{\Re \lambda \mid \Re \lambda \geq 0, \alpha_{\lambda}(w, b_f) \neq 0\}.$$ 

Otherwise, $\chi(x(t; w, f)) = 0$. □
References