J. Math. Anal. Appl. 386 (2012) 364–377

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiplicity results for a differential inclusion problem with non-standard growth ☆

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ARTICLE INFO

Article history:
Received 9 November 2010
Available online 10 August 2011
Submitted by P. Broadbridge

Keywords:
$p(x)$-Laplacian
Differential inclusions
Motreanu–Panagiotopoulos type functional
Critical points

ABSTRACT

In this paper we examine the multiplicity of solutions of a differential inclusion problem involving $p(x)$-Laplacian of the type

$$[P_\lambda]\quad \begin{cases} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in \partial j_1\big(x, u(x)\big) + \lambda \partial j_2\big(x, u(x)\big), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

By using the nonsmooth version of Ricceri variational principle we get three critical points of the corresponding energy Motreanu–Panagiotopoulos type functional, which are the solutions of (P_λ).

1. Introduction

In the past decade, the existence and multiplicity of solutions for elliptic type partial differential equations with different boundary value conditions have been widely investigated by many authors, which are usually reduced to the solutions of many of Dirichlet and Neumann type problems and obstacle problems with all kinds of nonlinearities. During the authors studied these problems, the variational methods to a class of non-differentiable functionals which is extended by Chang [7] often applied directly to prove some existence and multiplicity theorems. Later, Ricceri in his paper [27] introduced another variational principle for the problems whose corresponding energy functional is Gateaux differentiable. After that, the result of Ricceri has been extended by Marano and Motreanu [23] to a large class of non-differentiable functionals. From then on, many authors used this result to study all kinds of Dirichlet and Neumann boundary value problems involving the p-Laplacian with discontinuous nonlinearities.

The hemivariational and variational-hemivariational inequalities of Dirichlet and Neumann boundary value problems whose corresponding energy functional is called a Motreanu–Panagiotopoulos type functional have been considered by many authors, such as S.A. Marano and N.S. Papageorgiou in [24] by the extended variational methods due to Chang [7] to examine the following elliptic variational-hemivariational inequality

$$(P_1)\quad \begin{cases} -\Delta u \in \partial J(x,u) - \partial G(x,u), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

☆ Supported by National Nature Science Foundation of China (10971194), Natural Science Foundation of Zhejiang Province (Y7080008).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.08.015

in which the authors get two existence results for (P_1), the first basically through the nonsmooth mountain pass theorem, while another is given through appropriate assumptions on $J(x,u)$ and $G(x,u)$ such that the energy functional possesses a global minimum, which turn out to be a critical point. Besides, similarly to [24], M.E. Filippakis and N.S. Papageorgiou in [14] examined a resonant variational inequality driven by the p-Laplacian and with a nonsmooth potential, and also through the nonsmooth critical point theory for Motreanu–Panagiotopoulos type functional to obtain the existence of a nontrivial solution and nontrivial positive solutions. The nonsmooth version of Ricceri variational principle is also another important tool during the authors considering the hemivariational and variational-hemivariational inequalities with discontinuous nonlinearities. For example, the following differential inclusion problem (P_2)

$$(P_2)\quad \begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F\big(u(x)\big) + \mu\beta(x)\partial G\big(u(x)\big), & \text{in } \mathbb{R}^N,\\ u(x)\to 0, & \text{as } |x|\to\infty, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. A. Kristály, W. Marzantowicz and C. Varga in [17], as an applications of nonsmooth three critical points theorems, studied the three solutions of (P_2) when $p > N \geqslant 2$. Moreover, G. Bonanno and N. Giovannelli in [4] deal with an eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities. When $p > N$, through nonsmooth version of B. Ricceri's three existence theorem which is extended by Marano and Motreanu, the authors also proved the multiplicity result. There are a lot of people treated the elliptic hemivariational inequalities with the corresponding energy functional, i.e., Motreanu–Panagiotopoulos type functional. We refer the readers to the references [30,5,3, 20,19,16].

In the latest years the study of nonlinear partial differential equations with non-standard growth conditions has been the object of increasing amount of attention. The problems of this type are studied in the behavior of electrorheological fluids (see M. Růžička [29]), and of other phenomena related to image processing, elasticity and the flow in porous media (see V.V. Zhikov [31]). For the application backgrounds we also can trace the book of L. Diening [11] and the papers of E. Acerbi and G. Mingione [1,2], and M. Mihăilescu, V. Rădulescu [25]. Therefore, it is no surprised that the nonsmooth version of Ricceri variational principle which is established by Marano and Motreanu is used to study the various mathematical problems with $p(x)$-growth conditions. The purpose of the present paper is by the extension of the three critical points theorem of Ricceri to discuss the existence of solutions of the following $p(x)$-Laplacian equation

$$(P_\lambda)\quad \begin{cases} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in \partial j_1\big(x,u(x)\big) + \lambda\partial j_2\big(x,u(x)\big), & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian operator, $\lambda > 0$, $\Omega \subset R^N$ $(N > 2)$ is a nonempty bounded domain with a boundary $\partial\Omega$ of class C^1, $p(x) > 0$, $p(x) \in C(\bar\Omega)$ with $1 < p^- = \inf_{x\in\Omega} p(x)$, $p^+ = \sup_{x\in\Omega} p(x)$, $V(x) \in L^\infty(\Omega)$ is a function possibly changing sign, $j_1(x,\zeta)$ and $j_2(x,\zeta)$ are locally Lipschitz functions in the ζ-variable integrand (in general it can be nonsmooth), and $\partial j_1(x,\zeta)$ and $\partial j_2(x,\zeta)$ are the subdifferential with respect to the ζ-variable in the sense of Clarke [8], and n is the outward unit normal on $\partial\Omega$.

The above mentioned variational principle is also widely used to the variational problems with $p(x)$-growth conditions, such as X.L. Fan and S.G. Deng in [12] studied $p(x)$-Laplacian equation with Neumann, Dirichlet problem:

$$-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega,$$

where the function satisfies $a(x) \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_{x\in\Omega} = a_0 > 0$. By using the variational principle of Ricceri and the local Mountain pass lemma one gets the multiplicity of solutions of the problem with $\lambda = 1$. While, through the same variational principle, F. Cammarotoa, A. Chinnì and B. Di Bella in [6] studied the problem when λ is an arbitrary positive constant. More generality, G.W. Dai in [9] applied the version of nonsmooth three critical points theorem to examine the Neumann type differential inclusion problems involving $p(x)$-Laplacian:

$$(P_3)\quad \begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + \mu|u|^{p(x)-2}u = \lambda\partial F(x,u), & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} = \varphi, & \text{on } \partial\Omega. \end{cases}$$

Besides, by the same theory, he and his partner W.L. Liu in [10] also considered the Dirichlet problem with nonsmooth potential when $\mu = 0$ in (P_3). For other results, by using the B. Ricceri's three critical points theorem or the nonsmooth version of B. Ricceri's variational principle to consider the quasilinear elliptic equation and elliptic systems involving $p(x)$-Laplacian, we refer the readers to the references [21,22,15].

We must point out that the $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian. For example, it is inhomogeneous and, in general, it does not have the "first eigenvalue". In other words, the infimum of the eigenvalues of $p(x)$-Laplacian equals 0. Moreover, the compact theorem of the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ has a more strict requirement. Therefore, to get the three solutions for (P_λ) by the nonsmooth version of Ricceri variational principle, we have to get over several difficulties, and the contribution of this paper can be briefly described as follows:

1. One of the aims of the present paper is to improve and generalize the results of [17] to the case of variable exponent $p(x)$-Laplacian. Since the property of the first eigenvalue of $p(x)$-Laplacian is not the same as the p-Laplacian, namely the first eigenvalue is not isolated (see [13]), therefore, the first difficulty is we cannot use the eigenvalue property of $p(x)$-Laplacian.

2. We study problem (P_λ) from a more extensive viewpoint. We want to give a progression of the results of [12,6] to the nonsmooth case. Unfortunately, the technique and assumptions on nonlinearities in [6] is not applicable here since the discontinuous nonlinearities in (P_λ). Here, we use another technique which is simpler and more direct in this paper. Besides, the indefinite weight function $V(x)$ in (P_λ) is another difficulty.

3. We note that there is a key assumption on the exponent that $p > N$ or $p^- > N$, such as in the references [17,4,6,9] and so on. For the detailed reason of this assumption see Remark 3.1. Here, we retreat this restriction on variable exponent $p(x)$.

