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ScienceDirect
J. Math. Anal. Appl. 335 (2007) 213–224
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Splitting of liftings in products of probability spaces II ☆

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Received 18 November 2005

Available online 31 January 2007

Submitted by Steven G. Krantz

Abstract

For a probability measure R on a product of two probability spaces that is absolutely continuous with respect to the product measure we prove the existence of liftings subordinated to a regular conditional probability and the existence of a lifting for R with lifted sections which satisfies in addition a rectangle formula. These results improve essentially some of the results from the former work of the authors [W. Strauss, N.D. Macheras, K. Musiał, Splitting of liftings in products of probability spaces, Ann. Probab. 32 (2004) 2389–2408], by weakening considerably the assumptions and by presenting more direct and shorter proofs. In comparison with [W. Strauss, N.D. Macheras, K. Musiał, Splitting of liftings in products of probability spaces, Ann. Probab. 32 (2004) 2389–2408] it is crucial for applications intended that we can now prescribe one of the factor liftings completely freely. We demonstrate the latter by applications to τ-additive measures, transfer of strong liftings, and stochastic processes.

Keywords: Product liftings and densities; Regular conditional probability; Strong liftings; Product measures; τ-Additive measures

☆ The authors were partially supported by KBN Grant 5 P03A 016 21 and by NATO Grant PST.CLG.977272.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.01.057

1. Introduction

We prove that if $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ are arbitrary probability spaces and R is an arbitrary probability on $\mathfrak{A} \otimes \mathfrak{B}$ with marginals P and Q such that R is $P \otimes Q$-continuous, then for given lifting ρ on the complete space $(Y, \widehat{\mathfrak{B}}, \widehat{Q})$ there exist a lifting ρ' on $(Y, \widehat{\mathfrak{B}}, \widehat{Q})$, a P-continuous product regular conditional probability $\{S_y\colon\ y \in Y\}$ on $\mathfrak{A}$ with respect to $\widehat{\mathfrak{B}}$, a lifting ξ on the R-complete space $(X \times Y, \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}, \widehat{R})$ and a collection of liftings $\{\xi_y\colon\ y \in Y\}$ on the complete spaces $(X, \widehat{\mathfrak{A}}, \widehat{S_y})$, $y \in Y$, possessing the *section property*

$[SP(\xi)]$: $[\xi(E)]^y = \xi_y([\xi(E)]^y)$ for all $E \in \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}$ and all $y \in Y$,

as well as the *rectangle formula* in the following two versions (RF) and (RF'):

$[RF'(\xi, \rho)]$: There exist $N \in (\widehat{\mathfrak{B}})_0$ and $y_0 \in N^c$ such that for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$,

(a) $\xi(A \times B) = \bigcup_{y \in \rho(B) \cup N} [\xi_y(A) \times \{y\}]$ if $y_0 \in \rho(B)$,

(b) $\xi(A \times B) = \bigcup_{y \in \rho(B) \cap N^c} [\xi_y(A) \times \{y\}]$ if $y_0 \notin \rho(B)$;

$[RF(\xi, \rho')]$: $\xi(A \times B) = \bigcup_{y \in \rho'(B)} [\xi_y(A) \times \{y\}]$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

In the sequel we will use the above notation also in case when ξ and ξ_y are densities only. It is known from [10] that a product formula like (RF) can only be achieved for a probability R which is absolutely continuous with respect to $P \otimes Q$. The above result generalizes the main positive result from [6], proved there for $R = P \otimes Q$. It improves also Theorem 2.6 from [10], where the lifting ρ could only be prescribed arbitrarily under a stronger assumption, and gives a direct proof of it. But note that the free choice of the lifting ρ is crucial for the applications below.

We give applications to τ-additive absolutely continuous measures, see Theorem 4.2, which improve the main result from [7], proved there for $R = P \otimes Q$.

We give applications also to strong liftings, see Theorems 5.1 and 5.2, which improve Theorem 3 of Section 3 from [5] as well as the main result from [4], respectively, both proved there for $R = P \otimes Q$. The strong lifting results are related to the problem of transfer of the strong lifting property from the marginals P and Q to R, and give, for a wide class of topological probability spaces, a positive solution of the problem. Finally, we present an application to stochastic processes.

2. Preliminaries

If $(Z, \mathfrak{Z}, S)$ is a probability space, then we denote by $\widehat{\mathfrak{Z}}$ the completion of $\mathfrak{Z}$ with respect to S and by $\widehat{S}$ the completion of S. We write $\mathcal{L}^\infty(S) := \mathcal{L}^\infty(Z, \mathfrak{Z}, S)$ for the space of bounded $\mathfrak{Z}$-measurable real valued functions. Functions equal a.e. are not identified.

We use the notion of *lower density* (but we call it a *density*) and *lifting* in the sense of [3]. $\Lambda(S)$ denotes the system of all liftings on $(Z, \mathfrak{Z}, S)$. Similarly, $\vartheta(S)$ is the collection of all lower densities on $(Z, \mathfrak{Z}, S)$.

Throughout what follows $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ are arbitrary probability spaces. $\mathfrak{A} \times \mathfrak{B}$ is the product algebra generated by $\mathfrak{A}$ and $\mathfrak{B}$ in $X \times Y$, $\mathfrak{A} \otimes \mathfrak{B} := \sigma(\mathfrak{A} \times \mathfrak{B})$ is the product σ-algebra generated by $\mathfrak{A} \times \mathfrak{B}$ and $P \otimes Q$ is the direct product of P and Q. The completion of $\mathfrak{A} \otimes \mathfrak{B}$ with respect to $P \otimes Q$ is denoted by $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ and by $P \widehat{\otimes} Q$ is denoted the completion of $P \otimes Q$. R is a probability measure on $\mathfrak{A} \otimes \mathfrak{B}$, such that P and Q are the marginals of R. We write $\mathfrak{A}_0 = \{A \in \mathfrak{A}: P(A) = 0\}$ and $\mathfrak{B}_0 = \{B \in \mathfrak{B}: Q(B) = 0\}$. $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{B}}$ are reserved for the completions of $\mathfrak{A}$ and $\mathfrak{B}$ with respect to P and Q, respectively.