This paper is divided into three sections, in Section 2, we introduce some basic properties of the generalized Lebesgue space $L^{p(x)}(\Omega)$ and the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$, and the generalized gradient of locally Lipschitz function. In Section 3, we give suitable hypotheses on $j_1(x,\zeta)$ and $j_2(x,\zeta)$, and use the nonsmooth three critical points theorems to prove the existence results for problem (P_λ).

2. Mathematical preliminaries

In this section we introduce some auxiliary results which is necessary for us to discuss problem (P_λ). Set

$$C_+(\bar{\Omega}) = \{h \mid h(x) \in C(\Omega),\ h(x) > 1,\ \text{for any } x \in \bar{\Omega}\}.$$

Firstly, define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{u \;\middle|\; u \text{ is a measurable real-valued function } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}$$

with the norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\};$$

and generalized Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Then $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces. For brevity we shall write $X = W^{1,p(x)}(\Omega)$. For $V(x) \in L^\infty(\Omega)$, we set $V^+ = \max\{V(x), 0\}$ and $V^- = \max\{-V(x), 0\}$. Thus we have $V(x) = V^+(x) - V^-(x)$. Define $J : X \to R$ by

$$J(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V^+|u|^{p(x)}\right) dx, \quad \forall u \in X.$$

Then J is even, $J \in C^1(X, R)$ and

$$\langle J'(u), v\rangle_X = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + V^+(x)|u|^{p(x)-2}uv\,dx, \quad \forall u, v \in X,$$

where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X^* and X.

Proposition 2.1. *(See [12].) The function $J : X \to R$ is convex. The mapping $J' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \limsup_{n\to\infty}\langle J'(u_n), u_n - u\rangle \leqslant 0, \quad \text{implies } u_n \to u.$$

Proposition 2.2. *(See [12].)* (1) *If $q(x) \in C_+(\bar{\Omega})$ and $q(x) \leqslant p^*(x)$, $\forall x \in \bar{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geqslant N. \end{cases}$$

(2) *If $p_1(x), p_2(x) \in C_+(\bar{\Omega})$, and $p_1(x) \leqslant p_2(x)$, $\forall x \in \bar{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the imbedding is continuous.*

Let X be a Banach space and X^* its topological dual. A function $\varphi : X \to R$ is said to be locally Lipschitz, if for every $u \in X$ there exist a neighborhood U of u and a constant $K > 0$, such that $|\varphi(v) - \varphi(w)| \leqslant K|v - w|$ for every $v, w \in U$. From convex analysis we know that a proper, convex and lower semicontinuous function $g : X \to \bar{R} = R \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} g = \{u \in X\colon\ g(u) < +\infty\}$. For each $h \in X$, we define the generalized directional derivative of φ at u in the direction h by

$$\varphi^\circ(u,h) = \limsup_{w \to u,\, t \to 0} \frac{\varphi(w + th) - \varphi(w)}{t}.$$

It is easy to check that the function $X \ni h \to \varphi^\circ(u,h)$ is sublinear and continuous, by the Hahn–Banach theorem it is the support function of a nonempty, convex and W^*-compact set $\partial\varphi(u) \subseteq X^*$, defined by

$$\partial\varphi(u) = \bigl\{u^* \in X^*\colon \langle u^*, h\rangle_X \leqslant \varphi^\circ(u,h);\ \forall h \in X\bigr\}.$$

The set $\partial\varphi(u)$ is known as the subdifferential of φ at x.

Proposition 2.3. *(See [8].) Let $f, g : X \to R$ be two locally Lipschitz functions. Then*

(1) $f^\circ(u;h) = \max\{\langle \xi, h\rangle\colon\ \xi \in \partial f(u)\}$;
(2) $(f+g)^\circ(u;h) \leqslant f^\circ(u;h) + g^\circ(u;h)$;
(3) *Let $j : X \to R$ be a continuously differentiable function. Then $\partial j(u) = \{j'(u)\}$, $j^\circ(u;h)$ coincides with $\langle j'(u), h\rangle_X$ and $(f+j)^\circ(u;h) = f^\circ(u;h) + \langle j'(u), h\rangle_X$ for all $u, h \in X$;*
(4) *$(-f)^\circ(u;h) = f^\circ(u;-h)$, and $f^\circ(u;\kappa h) = \kappa f^\circ(u;h)$ for every $\kappa > 0$;*
(5) *The function $(u,h) \to f^\circ(u;h)$ is upper semicontinuous.*

Let I be a function on X satisfying the following structure hypothesis (I is called a Motreanu–Panagiotopoulos type functional):

(**H**) $I = \Phi + \Psi$, where $\Phi : X \to R$ is locally Lipschitz while $\Psi : X \to R \cup \{+\infty\}$ is convex, proper, and lower semicontinuous.

We say that $u \in X$ is a critical point of I if it fulfills the inequality

$$\Phi^\circ(u; v - u) + \Psi(v) - \Psi(u) \geqslant 0, \quad \forall v \in X.$$

Set $K := \{u \in X \mid u \text{ is a critical point of } I\}$ and $K_c = K \cap I^{-1}(c)$. A number $c \in R$ such that $K_c \neq 0$ is called a critical value of I.

Definition 2.4. (See [26].) $I = \Phi + \Psi$ is said to satisfy the Palais–Smale condition at level $c \in R$ (shortly, $(PS)_c$) if every sequence $\{u_n\}$ in X satisfying $I(u) \to c$ and

$$\Phi^\circ(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\}$ in $[0,\infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in R$, I is said to satisfy the Palais–Smale condition (shortly, (PS)).

Proposition 2.5. *(See [23, Theorem B].) Let X be a separable and reflexive Banach space, let $I_1 := \Phi_1 + \Psi_1$ and $I_2 := \Phi_2$ be like in* (**H**)*, let Λ be a real interval. Suppose that*:

(b_1) *Φ_1 is weakly sequentially lower semicontinuous while Φ_2 is weakly sequentially continuous.*
(b_2) *For every $\lambda \in \Lambda$ the function $I_1 + \lambda I_2$ fulfills $(PS)_c$, $c \in R$, together with $\lim_{\|u\| \to +\infty}(I_1(u) + \lambda I_2(u)) = +\infty$.*
(b_3) *There exists a continuous concave function $h : \Lambda \to R$ satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} \bigl(I_1(u) + \lambda I_2(u) + h(\lambda)\bigr) < \inf_{u \in X} \sup_{\lambda \in \Lambda} \bigl(I_1(u) + \lambda I_2(u) + h(\lambda)\bigr).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X. Moreover; if $\Psi_1 \equiv 0$, then there exist an open interval $\Lambda_1 \subseteq \Lambda$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X having norms less than $\sigma > 0$.

Proposition 2.6. *(See [28].) Let X be a non-empty set and Φ, Ψ two real functionals on X. Assume that there are $\gamma > 0$, $u_0, u_1 \in X$, such that*

$$\Phi(u_0)=\Psi(u_0)=0, \qquad \Phi(u_1)>\gamma,$$
$$\sup_{u\in\Phi^{-1}((-\infty,\gamma])}\Psi(u)<\gamma\frac{\Psi(u_1)}{\Phi(u_1)}.$$

Then, for each ρ satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,\gamma])}\Psi(u)<\rho<\gamma\frac{\Psi(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geqslant 0}\inf_{u\in X}\big(\Phi(u)+\lambda\big(\rho-\Psi(u)\big)\big)<\inf_{u\in X}\sup_{\lambda\geqslant 0}\big(\Phi(u)+\lambda\big(\rho-\Psi(u)\big)\big).$$

3. Main result

In this section we prove the existence theorem for problem (P_λ). We first fix some notation. The energy functional $\varphi: X\to R$ corresponding to problem (P_λ) is given by

$$\begin{aligned}\varphi(u)&=\int_\Omega\frac{1}{p(x)}\big(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\big)\,dx-\int_\Omega j_1(x,u)\,dx-\lambda\int_\Omega j_2\big(x,u(x)\big)\,dx\\&=\int_\Omega\frac{1}{p(x)}\big(|\nabla u|^{p(x)}+V^+(x)|u|^{p(x)}\big)\,dx-\int_\Omega j_1(x,u)\,dx-\lambda\left\{\int_\Omega j_2\big(x,u(x)\big)\,dx+\frac{1}{\lambda}\int_\Omega\frac{1}{p(x)}V^-(x)|u|^{p(x)}\,dx\right\}.\end{aligned}\tag{3.1}$$

Set

$$\begin{aligned}&A_1(u)=-\int_\Omega j_1(x,u)\,dx; \qquad J(u)=\int_\Omega\frac{1}{p(x)}\big(|\nabla u|^{p(x)}+V^+|u|^{p(x)}\big)\,dx;\\&A_2(u)=-\int_\Omega j_2(x,u)\,dx; \qquad L(u)=-\frac{1}{\lambda}\int_\Omega\frac{1}{p(x)}V^-|u|^{p(x)}\,dx,\end{aligned}\tag{3.2}$$

and

$$h_1=A_1+J, \qquad h_2=A_2+L,\tag{3.3}$$

then, under these notations, $\varphi=h_1+\lambda h_2$. Let $V_{p(x)}=\{u\in W^{1,p(x)}(\Omega)\colon \int_\Omega u\,dx=0\}$, then $V_{p(x)}$ is a closed linear subspace of $W^{1,p(x)}(\Omega)$ with codimension 1, and we have $W^{1,p(x)}(\Omega)=\mathbb{R}\oplus V_{p(x)}$ (see [13]). If we define the norm by

$$\|u\|'=\inf\left\{\lambda>0\colon \int_\Omega\left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)}+V^+(x)\left|\frac{u}{\lambda}\right|^{p(x)}\right)dx\leqslant 1\right\},$$

by Proposition 2.6 in [13], it is easy to see that $\|u\|'$ is an equivalent norm in $V_{p(x)}$. Thereafter, we will also use $\|\cdot\|$ to denote the equivalent norm $\|\cdot\|'$ in $V_{p(x)}$.

It is easy to see that, when $u\in V_{p(x)}$, we have

(i) if $\|u\|<1$, then $\frac{1}{p^+}\|u\|^{p^+}\leqslant J(u)\leqslant\frac{1}{p^-}\|u\|^{p^-}$;
(ii) if $\|u\|>1$, then $\frac{1}{p^+}\|u\|^{p^-}\leqslant J(u)\leqslant\frac{1}{p^-}\|u\|^{p^+}$.

Moreover, put

$$\lambda_1:=\inf_{u\in V_{p(x)}\setminus\{0\},\ \|u\|>1}\frac{\int_\Omega\frac{1}{p(x)}(|\nabla u|^{p(x)}+V^+|u|^{p(x)})\,dx}{\int_\Omega|u|^{q(x)}\,dx}.$$

If $q^+<p(x)$, $\forall x\in\Omega$, by continuous embedding of X in $L^{p^-}(\Omega)$ and, by Proposition 2.2(2), the above inequalities of (i), (ii), we have that $\lambda_1>0$.

Our assumptions on the data of (P_λ) are the following:

$H(j_1): j_1:\Omega\times R\to R$ is a function such that $j_1(x,\zeta)$ satisfies $j_1(x,0)=0$ and also

$(j_1)_1$ for all $\zeta \in R$, $\Omega \ni x \mapsto j_1(x,\zeta) \in R$ is measurable;
$(j_1)_2$ for almost all $x \in \Omega$, $R \ni \zeta \mapsto j_1(x,\zeta) \in R$ is locally Lipschitz;
$(j_1)_3$ for almost all $x \in \Omega$, all $\zeta \in R$ and all $w \in \partial j_1(x,\zeta)$, we have

$$|w| \leqslant a(x)|\zeta|^{p^- -1}, \quad \text{with } a(x) \in L^\infty(\Omega)_+;$$

$(j_1)_4$ there exist $q(x), s(x) \in C_+(\bar{\Omega})$ satisfying $q^+ < p(x) < s^-$, $\forall x \in \Omega$, such that

$$\limsup_{|\zeta|\to 0} \frac{j_1(x,\zeta)}{|\zeta|^{q(x)}} < -2\lambda_1,$$

and

$$\limsup_{|\zeta|\to \infty} \frac{j_1(x,\zeta) - \hat{a}(x)|\zeta|^{p^-}}{|\zeta|^{s^-}} < 0, \quad \text{with } \hat{a}(x) \in L^\infty(\Omega)_+,$$

uniformly for almost all $x \in \Omega$.

$H(j_2)$: $j_2 : \Omega \times R \to R$ is a function, such that $j_2(x,\zeta)$ satisfies $j_2(x,0) = 0$ and also
$(j_2)_1$ for all $\zeta \in R$, $\Omega \ni x \mapsto j_2(x,\zeta) \in R$ is measurable;
$(j_2)_2$ for almost all $x \in \Omega$, $R \ni \zeta \mapsto j_2(x,\zeta) \in R$ is locally Lipschitz;
$(j_2)_3$ for almost all $x \in \Omega$, all $\zeta \in R$ and all $v(x) \in \partial j_2(x,\zeta)$, we have

$$|v| \leqslant b(x)|\zeta|^{p(x)-1} \quad \text{with } b(x) \in L^\infty(\Omega)_+;$$

$(j_2)_4$ there exists $\mathcal{K} > 0$, for all $0 < |\zeta| < \mathcal{K}$, such that $p(x)j_2(x,\zeta) > -\frac{1}{\lambda}V^-(x)|\zeta|^{p(x)}$, and

$$p(x)j_2(x,\zeta) + \frac{1}{\lambda}V^-(x)|\zeta|^{p(x)} = o\big(|\zeta|^{p^+}\big) \quad \big(|\zeta| \to 0\big); \qquad \limsup_{\zeta\to\infty} \frac{j_2(x,\zeta)}{|\zeta|^{p^+}} < 0,$$

uniformly for almost all $x \in \Omega$.

Remark 3.1. Recently papers on the three solutions or infinitely many solutions for the partial differential equations with non-standard growth conditions have been the object of increasing amount of attention. Many authors often give the crucial hypothesis on the variable exponent of $N < p^-$, which is necessary of the compact imbedding form $W^{1,p(x)}(\Omega)$ to $C^0(\Omega)$ or $L^\infty(\Omega)$. Besides, due to the complexity of problem which they considered, this assumption may be more convenient for them to deal with the problems. In the present paper we abandon this restriction on $p(x)$. Moreover, when $N < p^-$, the next lemma is our appendant result, we list it below.

Lemma 3.1. *For every $p(x) \in C_+(\bar{\Omega})$ with $N < p^-$, there exists a function $r(x) \in C_+(\bar{\Omega})$ satisfying $r^+ < p(x) < r^*(x)$, $\forall x \in \Omega$.*

Proof. Let us suppose that $\mu : \Omega \to R$ is a continuous function, satisfying $0 < \mu(x) < \min\{\ln p^-, \ln(p^-/p^+ + 1)\}$, $\forall x \in \Omega$. Define a new function $f(t) = p^- \exp(-t)$, it is easy to see that $f(t)$ is a decreasing function. Set $r(x) = p^- \exp(-\mu(x))$, then the compound function $r(x)$ is of $C(\bar{\Omega})$. By the properties of decreasing function and $\mu(x) < \ln p^-$, $\forall x \in \Omega$, one has $f(\ln p^-) < f(\mu(x))$, note that $f(\ln p^-) = 1$, it follows that $f(\mu(x)) > 1$, $\forall x \in \Omega$. Therefore $r(x) \in C_+(\bar{\Omega})$. Besides, by $\mu(x) > 0$, we have $r(x) < p^-$, $\forall x \in \Omega$, it follows that $r^+ < p^-$. Next we show $p(x) < r^*(x)$, $\forall x \in \Omega$. If $r(x) \geqslant N$, then $r^*(x) = \infty$, it is obvious that $p(x) < r^*(x)$. If $r(x) < N$, then from $\mu(x) < \ln(p^-/p^+ + 1)$ and keeping in mind $N < p^-$, we get

$$\begin{aligned}
&\exp\big(\mu(x)\big) < \frac{p^-}{p^+} + 1 < \frac{Np^- + p^- p(x)}{Np(x)} \\
&\quad \Longrightarrow \quad \exp\big(-\mu(x)\big)\big(Np^- + p^- p(x)\big) > Np(x) \\
&\quad \Longrightarrow \quad Np^- \exp\big(-\mu(x)\big) > p(x)\big(N - p^- \exp\big(-\mu(x)\big)\big),
\end{aligned}$$

namely, $Nr(x) > p(x)(N - r(x))$, since $r(x) < N$, then $p(x) < r^*(x)$. This completes the proof. $\square$

We now state our main result of this section.

Theorem 3.1. *If hypotheses $H(j_1)$ and $H(j_2)$ hold, then there exists $\Lambda_1 \subseteq [0, +\infty)$, such that for each $\lambda \in \Lambda_1$, problem (P_λ) has at least three nontrivial solutions in $V_{p(x)}$.*

Before proving Theorem 3.1, we first prove the following lemmas which are useful for the proof of this theorem.