By $(X \times Y, \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}, \widehat{R})$ we denote the completion of the probability space $(X \times Y, \mathfrak{A} \otimes \mathfrak{B}, R)$. $(X, \mathcal{T}, \mathfrak{A}, P)$ is a *topological measure space* if $(X, \mathfrak{A}, P)$ is a measure space and $\mathcal{T}$ is a topology on X such that $\mathfrak{A} \supseteq \mathcal{T}$. $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X. The Baire σ-algebra of $(X, \mathcal{T})$, i.e. the σ-algebra generated by the system of all co-zero sets of X, is denoted by $\mathfrak{B}_0(X)$. A measure μ on $\mathfrak{B}_0(X)$ is called *completion regular*, if for every Borel set B there exist sets $A_1, A_2 \in \mathfrak{B}_0(X)$ such that $A_1 \subseteq B \subseteq A_2$ and $\mu(A_2 \setminus A_1) = 0$. A measure μ on $\mathfrak{B}(X)$ is called completion regular, if its restriction to $\mathfrak{B}_0(X)$ is completion regular.

Unexplained notation and terminology concerning liftings and densities come from [9].

Definition 1. Assume that for every $y \in Y$ there is a probability S_y on $\mathfrak{A}$ such that:

(D1) For every $A \in \mathfrak{A}$ the map $y \to S_y(A)$ is $\mathfrak{B}$-measurable;
(D2) $R(A \times B) = \int_B S_y(A)\, dQ(y)$ for all $A \in \mathfrak{A}$ and all $B \in \mathfrak{B}$.

Then the family $\{S_y: y \in Y\}$ is called a *product regular conditional probability* (product r.c.p. for short) on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$ (cf. [2]). A product r.c.p. $\{S_y: y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$ is said to be *absolutely continuous* with respect to P (or P-continuous), if

$$\forall A \in \mathfrak{A} \quad \bigl[P(A) = 0 \Rightarrow \forall y \in Y,\ S_y(A) = 0\bigr].$$

The completion of $\mathfrak{A}$ with respect to S_y is denoted by $\widehat{\mathfrak{A}}_y$. Clearly $\widehat{\mathfrak{A}} \subseteq \widehat{\mathfrak{A}}_y$, if $S_y \ll P$.

It is known from [10, p. 2390] that $R \ll P \otimes Q$ if and only if there exists a P-continuous product r.c.p. $\{S_y: y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$.

If ρ is a density in $\vartheta(\widehat{Q})$, we say that ρ and a product r.c.p. $\{S_y: y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$ satisfy the condition $IT(\rho)$, if

$[IT(\rho)]$: $R(A \times B) = 0 \Rightarrow S_y(A) = 0$ for all $y \in \rho(B)$ or $Q(B) = 0$.

Condition $IT(\rho)$ traces back to A. and C. Ionescu-Tulcea [3, p. 115]. Clearly, $IT(\rho)$ yields the absolute continuity of $\{S_y: y \in Y\}$ with respect to P.

A probability measure μ on a topological measure space is τ-*additive* if

$$\mu\Bigl(\bigcup \mathcal{G}\Bigr) = \sup\bigl\{\mu(G): G \in \mathcal{G}\bigr\},$$

whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets.

The probability measure μ is *Radon* if for each $A \in \mathfrak{A}$, we have

$$\mu(A) = \sup\bigl\{\mu(K): K \subseteq A,\ K \text{ compact}\bigr\}.$$

If μ is a Radon probability measure then $(X, \mathcal{T}, \mathfrak{A}, \mu)$ is called a *Radon probability space*.

3. Arbitrary absolutely continuous measures

We have proven in [6] that given complete probability spaces $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ and a lifting $\rho \in \Lambda(Q)$ there exist liftings $\sigma \in \Lambda(P)$ and $\varphi \in \Lambda(P \mathbin{\widehat{\otimes}} Q)$ such that $\varphi(A \times B) = \sigma(A) \times \rho(B)$ whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, and $[\varphi(E)]^y = \sigma([\varphi(E)]^y)$ for all $E \in \mathfrak{A} \mathbin{\widehat{\otimes}} \mathfrak{B}$ and

$y \in Y$. That result has been generalized in [10, Theorem 2.6] to the case of measures defined on the product σ-algebra which are absolutely continuous with respect to the product measure $P \otimes Q$. We are going to present now an improvement of the result of [10] with a shorter and more direct proof. We present it not just for completeness but because we need an essential part of the new proof in order to prove the main result of our paper.

As a preparation we give the next two results. The following lemma is of independent interest, since it produces rectangle formulas for densities.

Lemma 3.1. *Let R be a probability measure defined on $\mathfrak{A} \otimes \mathfrak{B}$ with marginals P and Q, and let ρ be an arbitrary member of $\vartheta(\widehat{Q})$.*

If $R \ll P \otimes Q$, then there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$, and there exist $\eta_y \in \vartheta(\widehat{S}_y)$ for all $y \in Y$ as well as $\eta \in \vartheta(\widehat{R})$ and $\rho' \in \vartheta(\widehat{Q})$ such that the section property $SP(\eta)$, the property $IT(\rho')$ and the rectangle formulas $RF(\eta, \rho')$ as well as $RF'(\eta, \rho)$ hold true. In addition η has measurable x-sections:

$[MS(\eta)]$: $[\eta(E)]_x \in \widehat{\mathfrak{B}}$ *for all $E \in \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}$ and all $x \in X$.*

Proof. According to [6, Corollary 2.2], that holds true for arbitrary $(X, \mathfrak{A}, P)$, for given density $\rho \in \vartheta(\widehat{Q})$ there exist densities $\tau \in \vartheta(P)$ and $\varphi \in \vartheta(P \widehat{\otimes} Q)$ such that

$$\varphi(A \times B) = \tau(A) \times \rho(B) \quad \text{for each } A \times B \in \mathfrak{A} \times \mathfrak{B}, \tag{1}$$

$$\left[\varphi(E)\right]^y = \tau\left(\left[\varphi(E)\right]^y\right) \quad \text{for all } E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B} \text{ and all } y \in Y, \tag{2}$$

$$\left[\varphi(E)\right]_x \in \widehat{\mathfrak{B}} \quad \text{for all } x \in X \text{ and all } E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}. \tag{3}$$

If $R \ll P \otimes Q$, then $R(E) = \int_E f(x, y)\, d(P \otimes Q)$ for every $E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$, where f is the Radon–Nikodym derivative of R with respect to $P \otimes Q$. Hence

$$R(A \times B) = \int\limits_B \left(\int\limits_A f^y(x)\, dP(x) \right) dQ(y) \quad \text{for every } A \times B \in \mathfrak{A} \widehat{\otimes} \mathfrak{B},$$

and

$$Q(B) = \int\limits_B \left(\int\limits_X f^y(x)\, dP(x) \right) dQ(y) \quad \text{for every } B \in \mathfrak{B}.$$

We may and do assume that $\int_X f^y(x)\, dP(x) = 1$ for all $y \in Y$. Set $T_y(A) = \int_A f^y(x)\, dP(x)$ for all $y \in Y$. Since $f \in L^1(P \otimes Q)$, the Fubini Theorem yields the $\widehat{Q}$-measurability of the function $T_\cdot(A)$, when $A \in \mathfrak{A}$. Hence $\{T_y\colon y \in Y\}$ is a product r.c.p. on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$.