Lemma 3.2. *Since j_i are locally Lipschitz functions which satisfy $(j_i)_3$, then A_i in* (3.2) *are well defined and they are locally Lipschitz. Moreover, let E be a closed subspace of X and $A_i|_E$ the restriction of A_i to E, where $i = 1$ or 2. Then*

$$(A_i|_E)^\circ(u; v) \leqslant \int_\Omega (-j_i)^\circ\big(x, u(x); v(x)\big)\,dx, \quad \text{for all } u, v \in E.$$

Since the proof of the above lemma is similar to that of [18, Lemma 4.2], we shall omit it here.

Lemma 3.3. *For any $\varepsilon > 0$, and $q(x)$ as mentioned in $(j_1)_4$ there exists a $u_\varepsilon \in V_{p(x)}$ with $\|u_\varepsilon\| > 1$, such that*

$$\frac{1}{p^+}\|u_\varepsilon\|^{p^-} + \lambda_1 \int_\Omega |u_\varepsilon|^{q(x)}\,dx \geqslant \frac{\varepsilon + 2\lambda_1}{p^+(\lambda_1+\varepsilon)}\|u_\varepsilon\|^{p^-}.$$

Proof. Since $\lambda_1 := \inf_{u\in V_{p(x)}\setminus\{0\},\|u\|>1} \frac{\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + V^+|u|^{p(x)})\,dx}{\int_\Omega |u|^{q(x)}\,dx}$, then by the definition of infimum, for every $\varepsilon > 0$, there exists a $u_\varepsilon \in V_{p(x)}$ with $\|u_\varepsilon\| > 1$, such that

$$\frac{\int_\Omega \frac{1}{p(x)}(|\nabla u_\varepsilon|^{p(x)} + V^+|u_\varepsilon|^{p(x)})\,dx}{\int_\Omega |u_\varepsilon|^{q(x)}\,dx} < \lambda_1 + \varepsilon.$$

It follows that

$$\frac{\lambda_1}{\lambda_1+\varepsilon} < \frac{\lambda_1\int_\Omega |u_\varepsilon|^{q(x)}\,dx}{\int_\Omega \frac{1}{p(x)}(|\nabla u_\varepsilon|^{p(x)} + V^+|u_\varepsilon|^{p(x)})\,dx} \leqslant \frac{\lambda_1\int_\Omega |u_\varepsilon|^{q(x)}\,dx}{\frac{1}{p^+}\|u_\varepsilon\|^{p^-}}$$

therefore, we get

$$\frac{1}{p^+}\|u_\varepsilon\|^{p^-} + \lambda_1 \int_\Omega |u_\varepsilon|^{q(x)}\,dx \geqslant \frac{1}{p^+}\left(1 + \frac{\lambda_1}{\lambda_1+\varepsilon}\right)\|u_\varepsilon\|^{p^-}.$$

This completes the proof. $\square$

Lemma 3.4. *There exist a $u^\star \in V_{p(x)}$ with $u^\star \neq 0$ and, $\gamma^\star > 0$ such that $h_1(u^\star) > \gamma^\star$, where $h_1 = J + A_1$ is as mentioned in* (3.3).

Proof. Case I: By the virtue of assumptions of $(j_1)_4$, there exists a $\delta_0 > 0$ small enough (without loss of generality we assume $\delta_0 < 1$), such that for almost all $x \in \Omega$, one has

$$j_1(x,\zeta) \leqslant -2\lambda_1|\zeta|^{q(x)}, \quad \forall|\zeta| \leqslant \delta_0, \tag{3.4}$$

besides by $(j_1)_4$, we know that there exists $\tau > 0$ such that

$$\limsup_{|\zeta|\to\infty} \frac{j_1(x,\zeta) - \hat{a}(x)|\zeta|^{p^-}}{|\zeta|^{s^-}} < -2\tau,$$

uniformly for almost all $x \in \Omega$. So we can find an $M > 1$ large enough such that for almost all $x \in \Omega$, all ζ such that $|\zeta| \geqslant M$, we have

$$\limsup_{|\zeta|\to\infty} \frac{j_1(x,\zeta) - \hat{a}(x)|\zeta|^{p^-}}{|\zeta|^{s^-}} < -\tau,$$

it immediately follows

$$j_1(x,\zeta) \leqslant \hat{a}(x)|\zeta|^{p^-} - \tau|\zeta|^{s^-}, \quad \forall|\zeta| > M. \tag{3.5}$$

On the other hand, since $j_1(x,0) = 0$ and by $(j_1)_3$, then from the Lebourg mean value theorem (see [3]) for almost all $x \in \Omega$

$$j_1(x,\zeta) \leqslant a(x)|\zeta|^{p^-}, \quad \forall\delta_0 < |\zeta| \leqslant M. \tag{3.6}$$

Note that $a(x) \in L^\infty(\Omega)_+$, $p^- > q^+$ and, $\delta_0 < 1$ imply $-\frac{2\lambda_1}{\delta_0^{p^-}} < -2\lambda_1$, for almost all $x \in \Omega$, from (3.4) and (3.6) we have

$$j_1(x,\zeta) \leqslant -2\lambda_1|\zeta|^{q(x)} + \left(a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-}, \quad \forall|\zeta| \leqslant M, \tag{3.7}$$

when $|\zeta| > M > \delta_0$. Since $\delta_0 < 1$ and $p^- > q^+$, then $(\frac{|\zeta|}{\delta_0})^{p^-} > |\zeta|^{q(x)}$, which shows that

$$-2\lambda_1|\zeta|^{q(x)} + \left(a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-} > 0. \tag{3.8}$$

Thus through (3.5), for almost all $x \in \Omega$, and (3.8) one has

$$j_1(x,\zeta) \leqslant \left(a(x) + \hat{a}(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-} - \tau|\zeta|^{s^-} - 2\lambda_1|\zeta|^{q(x)}, \quad \forall|\zeta| > M. \tag{3.9}$$

Similarly to the above, note $q^+ < p^- < s^-$, from (3.7) it follows that

$$\begin{aligned} j_1(x,\zeta) &\leqslant -2\lambda_1|\zeta|^{q(x)} + \left(a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-} - \frac{a(x)}{M^{p^-}}|\zeta|^{p^-} - \frac{\tau}{M^{s^-}}|\zeta|^{s^-} + a(x)\left|\frac{\zeta}{M}\right|^{q(x)} + \tau\left|\frac{\zeta}{M}\right|^{q(x)} \\ &\leqslant \left(-2\lambda_1 + \frac{a(x)}{M^{q(x)}} + \frac{\tau}{M^{q(x)}}\right)|\zeta|^{q(x)} - \frac{\tau}{M^{s^-}}|\zeta|^{s^-} + \left(a(x) - \frac{a(x)}{M^{p^-}} + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-}, \quad \forall|\zeta| \leqslant M. \end{aligned} \tag{3.10}$$

Therefore, note that $M > 1$, from (3.9) and (3.10), for almost all $x \in \Omega$ and all $\zeta \in R$, it follows that

$$j_1(x,\zeta) \leqslant \left(-2\lambda_1 + \frac{a(x)+\tau}{M^{q(x)}}\right)|\zeta|^{q(x)} + \left(a(x) + \hat{a}(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-} - \frac{\tau}{M^{s^-}}|\zeta|^{s^-},$$

we can choose M_0 so large that $\frac{a(x)+\tau}{M_0^{q(x)}} < \lambda_1$, without loss of generality we suppose that $a(x) > \hat{a}(x) > 0$, $\forall x \in \Omega$. Then the above inequality becomes

$$j_1(x,\zeta) \leqslant -\lambda_1|\zeta|^{q(x)} + \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|\zeta|^{p^-} - \frac{\tau}{M_0^{s^-}}|\zeta|^{s^-}. \tag{3.11}$$

Therefore through the inequality (3.11), for any $u \in V_{p(x)}$ with $\|u\| > 1$, we have

$$\begin{aligned} h_1(u) &= J(u) + A_1(u) \\ &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V^+|u|^{p(x)}\right)dx - \int_\Omega j_1(x,u)\,dx \\ &\geqslant \frac{1}{p^+}\|u\|^{p^-} + \lambda_1\int_\Omega |u|^{q(x)}\,dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}\,dx - \int_\Omega \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-}\,dx. \end{aligned} \tag{3.12}$$

Since the imbedding from X to L^{p^-} is compact, then there is a constant $c_0 > 0$, such that $|u|_{p^-} < c_0\|u\|$, $\forall u \in X$. Besides from Lemma 3.3, for an $\varepsilon_0 > 0$ small enough we make sure that

$$\frac{\lambda_1}{p^+(\lambda_1 + \varepsilon_0)} > \left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)c_0.$$