Put $H := \varphi(\{f > 0\})$. It follows that $H \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ and $\varphi(H) = H$. It is also obvious that $(P \widehat{\otimes} Q)(E \cap H) = 0$ implies $\widehat{R}(E) = 0$ and $\widehat{R}(H) = 1$. Moreover $H^y \in \widehat{\mathfrak{A}}$ and $\tau(H^y) = H^y$ for all $y \in Y$, by condition (2).

Next put $N := \{y \in Y\colon \widehat{T}_y(H^y) < 1\}$. Then $N \in (\widehat{\mathfrak{B}})_0$. Notice that H^y is a support of T_y for all $y \notin N$, since for each such y we have $T_y(H^y) = 1$ and $P(A \cap H^y) = 0$ implies $T_y(A) = 0$ (because $T_y \ll P$). Choose an arbitrary $y_0 \in N^c$ and modify $\{T_y\colon y \in Y\}$ to a product r.c.p. on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$ by means of $S_y := T_y$ if $y \in N^c$ and $S_y := T_{y_0}$ if $y \in N$. For all $A \in \widehat{\mathfrak{A}}_y$ and $y \notin N$, define

$$\eta_y(A) := \begin{cases} X & \text{if } A = X \text{ a.e. } (\widehat{S}_y), \\ \tau(A \cap H^y) & \text{otherwise.} \end{cases}$$

Then $\eta_y \in \vartheta(\widehat{S}_y)$ for all $y \notin N$ (cf. [1]). For $y \in N$ take $\eta_y := \eta_{y_0} \in \vartheta(\widehat{S}_y)$.

Define

$$\eta(E) := \big[\varphi(E \cap H) \cap \big(X \times N^c\big)\big] \cup \big\{(x, y) \in X \times N\colon (x, y_0) \in \varphi(E \cap H)\big\}$$

whenever $E \in \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}$.

The density properties of η are straightforward to verify. It is also easy to see that η satisfies conditions $SP(\eta)$ and $MS(\eta)$ of the lemma.

To verify condition $RF'(\eta, \rho)$, take $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Applying condition (1), for each $y \in N^c \cap \rho(B)$, we get

$$\big[\eta(A \times B)\big]^y = \big[\varphi\big((A \times B) \cap H\big)\big]^y = \big[\tau(A) \times \rho(B)\big]^y \cap H^y = \tau(A) \cap H^y,$$

while for each $y \in N$ and $y_0 \in \rho(B)$, we get

$$\big[\eta(A \times B)\big]^y = \big[\varphi\big((A \times B) \cap H\big)\big]^{y_0} = \big[\tau(A) \times \rho(B)\big]^{y_0} \cap H^{y_0} = \tau(A) \cap H^{y_0}.$$

Consequently, if $y_0 \in \rho(B)$, then

$$\begin{aligned}
\eta(A \times B) &= \bigcup_{y \in Y} \big[\big[\psi(A \times B)\big]^y \times \{y\}\big] \\
&= \bigcup_{y \in \rho(B) \cap N^c} \big[\eta_y(A) \times \{y\}\big] \cup \bigcup_{y \in N} \big[\eta_{y_0}(A) \times \{y\}\big] \\
&= \bigcup_{y \in \rho(B) \cup N} \big[\eta_y(A) \times \{y\}\big],
\end{aligned}$$

i.e. condition (a) of $RF'(\eta, \rho)$ holds true. If $y_0 \notin \rho(B)$, we get in the same way

$$\eta(A \times B) = \bigcup_{y \in \rho(B) \cap N^c} \big[\eta_y(A) \times \{y\}\big],$$

i.e. condition (b) of $RF'(\eta, \rho)$ holds true.

To show condition $RF(\eta, \rho')$, put $\rho'(B) := [\rho(B) \cap N^c] \cup N$ if $y_0 \in \rho(B)$ and $\rho'(B) := \rho(B) \cap N^c$ otherwise. It is now straightforward to verify that $\rho' \in \vartheta(\widehat{Q})$, hence condition $RF(\eta, \rho')$ follows immediately from $RF'(\eta, \rho)$.

If $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, then $R(A \times B) = 0$ yields $\eta(A \times B) = \emptyset$, hence $RF(\eta, \rho')$ yields $\eta_y(A) = \emptyset$ for all $y \in \rho'(B)$ or $\rho'(B) = \emptyset$. But $\rho'(B) = \emptyset$ implies $Q(B) = 0$. If $Q(B) > 0$, then $\eta_y(A) = \emptyset$ for all $y \in \rho'(B)$, implying $S_y(A) = 0$ for all $y \in \rho'(B)$. So the property $IT(\rho')$ holds true.

This completes the proof of the lemma. □

The following result improves Theorem 2.5. from [10].