Remind that $a(x) \in L^\infty(\Omega)_+$ where $\|\cdot\|_\infty$ denotes the norm of $L^\infty(\Omega)$, there are a $u_{\varepsilon_0} \in V_{p(x)}$ with $\|u_{\varepsilon_0}\| > 1$, gathering the inequality (3.12) and the above inequality, one has

$$\begin{aligned} h_1(u_{\varepsilon_0}) &= J(u_{\varepsilon_0}) + A_1(u_{\varepsilon_0}) \\ &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_{\varepsilon_0}|^{p(x)} + V^+|u_{\varepsilon_0}|^{p(x)}\right)dx - \int_\Omega j_1(x,u_{\varepsilon_0})\,dx \\ &\geqslant \frac{1}{p^+}\|u_{\varepsilon_0}\|^{p^-} + \lambda_1\int_\Omega |u_{\varepsilon_0}|^{q(x)}\,dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u_{\varepsilon_0}|^{s^-}\,dx - \int_\Omega \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u_{\varepsilon_0}|^{p^-}\,dx \\ &\geqslant \frac{1}{p^+}\left(1 + \frac{\lambda_1}{\lambda_1 + \varepsilon_0}\right)\|u_{\varepsilon_0}\|^{p^-} + \int_\Omega \frac{\tau}{M_0^{s^-}}|u_{\varepsilon_0}|^{s^-}\,dx - \left(2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}}\right)c_0\|u_{\varepsilon_0}\|^{p^-} \\ &\geqslant \frac{1}{p^+}\|u_{\varepsilon_0}\|^{p^-}. \end{aligned} \tag{3.13}$$

Defining $u_1 = u_{\varepsilon_0}$ and a constant $0 < \gamma_1 < \frac{1}{p^+}$, through (3.13) we have $h_1(u_1) > \gamma_1$.

Case II: We also can find a $u_2 \in V_{p(x)}$ with $\|u_2\| < 1$, and $\gamma_2 > 0$ satisfying $h_1(u_2) > \gamma_2$. In fact, similarly to (3.12) and (3.13), using (3.11), when $\|u\| < 1$, we get

$$h_1(u) \geqslant \frac{1}{p^+}\|u\|^{p^+} + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}\,dx + \int_\Omega \lambda_1|u|^{q(x)} - \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-}\,dx. \tag{3.14}$$

Set

$$K = \left\{\zeta \in R \;\middle|\; |\zeta| < \min\left(1, \left(\frac{\lambda_1\delta_0^{p^-}}{2\|a\|_\infty\delta_0^{p^-} + 2\lambda_1}\right)^{\frac{1}{p^- - q^+}}\right)\right\},$$

then in $V_{p(x)}$ we can choose a $u_2 \in K$, with $\|u_2\| < 1$, satisfying $\lambda_1|u_2|^{q(x)} - (2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}})|u_2|^{p^-} > 0$. By (3.14) we have

$$h_1(u_2) \geqslant \frac{1}{p^+}\|u_2\|^{p^+} + \frac{\tau}{M_0^{s^-}}\int_\Omega |u_2|^{s^-}\,dx \geqslant \frac{1}{p^+}\|u_2\|^{p^+} > 0.$$

From the above inequality we can find a $\gamma_2 > 0$ satisfying $h_1(u_2) > \gamma_2$. Combining the above two cases, there exist a $u^\star \in V_{p(x)}$ with $u^\star \neq 0$ and, $\gamma^\star > 0$ such that $h_1(u^\star) > \gamma^\star$. Thus we have completed the proof. $\square$

Lemma 3.5. *There exists a $\gamma > 0$ with $\gamma < \gamma^\star$ such that*

$$\sup_{u \in h_1^{-1}((-\infty,\gamma])\cap V_{p(x)}} \big(-h_2(u)\big) < \gamma\frac{-h_2(u^\star)}{h_1(u^\star)},$$

where $u^\star$, $\gamma^\star$ and $h_2 = A_2 + L$ are as mentioned in Lemma 3.4 *and in* (3.3) *respectively.*

Proof. Firstly, from the assumptions of $(j_2)_4$, for $\forall\varepsilon > 0$, $\exists\delta' > 0$, $\forall 0 < |\zeta| \leqslant \delta_1 < \min\{\delta', 1\}$, for almost all $x \in \Omega$, we have

$$j_2(x,\zeta) \leqslant -\frac{1}{\lambda p(x)}V^-|\zeta|^{p(x)} + \frac{1}{p(x)}\varepsilon|\zeta|^{p^+}. \tag{3.15}$$

Again, by $(j_2)_4$, for the above $\varepsilon > 0$, there exists a $C > 1$ large enough, for almost all $x \in \Omega$, such that

$$j_2(x,\zeta) \leqslant \varepsilon|\zeta|^{p^+}, \quad \forall|\zeta| > C. \tag{3.16}$$

Secondly, since $j_2(x,0) = 0$ and by $(j_2)_3$, and note $b(x) \in L^\infty(\Omega)_+$, then similarly to the proof of Lemma 3.4, for almost all $x \in \Omega$, through the Lebourg mean value theorem

$$j_2(x,\zeta) \leqslant c(x)|\zeta|^{\alpha(x)}, \quad \forall\delta_1 < |\zeta| \leqslant C, \tag{3.17}$$

where $c(x) \in L^\infty(\Omega)_+$ and $p^+ < \alpha(x) < p^*(x)$, $\forall x \in \Omega$. Gathering (3.16) and (3.17), for almost all $x \in \Omega$, it follows that

$$j_2(x,\zeta) \leqslant c(x)|\zeta|^{\alpha(x)} + \varepsilon|\zeta|^{p^+}, \quad \forall|\zeta| > \delta_1. \tag{3.18}$$

Keeping in mind that $p^+ < \alpha(x)$, $\forall x \in \Omega$, and $\delta_1 < 1$, (3.15) and (3.18) lead to

$$\begin{aligned} j_2(x,\zeta) &\leqslant c(x)|\zeta|^{\alpha(x)} + \varepsilon|\zeta|^{p^+} + \frac{1}{p(x)}\varepsilon|\zeta|^{p^+} - \frac{1}{\lambda p(x)}V^-\left|\frac{\zeta}{\delta_1}\right|^{p(x)} + \frac{1}{\lambda p(x)}V^-\left|\frac{\zeta}{\delta_1}\right|^{\alpha(x)} \\ &\leqslant -\frac{1}{\lambda p(x)}V^-\left|\frac{\zeta}{\delta_1}\right|^{p(x)} + \left(1 + \frac{1}{p(x)}\right)\varepsilon|\zeta|^{p^+} + \left(c(x) + \frac{V^-}{\lambda p(x)|\delta_1|^{\alpha(x)}}\right)|\zeta|^{\alpha(x)}, \quad \forall|\zeta| > \delta_1 \end{aligned} \tag{3.19}$$

and $j_2(x,\zeta) \leqslant -\frac{1}{\lambda p(x)}V^-|\zeta|^{p(x)} + \frac{1}{p(x)}\varepsilon|\zeta|^{p^+} + \varepsilon|\zeta|^{p^+} + c(x)|\zeta|^{\alpha(x)}$, $\forall|\zeta| \leqslant \delta_1$. Therefore, through the above two inequalities, for almost all $x \in \Omega$ and all $\zeta \in R$, we show that

$$j_2(x,\zeta) + \frac{1}{\lambda p(x)}V^-|\zeta|^{p(x)} \leqslant \left(1 + \frac{1}{p(x)}\right)\varepsilon|\zeta|^{p^+} + \left(c(x) + \frac{V^-}{\lambda p(x)|\delta_1|^{\alpha(x)}}\right)|\zeta|^{\alpha(x)}. \tag{3.20}$$

Define the function $g : [0,+\infty) \to \mathbb{R}$ by

$$g(t) = \sup\big\{-h_2(u)\colon u \in V_{p(x)} \text{ with } \|u\|^{p^+} \leqslant \eta t\big\},$$

where η is an arbitrary constant satisfying $\eta > 1$. Pay attention to the $\alpha(x) < p^*(x)$, $\forall x \in \Omega$, then from the compact imbedding of X to $L^{\alpha(x)}(\Omega)$ and the continuous imbedding to $L^{p^+}(\Omega)$, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$|u|_{p^+} < c_1\|u\|; \qquad |u|_{\alpha(x)} < c_2\|u\|, \quad \forall u \in X. \tag{3.21}$$

Since $h_2(u) = A_2(u) + L(u)$ (here, A_2 and L are stated in (3.2)), then from (3.20) and (3.21), we have

$$\begin{aligned} g(t) &\leqslant \left(1+\frac{1}{p^-}\right)\varepsilon|u|_{p^+}^{p^+} + c_3 \max\left\{|u|_{\alpha(x)}^{\alpha^+}, |u|_{\alpha(x)}^{\alpha^-}\right\} \\ &\leqslant \left(1+\frac{1}{p^-}\right)c_1^{p^+}\varepsilon\eta t + c_3 \max\left\{c_2^{\alpha^+}\eta^{\frac{\alpha^+}{p^+}}t^{\frac{\alpha^+}{p^+}}, c_2^{\alpha^-}\eta^{\frac{\alpha^-}{p^+}}t^{\frac{\alpha^-}{p^+}}\right\}, \end{aligned} \tag{3.22}$$

where $c_3 = (\|c\|_\infty + \frac{\|V\|_\infty}{\lambda p^-|\delta_1|^{\alpha^+}})$. On the other hand, by virtue of $(j_2)_4$, $g(t) > 0$ for $t > 0$. Furthermore, due to $\alpha^- > p^+$ and the arbitrariness of $\varepsilon > 0$, we deduce

$$\lim_{t\to 0^+}\frac{g(t)}{t} = 0. \tag{3.23}$$

By Lemma 3.4, we know $h_1(u^\star) > 0$, it is obvious that $u^\star \neq 0$. Therefore also by $(j_2)_4$, it shows that $-h_2(u^\star) > 0$. Since from (3.23), for $\frac{-h_2(u^\star)}{h_1(u^\star)} > 0$, there exists a $t_0 > 0$ such that

$$\frac{g(t)}{t} < \frac{-h_2(u^\star)}{h_1(u^\star)}, \quad \forall t < t_0,$$

namely

$$\sup_{u\in\{\|u\|^{p^+}\leqslant\eta t\}\cap V_{p(x)}} -h_2(u) < t\frac{-h_2(u^\star)}{h_1(u^\star)}. \tag{3.24}$$

From the proof of Lemma 3.4, it is easy to get h_1 being weakly coercive, i.e. $h_1 \to +\infty$ as $\|u\| \to \infty$. In fact, when $\|u\| \to +\infty$, without loss of generality we may assume $|u| \to +\infty$. Since $p^- < s^-$, then by virtue of the Young inequality, we can get

$$\begin{aligned} \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-} &\leqslant \varepsilon\left||u|^{p^-}\right|^{s^-/p^-} + \varepsilon^{-(p^-)/(s^--p^-)}c_4^{s^-/(s^--p^-)} \\ &\leqslant \varepsilon|u|^{s^-} + c_5, \end{aligned} \tag{3.25}$$

for some $c_4 = 2\|a\|_\infty + \frac{2\lambda_1}{\delta_0^{p^-}} > 0$, $c_5 = c_5(c_4, \varepsilon) > 0$. Therefore by (3.12), we know that

$$h_1(u) \geqslant \frac{1}{p^+}\|u\|^{p^-} + \lambda_1\int_\Omega |u|^{q(x)}\,dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}\,dx - \int_\Omega \varepsilon|u|^{s^-}\,dx + c_4|\Omega|. \tag{3.26}$$

Let $\varepsilon < \frac{\tau}{M_0^{s^-}}$, then from (3.26) it is easy to see the coercivity of h_1.

Now, we choose a constant γ with $0 < \gamma < \min\{t_0, \gamma^\star\}$ (where $\gamma^\star$ is the one in Lemma 3.4). If $h_1(u) \leqslant \gamma$, then by the coercivity of h_1, there exists a constant $c_6 > 1$ such that $\|u\|^{p^+} \leqslant c_6\gamma$, therefore, we have

$$h_1^{-1}\big((-\infty,\gamma]\big) \cap V_{p(x)} \subseteq \left\{u \in V_{p(x)},\ \|u\|^{p^+} \leqslant c_6\gamma\right\},$$

from which it follows that

$$\sup_{u\in h_1^{-1}((-\infty,\gamma])\cap V_{p(x)}} -h_2(u) \leqslant \sup_{u\in\{\|u\|^{p^+}\leqslant c_6\gamma\}\cap V_{p(x)}} -h_2(u). \tag{3.27}$$

Because of $\gamma < t_0$ and $c_6 > 1$, the inequality (3.24) and the arbitrariness of $\eta > 1$ deduce that

$$\sup_{u\in\{\|u\|^{p^+}\leqslant c_6\gamma\}\cap V_{p(x)}} -h_2(u) < \gamma\frac{-h_2(u^\star)}{h_1(u^\star)}. \tag{3.28}$$

Combining (3.27) and (3.28) we obtain the desired inequality of this lemma, and the proof is complete. □

By the assumptions of $H(j_1)$ and $H(j_2)$, the functions h_1 and h_2 turn out to be locally Lipschitz. Here, we consider the indicator function of the closed subspace $V_{p(x)}$, i.e., $\psi_1 : X \to (-\infty, +\infty]$,

$$\psi_1(u) = \begin{cases} 0, & \text{if } u \in V_{p(x)}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where ψ_1 is evidently convex, proper, and lower semicontinuous. Write

$$I_1(u) := h_1(u) + \psi_1(u) \quad \text{as well as} \quad I_2(u) := h_2(u), \quad u \in X.$$

Obviously, I_2 and I_2 satisfy condition (**H**) of Section 2. Therefore, for every $\lambda > 0$ the function $I_1 + \lambda I_2$ complies with (**H**) too. Now, we will prove the key lemma of this paper.

Lemma 3.6. *If the hypotheses of $H(j_1)$ and $H(j_2)$ hold, then for every $\lambda > 0$ the function $I_1 + \lambda I_2$ fulfills $(PS)_c$ in $V_{p(x)}$, $c \in \mathbb{R}$, and*

$$\lim_{\|u\| \to +\infty} \big(I_1(u) + \lambda I_2(u)\big) = +\infty, \quad \forall u \in V_{p(x)}.$$

Proof. Firstly, we will prove that $h_1 + \lambda h_2$ is weakly coercive, for every $\lambda > 0$. Indeed, for every $u \in V_{p(x)}$ and without loss of generality we assume $\|u\| > 1$, because of the definition of ψ_1 and (3.12) we have that

$$\begin{aligned} I_1(u) + \lambda I_2(u) &= \int_\Omega \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + V^+|u|^{p(x)}\big)\,dx - \int_\Omega j_1(x,u)\,dx - \lambda \int_\Omega j_2(x,u)\,dx - \int_\Omega \frac{1}{p(x)} V^-|u|^{p(x)}\,dx \\ &\geqslant \frac{1}{p^+}\|u\|^{p^-} + \lambda_1 \int_\Omega |u|^{q(x)}\,dx + \int_\Omega \frac{\tau}{M_0^{s^-}}|u|^{s^-}\,dx - \int_\Omega \left(2a(x) + \frac{2\lambda_1}{\delta_0^{p^-}}\right)|u|^{p^-}\,dx - \lambda \int_\Omega j_2(x,u)\,dx \\ &\quad - \frac{\|V\|_\infty}{p^-}\int_\Omega |u|^{p(x)}\,dx. \end{aligned} \tag{3.29}$$

Similarly to (3.25), note $s^- > p^+$, through Young inequality one has

$$\begin{aligned} \frac{\|V\|_\infty}{p^-}|u|^{p^+} &\leqslant \varepsilon\big||u|^{p^+}\big|^{s^-/p^+} + \varepsilon^{-(p^+)/(s^- - p^+)}\left(\frac{\|V\|_\infty}{p^-}\right)^{s^-/(s^- - p^+)} \\ &\leqslant \varepsilon|u|^{s^-} + c_7, \end{aligned} \tag{3.30}$$

where $c_7 > 0$ is a constant. On one hand, by assumption of $(j_2)_4$, there exists an $M_1 > 0$, for almost all $x \in \Omega$ and all $|\zeta| > M_1$, such that

$$j_1(x,\zeta) < 0.$$

On the other hand, from the Lebourg mean value theorem (see [4]), for almost all $x \in \Omega$ and all $\zeta \in R$, we have

$$\big|j_2(x,\zeta) - j_2(x,0)\big| \leqslant \big|v_1(x,\zeta)\big||\zeta|, \tag{3.31}$$

for some $v_1 \in \partial j(x, t\zeta)$ with $0 < t < 1$. On account of $(j_2)_3$ besides $j_2(x,0) = 0$, then for almost all $x \in \Omega$ and all ζ such that $|\zeta| \leqslant M_1$, through (3.31) it follows that

$$\big|j_2(x,\zeta)\big| \leqslant c_8,$$

where $c_8 = c_8(M_1, \|b\|_\infty, \beta) > 0$. Thus through this discussion, it follows that

$$\int_\Omega j_2(x,u)\,dx = \int_{\{|u| > M_1\}} j_2(x,u)\,dx + \int_{\{|u| \leqslant M_1\}} j_2(x,u)\,dx \leqslant c_8|\Omega|. \tag{3.32}$$