Proposition 3.2. *Let R be a probability measure defined on $\mathfrak{A} \otimes \mathfrak{B}$ with marginals P and Q, and let $\rho \in \vartheta(\widehat{Q})$ be arbitrary.*

If $R \ll P \otimes Q$, then there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$, and there exist $\psi_y \in \vartheta(\widehat{S}_y)$ for all $y \in Y$, as well as $\psi \in \vartheta(\widehat{R})$ and $\rho' \in \vartheta(\widehat{Q})$ such that the properties $SP(\psi)$, $MS(\psi)$, $IT(\rho')$ and the full section property

$[FS(\psi)]$: $\widehat{S}_y([\psi(E)]^y \cup [\psi(E^c)]^y) = 1$ *for all $E \in \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}$ and all $y \in Y$,* *hold true. In addition we have the following two properties*:

(i) *There exist* $N \in (\widehat{\mathfrak{B}})_0$ *and* $y_0 \in N^c$ *such that for all* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{B}$,
   (a) $\psi(A \times B) \supseteq \bigcup_{y \in \rho(B) \cup N}[\psi_y(A) \times \{y\}]$ *if* $y_0 \in \rho(B)$;
   (b) $\psi(A \times B) \supseteq \bigcup_{y \in \rho(B) \cap N^c}[\psi_y(A) \times \{y\}]$ *if* $y_0 \notin \rho(B)$,

(ii) $\psi(A \times B) \supseteq \bigcup_{y \in \rho'(B)}[\psi_y(A) \times \{y\}]$ *for all* $A \in \mathfrak{A}$ *and all* $B \in \mathfrak{B}$.

Proof. There exist $\eta \in \vartheta(\widehat{R})$ and $\eta_y \in \vartheta(\widehat{S}_y)$ for all $y \in Y$, satisfying the thesis of Lemma 3.1. Let

$$\Phi := \{\bar{\psi} \in \vartheta(\widehat{R})\colon \forall y \in Y\ \forall E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}\ [\bar{\psi}(E)]^y \subseteq \eta_y([\bar{\psi}(E)]^y) \text{ and } \forall E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}\ \eta(E) \subseteq \bar{\psi}(E)\}.$$

Notice first that $\Phi \neq \emptyset$ since $\eta \in \Phi$.

We consider Φ with inclusion as the partial order: $\bar{\psi}_1 \leqslant \bar{\psi}_2$ if $\bar{\psi}_1(E) \subseteq \bar{\psi}_2(E)$ for each $E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$. Following the arguments of the proof of Theorem 3.5 from [10], we get a maximal element ψ in Φ satisfying conditions $MS(\psi)$, $SP(\psi)$ and $FS(\psi)$ of the theorem. Setting $\psi_y = \eta_y$, for every $y \in Y$, we get immediately condition (i) from the property $\eta(E) \subseteq \psi(E)$ for all $E \in \mathfrak{A} \mathbin{\widehat{\otimes}} \mathfrak{B}$, and from condition $RF'(\eta, \rho)$ of Lemma 3.1. Condition (ii) and property $IT(\rho')$ follow in the same way as condition $RF(\eta, \rho')$ and property $IT(\rho')$, respectively, of Lemma 3.1. □

Theorem 3.3. *Let* R *be a probability measure defined on* $\mathfrak{A} \otimes \mathfrak{B}$ *with marginals* P *and* Q, *and let* $\rho \in \Lambda(\widehat{Q})$ *be an arbitrary lifting.*

If $R \ll P \otimes Q$, *then there exists a product r.c.p.* $\{S_y\colon y \in Y\}$ *on* $\mathfrak{A}$ *for* R *with respect to* $\widehat{\mathfrak{B}}$, *and there exist* $\xi \in \Lambda(\widehat{R})$, $\rho' \in \Lambda(\widehat{Q})$ *and a collection of liftings* $\{\xi_y \in \Lambda(\widehat{S}_y)\colon y \in Y\}$ *such that the properties* $SP(\xi)$, $IT(\rho')$, $RF(\xi, \rho')$ *and* $RF'(\xi, \rho)$ *hold true.*

Proof. According to Proposition 3.2 there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$, and there exist densities $\psi_y \in \vartheta(\widehat{S}_y)$ for all $y \in Y$ and $\psi \in \vartheta(\widehat{R})$ such that the properties $SP(\psi)$ and $FS(\psi)$ hold true.

Let N and $y_0 \in N^c$ be as in Proposition 3.2 and take for each $y \in N^c$ a lifting $\xi_y \in \Lambda(\widehat{S}_y)$ such that $\xi_y(A) \supseteq \eta_y(A)$ for each $A \in \widehat{\mathfrak{A}}_y$ (for the existence of such a lifting see e.g. [11]). For each $y \in N$, take $\xi_y := \xi_{y_0}$.

Define $\xi \in \vartheta(\widehat{R})$ by setting for each $E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$ and each $y \in Y$,

$$[\xi(E)]^y = \xi_y([\psi(E)]^y). \tag{4}$$

Since $\psi(E) \subseteq \xi(E)$ for all $E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$, we get the $\widehat{R}$-measurability of $\xi(E)$ and $\xi(E) = E$ a.e. $(\widehat{R})$. In order to prove that ξ is a lifting it suffices to show that we have always $\xi(E^c) = [\xi(E)]^c$. But this is a consequence of $FS(\psi)$ and (4) as we get for each y the equality $[\xi(E^c)]^y = ([\xi(E)]^y)^c$. This proves that $\xi \in \Lambda(\widehat{R})$.

To prove the validity of the rectangle property $RF'(\xi, \rho)$, let us fix an arbitrary $A \times B \in \mathfrak{A} \times \mathfrak{B}$. Applying condition (4) and assertion (i) of Proposition 3.2, we get

$$[\xi(A \times Y)]^y = \xi_y([\psi(A \times Y)]^y) \supseteq \xi_y(\psi_y(A)) = \xi_y(A) \quad \text{if } y \in N^c,$$

and

$$[\xi(A \times Y)]^y = \xi_y([\psi(A \times Y)]^y) \supseteq \xi_y(\psi_{y_0}(A)) = \xi_{y_0}(A) \quad \text{if } y \in N.$$

Hence $[\xi(A \times Y)]^y \supseteq \xi_y(A)$ for each $y \in Y$. Standard calculations involving the lifting properties of ξ and ξ_y yield

$$\left[\xi(A \times Y)\right]^y = \xi_y(A) \quad \text{for each } y \in Y. \tag{5}$$

In the same way for $A = X$ we get $[\xi(X \times B)]^y = X$ for each $y \in \rho(B) \cap N^c$, and $[\xi(X \times B)]^y = X$ for each $y \in N$ if $y_0 \in \rho(B)$, hence

$$\left[\xi(X \times B)\right]^y = X \quad \text{for each } y \in \left[\rho(B) \cap N^c\right] \cup N \text{ if } y_0 \in \rho(B), \tag{6}$$

and

$$\left[\xi(X \times B)\right]^y = X \quad \text{for each } y \in \rho(B) \cap N^c \text{ if } y_0 \notin \rho(B). \tag{7}$$

Consequently, if $y_0 \in \rho(B)$, applying conditions (5) and (6) we have

$$\begin{aligned}
\xi(A \times B) &= \xi(A \times Y) \cap \xi(X \times B) \\
&= \left(\bigcup_{y \in Y} \left[\xi_y(A) \times \{y\}\right]\right) \cap \left(\bigcup_{y \in \rho(B) \cup N} \left[X \times \{y\}\right]\right) \\
&= \bigcup_{y \in \rho(B) \cup N} \left[\xi_y(A) \times \{y\}\right],
\end{aligned}$$

hence condition (a) of $RF'(\xi, \rho)$ holds true.