Using (3.25), (3.30) and (3.32) to (3.29), it follows that

$$I_1(u) + \lambda I_2(u) \geqslant \frac{1}{p^+}\|u\|^{p^-} + \lambda_1 \int_\Omega |u|^{q(x)}\,dx + \frac{\tau}{M_0^{s^-}}\int_\Omega |u|^{s^-}\,dx - 2\varepsilon \int_\Omega |u|^{s^-}\,dx - (c_5 + c_7 + c_8)|\Omega|. \tag{3.33}$$

Choose $\varepsilon > 0$ such that $\varepsilon < \frac{\tau}{2M_0^{s^-}}$. Then from the inequality (3.33) it follows that $I_1 + \lambda I_2$ is weakly coercive in $V_{p(x)}$, for every $\lambda > 0$.

Now, we will prove that $I_1 + \lambda I_2$ fulfills $(PS)_c$, $c \in \mathbb{R}$. Let $\{u_n\} \subset V_{p(x)}$ be a sequence such that

$$I_1(u_n) + \lambda I_2(u_n) \to c \tag{3.34}$$

and for every $v \in V_{p(x)}$, we have

$$(h_1 + \lambda h_2)^\circ(u_n; v - u_n) + \psi_1(v) - \psi_1(u_n) \geqslant -\varepsilon_n\|v - u_n\|, \tag{3.35}$$

for a sequence $\{\varepsilon_n\}$ in $[0,+\infty)$ with $\varepsilon_n \to 0$. By the coerciveness of the function $I_1+\lambda I_2$, (3.34) implies that the sequence $\{u_n\}$ is bounded in $V_{p(x)}$. Therefore, there exists an element $u \in V_{p(x)}$ such that $\{u_n\}$ converges weakly to u in $V_{p(x)}$. Since J and L are continuous differentiable functions in (3.2), then by Proposition 2.3(2) and (3) we show that

$$\begin{aligned}(h_1+\lambda h_2)^\circ(u_n; v-u_n) &\leqslant h_1^\circ(u_n; v-u_n) + \mu h_2^\circ(u_n; v-u_n)\\ &= A_1^\circ(u_n; v-u_n) + \langle J'(u_n), v-u_n\rangle_X + \lambda A_2^\circ(u_n; v-u_n) + \lambda\langle L'(u_n), v-u_n\rangle_X,\end{aligned} \tag{3.36}$$

where $L' : X \to X^*$, and

$$\langle L'(u), v\rangle_X := -\frac{1}{\lambda}\int_\Omega V^-|u|^{p(x)-2}uv\,dx, \quad \forall u, v \in X.$$

Choosing in particular $v = u$ in (3.35) and the definition of ψ_1, (3.36) becomes

$$\langle J'(u_n), u_n-u\rangle_X \leqslant \varepsilon_n\|u-u_n\| + A_1^\circ(u_n; u-u_n) + \lambda A_2^\circ(u_n; u-u_n) + \lambda\langle L'(u_n), u-u_n\rangle_X.$$

Since the embedding $X \hookrightarrow L^{p(x)}(\Omega)$ is compact, then we have that $L : X \to R$ and $L' : X \to X^*$ are sequentially weakly-strongly continuous, namely, $u_n \rightharpoonup u$ in X, implies $L(u_n) \to L(u)$ and $L'(u_n) \to L'(u)$. Therefore, it follows that

$$\langle L'(u_n), u-u_n\rangle_X \to 0, \quad \text{as } n \to \infty. \tag{3.37}$$

On the other hand, also by compact imbedding of $X \hookrightarrow L^{p^-}(\Omega)$ and $X \hookrightarrow L^{p(x)}(\Omega)$, up to a subsequence, $\{u_n\}$ converges strongly to u in $L^{p^-}(\Omega)$ and $L^{p(x)}(\Omega)$. By virtue of Proposition 2.3, Lemma 3.2 and $(j_1)_3$, through Hö1der's inequality, one has

$$\begin{aligned}A_1^\circ(u_n; u-u_n) &\leqslant \int_\Omega (-j_1)^\circ\big(x, u_n(x); u-u_n\big)\,dx\\ &= \int_\Omega j_1^\circ\big(x, u_n(x); u_n-u\big)\,dx\\ &= \int_\Omega \max\big\{\langle\xi_n(x), u_n-u\rangle_X; \xi_n(x) \in \partial j_1\big(x, u_n(x)\big)\big\}\,dx\\ &\leqslant \int_\Omega a(x)|u_n|^{p^--1}|u_n-u|\,dx\\ &\leqslant \|a\|_\infty |u_n|_{p^-}^{p^--1}|u_n-u|_{p^-}.\end{aligned} \tag{3.38}$$

Due to $\{u_n\}$ is bounded in X, it follows that $\{u_n\}$ is bounded in $L^{p^-}(\Omega)$, and besides $\{u_n\}$ converges strongly to u in $L^{p^-}(\Omega)$. Therefore, $|u_n-u|_{p^-} \to 0$, and $A_1^\circ(u_n; u-u_n) \to 0$, as $n \to \infty$. Similarly to the above statement, by $(j_1)_3$, Proposition 2.3, Lemma 3.2 and Hö1der's inequality, we have

$$\begin{aligned}A_2^\circ(u_n; u-u_n) &\leqslant \int_\Omega (-j_2)^\circ\big(x, u_n(x); u-u_n\big)\,dx\\ &= \int_\Omega j_2^\circ\big(x, u_n(x); u_n-u\big)\,dx\\ &= \int_\Omega \max\big\{\langle\xi_n(x), u_n-u\rangle_X; \xi_n(x) \in \partial j_2\big(x, u_n(x)\big)\big\}\,dx,\\ &\leqslant \int_\Omega b(x)|u_n|^{p(x)-1}|u_n-u|\,dx\\ &\leqslant \|b\|_\infty |u_n|_{p(x)}^{p(x)-1}|u_n-u|_{p(x)},\end{aligned} \tag{3.39}$$

also by the compact imbedding of $X \hookrightarrow L^{p(x)}(\Omega)$, we obtain $A_2^\circ(u_n; u-u_n) \to 0$, as $n \to \infty$. Because of the sequence $\varepsilon_n \to 0$ as $n \to +\infty$, combining (3.37), (3.38) and (3.39), we show that

$$\limsup_{n\to+\infty}\langle J'(u_n), u_n-u\rangle_X \leqslant 0.$$

From Proposition 2.1 we have $u_n \to u$ as $n \to +\infty$. Thus the function $I_1+\lambda I_2$ fulfills $(PS)_c$ in $V_{p(x)}$, $c \in \mathbb{R}$. $\square$

Now we will prove the main result of this section.

Proof of Theorem 3.1. From the proofs of Lemmas 3.4 and 3.5, and the definition of ψ_1, we know that there exist a $\gamma > 0$ and $u^\star \in V_{p(x)}$ such that $I_1(u^\star) > \gamma$, moreover, keep in mind that $j_1(x,0) = j_2(x,0) = 0$. Thus we have $I_1(0) = -I_2(0) = 0$. Choose ρ satisfying

$$\sup_{u \in I_1^{-1}((-\infty,\gamma]) \cap V_{p(x)}} \bigl(-I_2(u)\bigr) < \rho < \gamma \frac{-I_2(u^\star)}{I_1(u^\star)},$$

by Proposition 2.6, we have

$$\sup_{\lambda \geqslant 0} \inf_{u \in V_{p(x)}} \bigl(I_1(u) + \lambda\bigl(\rho + I_2(u)\bigr)\bigr) < \inf_{u \in V_{p(x)}} \sup_{\lambda \geqslant 0} \bigl(I_1(u) + \lambda\bigl(\rho + I_2(u)\bigr)\bigr). \tag{3.40}$$

From the proof of Lemma 3.5, we see that $\rho > 0$. If we define $h : [0,+\infty) \to R$ by $h(\lambda) = \rho\lambda$ and $\Lambda = [0,+\infty)$, then h and the inequality (3.40) fulfill the condition of Proposition 2.5 (b_3). By standard results, the function h_1 is locally Lipschitz and weakly sequentially lower semicontinuous. Since $(j_2)_3$ holds and X is compactly imbedding in $L^{p(x)}(\Omega)$, the assertion remains true regarding h_2 too. So (b_1) of Proposition 2.5 is satisfied. Finally, Lemma 3.6 makes sure that Proposition 2.5(b_2) holds. Therefore, there is an open interval $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in $V_{p(x)}$. Then the energy functional $\varphi = h_1 + \lambda h_2$ corresponding to problem (P) has at least three critical points in $V_{p(x)}$. Suppose that $u_0 \in V_{p(x)}$ is a critical point of φ, then we have

$$(h_1 + \lambda h_2)^\circ(u_0; v - u_0) + \psi_1(v) - \psi_1(u_0) \geqslant 0, \quad \forall v \in V_{p(x)},$$

namely,

$$0 \leqslant A_1^\circ(u_0; v - u_0) + \bigl\langle J'(u_0), v - u_0\bigr\rangle_X + \lambda A_2^\circ(u_0; v - u_0) + \lambda\bigl\langle L'(u_0), v - u_0\bigr\rangle_X, \tag{3.41}$$

from (3.41) there exist $w_0(x) \in \partial j_1(x, u_0(x))$ and $v_0(x) \in \partial j_2(x, u_0(x))$ such that

$$J'(u_0) = V^-|u_0|^{p(x)-2}u_0 + w_0 + \lambda v_0. \tag{3.42}$$

Let any $\eta(x) \in C_0^\infty(\Omega)$, from (3.42) we have

$$\bigl\langle -\operatorname{div}\bigl(|\nabla u_0|^{p(x)-2}\nabla u_0\bigr), \eta\bigr\rangle_X = \bigl\langle -V|u_0|^{p(x)-2}u_0, \eta\bigr\rangle_X + \langle w_0, \eta\rangle_X + \lambda\langle v_0, \eta\rangle_X.$$

Recalling that the embedding $C_0^\infty(\Omega) \subseteq X$ is dense, we infer that

$$-\operatorname{div}\bigl(\bigl|\nabla u_0(x)\bigr|^{p(x)-2}\nabla u_0(x)\bigr) + V\bigl|u_0(x)\bigr|^{p(x)-2}u_0(x) \in \partial j_1\bigl(x, u_0(x)\bigr) + \lambda\partial j_2\bigl(x, u_0(x)\bigr).$$

This implies that u_0 is a solution of (P). Since $\psi_1 \equiv 0$ in $V_{p(x)}$, then, from Proposition 2.5, there exists $\Lambda_1 \subseteq [0,+\infty)$, such that for each $\lambda \in \Lambda_1$, problem (P_λ) has at least three nontrivial solutions in $V_{p(x)}$. $\square$

Example 3.1. As a simple example of nonsmooth locally Lipschitz potential functions satisfying hypotheses $H(j_1)$ and $H(j_2)$, we consider the following functions (for simplicity we drop the x-dependence):

$$j_1(\zeta) = \begin{cases} (-2\lambda_1 - \beta)|\zeta|^{q(x)}, & \text{if } |\zeta| \leqslant 1, \\ \hat{a}(x)|\zeta|^{p^-} - \ln|\zeta|^{s^-}, & \text{if } |\zeta| > 1 \end{cases}$$

and

$$j_2(\zeta) = \begin{cases} -\frac{1}{\lambda p(x)}V^-(x)|\zeta|^{p(x)} + \beta|\zeta|^{p(x)}, & \text{if } |\zeta| \leqslant 1, \\ -\beta\ln|\zeta|^{p^+}, & \text{if } |\zeta| > 1, \end{cases}$$

where $\beta > 0$, $\hat{a}(x), s^-$, and $q(x)$ are mentioned in the assumptions of $H(j_1)$ and $H(j_2)$. Evidently, $(j_1)_4$ and $(j_2)_4$ are satisfied. Besides, through Lebourg mean value theorem, the hypotheses $(j_1)_3$ and $(j_2)_3$ are satisfied. Moreover, it is easy to see that other assumptions in $H(j_1)$ and $H(j_2)$ are also satisfied.

Acknowledgments

I would like to express the gratitude to the anonymous referees for their very valuable observations.

References

[1] E. Acerbi, G. Mingione, Regularity results for stationary electrorheological fluids, Arch. Ration. Mech. Anal. 164 (2002) 213–259.
[2] E. Acerbi, G. Mingione, Regularity results for a class of functions with nonstandard growth, Arch. Ration. Mech. Anal. 156 (2001) 121–140.
[3] G.A. Afrouzi, S. Heidarkhani, Three solutions for a Dirichlet boundary value problem involving the p-Laplacian, Nonlinear Anal. 66 (2007) 2281–2288.
[4] G. Bonanno, N. Giovannelli, An eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities, J. Math. Anal. Appl. 308 (2005) 596–604.
[5] G. Bonanno, P. Candito, Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian, Arch. Math. 80 (2003) 424–429.
[6] F. Cammarotoa, A. Chinnì, B. Di Bella, Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian, Nonlinear Anal. 71 (2009) 4486–4492.
[7] K.C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981) 102–129.
[8] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[9] G.W. Dai, Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian, Nonlinear Anal. 70 (2009) 3755–3760.
[10] G.W. Dai, W.L. Liu, Three solutions for a differential inclusion problem involving the $p(x)$-Laplacian, Nonlinear Anal. 71 (2009) 5318–5326.
[11] L. Diening, Theoretical and numerical results for electrorheological fluids, PhD thesis, University of Frieburg, Germany, 2002.
[12] X.L. Fan, S.G. Deng, Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations, Nonlinear Anal. 67 (2007) 3064–3075.
[13] X.L. Fan, Eigenvalues of the $p(x)$-Laplacian Neumann problems, Nonlinear Anal. 67 (2007) 2982–2992.
[14] M.E. Filippakis, N.S. Papageorgiou, Solutions for nonlinear variational inequalities with a nonsmooth potential, Abstr. Appl. Anal. 2004 (8) (2004) 635–649.
[15] B. Ge, X.P. Xue, M.S. Guo, Three solutions to inequalities of Dirichlet problem driven by $p(x)$-Laplacian, Appl. Math. Mech. 31 (10) (2010) 1283–1292. (English edition).
[16] A. Iannizzotto, Three critical points for perturbed nonsmooth functionals and applications, Nonlinear Anal. 72 (2010) 1319–1338.
[17] A. Kristály, W. Marzantowicz, C. Varga, A non-smooth three critical points theorem with applications in differential inclusions, J. Global Optim. 46 (2010) 49–62.
[18] A. Kristály, Multiplicity results for an eigenvalue problem for hemi-variational inequalities in strip-like domains, Set-Valued Anal. 13 (1) (2005) 85–103.
[19] C. Li, C.L. Tang, Three solutions for a class of quasilinear elliptic systems involving the (p,q)-Laplacian, Nonlinear Anal. 69 (2008) 3322–3329.
[20] H. Lisei, Cs. Varga, Some applications to variational-hemivariational inequalities of the principle of symmetric criticality for Motreanu–Panagiotopoulos type functionals, J. Global Optim. 36 (2) (2006) 283–305.
[21] J.J. Liu, X.Y. Shi, Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x),q(x))$-Laplacian, Nonlinear Anal. 71 (2009) 550–557.
[22] Q. Liu, Existence of three solutions for $p(x)$-Laplacian equations, Nonlinear Anal. 68 (2008) 2119–2127.
[23] S.A. Marano, D. Motreanu, On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems, Nonlinear Anal. 48 (2002) 37–52.
[24] S.A. Marano, N.S. Papageorgiou, On some elliptic hemivariational and variational-hemivariational inequalities, Nonlinear Anal. 62 (2005) 757–774.
[25] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2006) 2625–2641.
[26] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Kluwer Academic Publishers, Dordrecht, 1999.
[27] B. Ricceri, On a three critical points theorem, Arch. Math. (Basel) 75 (2000) 220–226.
[28] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, in: Advanced Topics in Nonlinear Operator Theory, Math. Comput. Modelling 32 (1999) 1485–1494.
[29] M. Rủžička, Electrorheological Fluids Modeling and Mathematical Theory, Springer, 2000.
[30] G.Q. Zhang, S.Y. Liu, Three symmetric solutions for a class of elliptic equations involving the p-Laplacian with discontinuous nonlinearities in R^N, Nonlinear Anal. 67 (2007) 2232–2239.
[31] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR Izv. 9 (1987) 33–66.