Similarly, if $y_0 \notin \rho(B)$, applying condition (7), we get condition (b) of $RF'(\xi, \rho)$.

Conditions $RF(\xi, \rho')$ and $IT(\rho')$ follow as in Lemma 3.1. □

4. τ-Additive absolutely continuous measures

It is known (cf. Ressel [8]) that for arbitrary τ-additive topological probability spaces $(X, \mathcal{T}, \mathcal{B}(X), P)$ and $(Y, \mathcal{S}, \mathcal{B}(Y), Q)$ there exists exactly one τ-additive extension $P \otimes_\tau Q$ of the product measure $P \otimes Q$ to the Borel σ-algebra $\mathcal{B}(X \times Y)$ given by the formula

$$(P \otimes_\tau Q)(E) = \int_X Q(E_x)\, dP(x).$$

The following theorem is an extension of Theorem 3.3 to products of topological spaces. It also generalizes the main result proved in [7]. As a preparation we need the following result.

Lemma 4.1. *Let $(X, \mathcal{T}, \mathcal{B}(X), P)$ and $(Y, \mathcal{S}, \mathcal{B}(Y), Q)$ be topological probability spaces with τ-additive probability measures P and Q and let $P \otimes_\tau Q$ be the τ-additive extension of $P \otimes Q$ to the Borel σ-algebra $\mathcal{B}(X \times Y)$.*

If $\rho \in \vartheta(\widehat{Q})$, then there exist densities $\delta \in \vartheta(P)$ and $\zeta \in \vartheta(P \mathbin{\widehat{\otimes}_\tau} Q)$ such that $\zeta : \widehat{\mathcal{B}}_\tau(X \times Y) \to \mathcal{B}(X) \mathbin{\widehat{\otimes}} \mathcal{B}(Y)$, where $\widehat{\mathcal{B}}_\tau(X \times Y)$ is the completion of $\mathcal{B}(X \times Y)$ with respect to $P \otimes_\tau Q$, and

- (j) $[\zeta(E)]^y = \delta([\zeta(E)]^y)$ *for all* $E \in \widehat{\mathcal{B}}(X \times Y)$ *and* $y \in Y$;
- (jj) $[\zeta(E)]_x \in \widehat{\mathcal{B}}(X)$ *for all* $E \in \widehat{\mathcal{B}}(X \times Y)$ *and* $x \in X$;
- (jjj) $\zeta(A \times B) = \delta(A) \times \rho(B)$ *for all* $A \times B \in \mathcal{B}(X) \times \mathcal{B}(Y)$.

Proof. According to [6, Corollary 2.7], that holds true for arbitrary $(X, \mathfrak{A}, P)$, for given density $\rho \in \vartheta(\widehat{Q})$ there exist densities $\delta \in \vartheta(P)$ and $\varphi \in \vartheta(P \mathbin{\widehat{\otimes}} Q)$ satisfying conditions (j) and (jj) for all $E \in \mathcal{B}(X) \mathbin{\widehat{\otimes}} \mathcal{B}(Y)$, and condition (jjj). Then, applying the argument of the proof of Corollary 2.2 from [7], we show that the σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ is $P \mathbin{\widehat{\otimes}_\tau} Q$-dense in $\widehat{\mathcal{B}}_\tau(X \times Y)$.

Thus, if $H \in \widehat{\mathcal{B}}_\tau(X \times Y)$ is arbitrary, then there exists $E_H \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$ such that $P \mathbin{\widehat{\otimes}_\tau} Q(H \triangle E_H) = 0$. We define $\zeta : \widehat{\mathcal{B}}_\tau(X \times Y) \to \mathcal{B}(X) \mathbin{\widehat{\otimes}} \mathcal{B}(Y)$ by setting $\zeta(H) := \varphi(E_H)$. □

Theorem 4.2. *Let $(X, \mathcal{T}, \mathcal{B}(X), P)$ and $(Y, \mathcal{S}, \mathcal{B}(Y), Q)$ be topological probability spaces with τ-additive probability measures P and Q and let $P \otimes_\tau Q$ be the τ-additive extension of $P \otimes Q$ to the Borel σ-algebra $\mathcal{B}(X \times Y)$.*

If R is an arbitrary probability on $\mathcal{B}(X \times Y)$ with marginals P and Q such that $R \ll P \otimes_\tau Q$, and if $\rho \in \Lambda(\widehat{Q})$, then there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathcal{B}(X)$ for R with respect to $\widehat{\mathcal{B}}(Y)$ and there exist $\xi \in \Lambda(\widehat{R})$, $\rho' \in \Lambda(\widehat{Q})$ and a collection of liftings $\{\xi_y \in \Lambda(\widehat{S}_y)\colon y \in Y\}$ such that SP(ξ), IT(ρ'), RF$'(\xi, \rho)$ and RF(ξ, ρ') hold true, when $\widehat{\mathcal{B}}_R(X \times Y)$ (where $\widehat{\mathcal{B}}_R(X \times Y)$ is the completion of $\mathcal{B}(X \times Y)$ with respect to R) is put instead of $\mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$, $\mathcal{B}(X)$ instead of $\mathfrak{A}$, and $\mathcal{B}(Y)$ instead of $\mathfrak{B}$.

Proof. Choose a density f for $R \ll P \otimes_\tau Q$ with respect to $P \otimes_\tau Q$. It follows that there exists a set $N \in \mathcal{B}(Y)_0$ such that $\int_X f^y(x)\,dP(x) = 1$ if $y \notin N$. Let us set $S_y(A) = \int_A f^y(x)\,dP(x)$ if $y \notin N$ and $S_y(A) = S_{y_0}(A)$ otherwise.

Since $f \in L^1(P \otimes_\tau Q)$, the Fubini Theorem yields the $\widehat{Q}$-measurability of the function $S_{\cdot}(A)$, when $A \in \mathcal{B}(X)$. Hence $\{S_y\colon y \in Y\}$ is a product r.c.p. on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$.

Now, according to Lemma 4.1, for given $\rho \in \vartheta(\widehat{Q})$ there exist densities $\delta \in \vartheta(\widehat{P})$ and $\zeta \in \vartheta(P \mathbin{\widehat{\otimes}_\tau} Q)$, such that $\zeta : \widehat{\mathcal{B}}_\tau(X \times Y) \to \mathcal{B}(X) \mathbin{\widehat{\otimes}} \mathcal{B}(Y)$,

$$\zeta(A \times B) = \delta(A) \times \rho(B) \quad \text{for all } A \times B \in \widehat{\mathcal{B}}(X) \times \widehat{\mathcal{B}}(Y),$$

and

$$\big[\zeta(E)\big]^y = \delta\big(\big[\varphi(E)\big]^y\big) \quad \text{for all } E \in \widehat{\mathcal{B}}_\tau(X \times Y) \text{ and all } y \in Y.$$

Put $H := \zeta(\{f > 0\})$. The rest of the proof is similar to that of Lemma 3.2 and Theorem 3.3. □

Corollary 4.3. *Let $(X, \mathcal{T}, \mathcal{B}(X), P)$ and $(Y, \mathcal{S}, \mathcal{B}(Y), Q)$ be Radon probability spaces and let R be an arbitrary measure on $\mathcal{B}(X \times Y)$ with marginals P and Q.*

If $R \ll P \otimes_\tau Q$ and $\rho \in \Lambda(\widehat{Q})$, then R is a Radon measure and there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathcal{B}(X)$ for R with respect to $\widehat{\mathcal{B}}(Y)$ such that all S_y and R are Radon measures. Moreover, there exist $\xi \in \Lambda(\widehat{R})$, $\rho' \in \Lambda(\widehat{Q})$ and a collection of liftings $\{\xi_y \in \Lambda(\widehat{S}_y)\colon y \in Y\}$ such that the properties SP(ξ), IT(ρ'), RF$'(\xi, \rho)$ and RF(ξ, ρ') hold true, with $\widehat{\mathcal{B}}_R(X \times Y)$ instead of $\mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$, $\mathcal{B}(X)$ instead of $\mathfrak{A}$, and $\mathcal{B}(Y)$ instead of $\mathfrak{B}$.

Proof. Each Radon measure is τ-additive, hence applying Theorem 4.2 we get the existence of a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathcal{B}(X)$ with respect to $\widehat{\mathcal{B}}(Y)$, and the existence of the liftings ξ, ρ' and $\{\xi_y \in \Lambda(\widehat{S}_y)\colon y \in Y\}$ satisfying Theorem 4.2. Moreover according to Ressel [8, Theorem 1], there exists a unique Radon measure on $\mathcal{B}(X \times Y)$ extending $P \otimes Q$, hence the unique τ-additive extension $P \otimes_\tau Q$ of $P \otimes Q$ on $\mathcal{B}(X \times Y)$ is a Radon measure. Since P is a Radon measure and $S_y \ll P$ for all $y \in Y$, we get that all S_y are Radon measures. But since $R \ll P \otimes_\tau Q$, taking in account that $P \otimes_\tau Q$ is a Radon measure we get that R is Radon. □

5. Strong liftings

We say that a topological probability space $(X, \mathcal{T}, \mathfrak{A}, P)$ (or just the measure P) has the *universal strong density property* (*USDP* for short), if every density for P is almost strong (see [9]). Spaces possessing the *USDP* are e.g. those having a countable measurable network, in particular spaces with a second countable topology. In particular, polish spaces as well as locally compact metrizable spaces have the *USDP* (see [9]). The following result improves Theorem 3 of Section 3 from [5] proved there for $R = P \otimes Q$.

Theorem 5.1. *Let $(X, \mathcal{T}, \mathfrak{A}, P)$ be a topological probability space such that $\widehat{P}$ has the USDP, let $(Y, \mathcal{S}, \mathfrak{B}, Q)$ be a topological probability space admitting a strong lifting $\rho \in \Lambda(\widehat{Q})$, and let R be an arbitrary probability on $\mathfrak{A} \otimes \mathfrak{B}$ with marginals P and Q.*

If $R \ll P \otimes Q$, then there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$, and there exist an almost strong lifting $\xi \in \Lambda(\widehat{R})$ and a collection of strong liftings $\{\xi_y \in \Lambda(\widehat{S_y})\colon y \in Y\}$ such that the section property $SP(\xi)$ and the rectangle formula $RF'(\xi, \rho)$ hold true. Moreover $\mathcal{T} \times \mathcal{S} \subseteq \mathfrak{A} \widehat{\otimes}_R \mathfrak{B}$.

In particular if $\mathfrak{A} = \mathfrak{B}(X)$, $\mathfrak{B} = \mathfrak{B}(Y)$ and the measures P, Q are completion regular and $\mathfrak{B}_0(X \times Y) = \mathfrak{B}_0(X) \otimes \mathfrak{B}_0(Y)$, then R is completion regular too.

Proof. According to [6, Corollary 2.2] there exist densities $\tau \in \vartheta(\widehat{P})$ and $\varphi \in \vartheta(P \widehat{\otimes} Q)$ satisfying conditions (1) and (2). Since the measure $\widehat{P}$ has the *USDP*, it follows that the density τ is almost strong, i.e. there exists a null set $L \in \widehat{\mathfrak{A}}_0$ such that for each $G \in \mathcal{T}$, we have $\tau(G) \cup L \supseteq G$.

Let $\{S_y\colon y \in Y\}$, N, $y_0 \in N^c$, H, H^y for all $y \in Y$ as well as and η_y for all $y \in Y$ be as in the proof of Theorem 3.2.

(A) Let us fix an arbitrary $y \notin N$. Put $L_y := L \cup (X \setminus H^y)$. Then L_y is an $\widehat{S}_y$-null set and for every non-empty open subset G of X we have $\eta_y(G) = X \supseteq G$ if $G = X$ a.e. $(\widehat{S}_y)$ and $\eta_y(G) \cup L_y = \tau(G \cap H^y) \cup L_y = \tau(G) \cup L_y \supseteq G$ otherwise. Hence η_y is $\widehat{S}_y$-almost strong.

For every $x \in L_y$, let $\mathcal{E}_y(x) := \{G \in \mathcal{T}\colon x \in G\}$ and $\mathcal{F}_y(x)$ be the filter defined by

$$\mathcal{F}_y(x) := \bigl\{A \in \widehat{\mathfrak{A}}_y\colon \exists G \in \mathcal{E}_y(x)\ G \subseteq A \text{ a.e. } (\widehat{S}_y)\bigr\}.$$

Since all members of $\mathcal{E}_y(x)$ and consequently of $\mathcal{F}_y(x)$ are of positive measure, we see that $\mathcal{F}_y(x)$ is measure stable (i.e. $A \in \mathcal{F}_y(x)$ and $\widehat{S}_y(A \triangle B) = 0$ yields $B \in \mathcal{F}_y(x)$). Define the map $\bar{\eta}_y : \widehat{\mathfrak{A}}_y \to \widehat{\mathfrak{A}}_y$ by means of the formula

$$\bar{\eta}_y(A) := \bigl[\eta_y(A) \cap L_y^c\bigr] \cup \eta_y^*(A) \quad \text{for each } A \in \widehat{\mathfrak{A}}_y,$$

where $\eta_y^*(A) := \{x \in L_y\colon A \in \mathcal{F}_y(x)\}$.

It is straightforward to verify that $\bar{\eta}_y \in \vartheta(\widehat{S}_y)$.

Claim. *The density $\bar{\eta}_y$ is $\widehat{S}_y$-strong.*

Proof. Take $G \in \mathcal{T}$ and $x \in G$. If $x \in L_y$ then $G \in \mathcal{E}_y(x) \subseteq \mathcal{F}_y(x)$ hence $x \in \eta_y^*(G) \subseteq \bar{\eta}_y(G)$. If $x \in X \setminus L_y$ then $x \in \eta_y(G) \cap H^y \subseteq \bar{\eta}_y(G)$. Consequently, $\bar{\eta}_y$ is $\widehat{S}_y$-strong. This completes the proof of the claim. □

(B) For each $y \in N$, put $\bar{\eta}_y := \bar{\eta}_{y_0}$. It follows from the above claim that $\bar{\eta}_y$ is a strong density in $\vartheta(\widehat{S}_y)$.

Consequently, we get from (A) and (B) that for each $y \in Y$ the density $\bar{\eta}_y \in \vartheta(\widehat{S_y})$ is strong.

Then, take as ξ_y an arbitrary strong $\xi_y \in \Lambda(\widehat{S_y})$ such that $\bar{\eta}_y(A) \subseteq \xi_y(A)$ for all $A \in \mathfrak{A}$ and define ξ exactly as in the proof of Theorem 3.3. It follows (in the same way as in the proof of Theorem 3.3) that conditions $SP(\xi)$ and $RF'(\xi, \rho)$ hold true.

To show that ξ is almost strong, take $U_1 \times U_2 \in \mathcal{T} \times \mathcal{S}$. If $y_0 \in \rho(U_2)$, we can apply condition (a) of $RF'(\xi, \rho)$ to obtain

$$\begin{aligned}\xi(U_1 \times U_2) &= \bigcup_{y \in \rho(U_2) \cup N} \xi_y(U_1) \times \{y\} \supseteq \bigcup_{y \in \rho(U_2) \cup N} \big[U_1 \times \{y\}\big] \\ &= U_1 \times \big(\big[\rho(U_2) \cap N^c\big] \cup N\big) \supseteq U_1 \times \big[\rho(U_2) \cap N^c\big] \\ &\supseteq U_1 \times \big(U_2 \cap N^c\big).\end{aligned}$$

In the same way, if $y_0 \notin \rho(U_2)$, applying condition (b) of $RF'(\xi, \rho)$ we get $\eta(U_1 \times U_2) \supseteq U_1 \times (U_2 \cap N^c)$. Consequently, ξ is almost strong.

According to [9, Proposition 4.6], the almost strong lifting ξ can be modified into a strong lifting $\bar{\xi} \in \Lambda(\widehat{R})$.

It then follows that for each $U_1 \times U_2 \in \mathcal{T} \times \mathcal{S}$, we get

$$U_1 \times U_2 \in \tau_{\bar{\xi}} := \big\{E \in \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}\colon\ E \subseteq \bar{\xi}(E)\big\},$$

where $\tau_{\bar{\xi}}$ is one of the lifting topologies in $X \times Y$ associated with $\bar{\xi}$ (see [3] or [9]). Consequently, each element of $\mathcal{T} \times \mathcal{S}$ belongs to $\tau_{\bar{\xi}}$, and so $\mathcal{T} \times \mathcal{S} \subseteq \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$.

In particular, let $\mathfrak{A} = \mathfrak{B}(X)$, $\mathfrak{B} = \mathfrak{B}(Y)$, P, Q be completion regular and let $\mathfrak{B}_0(X \times Y) = \mathfrak{B}_0(X) \otimes \mathfrak{B}_0(Y)$. To show that R is completion regular, let us fix $A \times B \in \mathfrak{B}(X) \times \mathfrak{B}(Y)$. Since P, Q are completion regular, we get $A \times B \in \widehat{\mathfrak{B}}_0(X) \times \widehat{\mathfrak{B}}_0(Y)$, hence there exist $E, F \in \mathfrak{B}_0(X)$ and $G, M \in \mathfrak{B}_0(Y)$ such that $E \subseteq A \subseteq F$, $G \subseteq B \subseteq M$ and $P(F \setminus E) = 0 = Q(M \setminus G)$. Consequently, $E \times G, F \times M \in \mathfrak{B}_0(X \times Y)$, $E \times G \subseteq A \times B \subseteq F \times M$ and $P \otimes Q(F \times M \setminus E \times G) = 0$, hence $R(F \times M \setminus E \times G) = 0$, where the latter follows from the $P \otimes Q$-continuity of R. Therefore $A \times B \in \widehat{\mathfrak{B}}_{0R}(X \times Y)$, where $\widehat{\mathfrak{B}}_{0R}(X \times Y)$ is the completion of $\mathfrak{B}_0(X \times Y)$ with respect to R, hence $\mathfrak{B}(X) \mathbin{\widehat{\otimes}_R} \mathfrak{B}(Y) \subseteq \widehat{\mathfrak{B}}_{0R}(X \times Y)$. But from $\mathcal{T} \times \mathcal{S} \subseteq \mathfrak{B}(X) \mathbin{\widehat{\otimes}_R} \mathfrak{B}(Y)$ it follows that $\widehat{\mathfrak{B}}_R(X \times Y) = \mathfrak{B}(X) \mathbin{\widehat{\otimes}_R} \mathfrak{B}(Y)$, hence $\widehat{\mathfrak{B}}_R(X \times Y) \subseteq \widehat{\mathfrak{B}}_{0R}(X \times Y)$. Since the inverse inclusion is always true, we get $\widehat{\mathfrak{B}}_R(X \times Y) = \widehat{\mathfrak{B}}_{0R}(X \times Y)$, what proves that R is completion regular. This completes the proof. $\square$

It follows a corresponding result, where except of $\rho \in \Lambda(\widehat{Q})$ also a strong density $\tau \in \vartheta(\widehat{P})$ is given. In such a case we have to assume something more, since in general for given τ and ρ one cannot find strong liftings $\xi \in \Lambda(\widehat{R})$ and $\xi_y \in \Lambda(\widehat{S_y})$ for all $y \in Y$, satisfying conditions $SP(\xi)$ and $RF'(\xi, \eta)$ of Theorem 5.1 (see [5, Section 3, Remark 5]).

Theorem 5.2. *Let $(X, \mathcal{T}, \mathfrak{A}, P)$ be a topological probability space admitting a strong admissible density (see [6] or [9]) $\tau \in \vartheta(\widehat{P})$, let $(Y, \mathcal{S}, \mathfrak{B}, Q)$ be a topological probability space admitting a strong lifting $\rho \in \Lambda(\widehat{Q})$, and let R be an arbitrary probability on $\mathfrak{A} \otimes \mathfrak{B}$ with marginals P and Q.*

If $R \ll P \otimes Q$ then there exists a product r.c.p. $\{S_y\colon\ y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\widehat{\mathfrak{B}}$, and there exist a strong lifting $\xi \in \Lambda(\widehat{R})$ and a collection of strong liftings $\{\xi_y \in \Lambda(\widehat{S_y})\colon\ y \in Y\}$ such that the section property $SP(\xi)$ and the rectangle formula $RF'(\xi, \rho)$ hold true. Moreover $\mathcal{T} \times \mathcal{S} \subseteq \mathfrak{A} \mathbin{\widehat{\otimes}_R} \mathfrak{B}$.

Proof. It follows from [6, Corollary 2.2], that there exists a density $\varphi \in \vartheta(P \widehat{\otimes} Q)$ satisfying the conditions (1) and (2). It follows easily from condition (1) that φ is strong. Let N, $y_0 \in N^c$, H, H^y and η_y for all $y \in Y$, be as in the proof of Theorem 3.2. The rest of the proof is similar to that of Theorem 5.1. $\square$

Remark 5.3. The existence of a strong admissible density $\tau \in \vartheta(\widehat{P})$ is guaranteed by [4, Theorem 2.1] for non-atomic and strictly positive topological probability spaces $(X, \mathcal{T}, \mathfrak{A}, P)$ with a countable basis $\{B_n\colon n \in \mathbf{N}\}$ for the topology $\mathcal{T}$ such that $P(\partial B_n) = 0$ for all n (if $\mathcal{T}$ is regular, then it is metrizable and this condition is satisfied).

6. An application to stochastic processes

The next result is about measurable lifting modification of a measurable process in case of a probability measure R being not necessarily a product probability but still absolutely continuous with respect to the product measure. It improves Theorem 5.1 from [10]. For the terminology we apply in it we refer to [10, Section 5].

Theorem 6.1. *Let R be a probability with marginals P and Q such that $R \ll P \otimes Q$. Then there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$, which is absolutely continuous with respect to P such that for each bounded measurable stochastic process $\{\Phi_y\}_{y \in Y}$ on $(X, \mathfrak{A}, P)$ there exist a collection $\widetilde{\Psi} := \{\Psi_y\}_{y\in Y}$ of $\widehat{S}_y$-measurable functions Ψ_y on X and a collection of liftings $\sigma_y \in \Lambda(\widehat{S}_y)$, $y \in Y$, such that*

(i) *$\Phi_y = \Psi_y$ a.e. (S_y) for all $y \in Y$;*
(ii) *$\Psi_y = \sigma_y(\Psi_y)$ for all $y \in Y$;*
(iii) *the map $\widetilde{\Psi} : X \times Y \to (-\infty, +\infty)$ is $\widehat{R}$-measurable.*

Proof. In view of Theorem 3.3 there exists a product r.c.p. $\{S_y\colon y \in Y\}$ on $\mathfrak{A}$ for R with respect to $\mathfrak{B}$, which is absolutely continuous with respect to P, and there exist a lifting $\xi \in \Lambda(\widehat{R})$ and a family $\{\sigma_y \in \Lambda(\widehat{S}_y)\colon y \in Y\}$ of liftings such that given process $\widetilde{\Phi} = \{\Phi_y\}_{y\in Y}$, we have

$$\left[\xi(\widetilde{\Phi})\right]^y = \sigma_y\left(\left[\xi(\widetilde{\Phi})\right]^y\right) \quad \text{for all } y \in Y.$$

Since ξ is a lifting there exists $N_{\widetilde{\Phi}} \in \mathfrak{B}_0$ such that

$$\Phi_y = \left[\xi(\widetilde{\Phi})\right]^y \quad \text{a.e. } (S_y) \text{ for all } y \notin N_{\widetilde{\Phi}}.$$

We define now a collection $\widetilde{\Psi} := \{\Psi_y\}_{y\in Y}$ of S_y-measurable functions on X by setting

$$\Psi_y = \sigma_y(\Phi_y) \quad \text{for each } y \in Y.$$

Since $\xi(\widetilde{\Phi})$ is $\widehat{R}$-measurable, one can easily see that $\{\Psi_y\}_{y\in Y}$ satisfies the required conditions. $\square$

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