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On the first eigenvalue of the Dirichlet-to-Neumann operator on forms

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Abstract

We study a Dirichlet-to-Neumann eigenvalue problem for differential forms on a compact Riemannian manifold with smooth boundary. This problem is a natural generalization of the classical Dirichlet-to-Neumann (or Steklov) problem on functions. We derive a number of upper and lower bounds for the first eigenvalue in several contexts: many of these estimates will be sharp, and for some of them we characterize equality. We also relate these new eigenvalues with those of other operators, like the Hodge Laplacian or the biharmonic Steklov operator.

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1. Introduction

Let Ω be a compact, connected $(n + 1)$ -dimensional Riemannian domain with smooth boundary Σ^n . The Dirichlet-to-Neumann operator T , also called *Steklov operator*, acts on smooth functions on Σ in the following way. If $f \in C^\infty(\Sigma)$ and \hat{f} denotes the unique harmonic extension of f to Ω , then:

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$$Tf = -\frac{\partial \hat{f}}{\partial N}, \tag{1}$$

where N is the inner unit normal vector field on Σ . T defines a pseudo-differential operator on $C^\infty(\Sigma)$ which is known to be elliptic and self-adjoint; hence T has a discrete spectrum $0 = \nu_1 < \nu_2 \leq \nu_3 \leq \dots$. Note that the lowest eigenvalue is $\nu_1 = 0$, corresponding to the constant eigenfunctions; therefore, in our convention, the first positive eigenvalue of T will be denoted by ν_2 . There is a vast literature on eigenvalue estimates for the operator T ; directly related to our paper are the estimates given in [5] and [6].

In this paper, we consider a natural extension of the Dirichlet-to-Neumann operator T to an elliptic operator $T^{[p]}$ acting on differential forms of arbitrary degree p on the boundary Σ and then prove some geometric lower bounds for its first eigenvalue, given in terms of the second fundamental form of the boundary. We then estimate these new eigenvalues from above in terms of the isoperimetric ratio $\text{Vol}(\Sigma)/\text{Vol}(\Omega)$, and in terms of the eigenvalues of other differential operators, like the Hodge–Laplace operator on the boundary Σ and the biharmonic Steklov operator. In some cases we improve some known estimates. The operator $T^{[p]}$ seems to have interesting spectral properties which, we hope, justify the present work.

In the rest of the introduction we state the main results of the paper.

1.1. The definition of $T^{[p]}$

Let ω be a form of degree p on Σ^n , with $p = 0, 1, \dots, n$. Then there exists a unique p -form $\hat{\omega}$ on Ω such that:

$$\begin{cases} \Delta \hat{\omega} = 0, \\ J^* \hat{\omega} = \omega, \quad i_N \hat{\omega} = 0, \end{cases}$$

where J^* denotes the restriction of $\hat{\omega}$ to Σ , and i_N is the interior product of $\hat{\omega}$ with the inner unit normal vector field N . The form $\hat{\omega}$ will be called the *harmonic tangential extension* of ω . Its existence and uniqueness is proved, for example, in Schwarz [15]. We set:

$$T^{[p]} \omega = -i_N d\hat{\omega},$$

and then we have a linear operator $T^{[p]} : \Lambda^p(\Sigma) \rightarrow \Lambda^p(\Sigma)$, the *Dirichlet-to-Neumann operator on p -forms*, which reduces to the classical Dirichlet-to-Neumann operator acting on functions when $p = 0$, so that $T^{[0]} = T$. Here $\Lambda^p(\Sigma)$ denotes the vector bundle of differential p -forms on Σ .

We observe in Section 2 that $T^{[p]}$ is an elliptic self-adjoint pseudo-differential operator, with discrete spectrum

$$\nu_{1,p}(\Omega) \leq \nu_{2,p}(\Omega) \leq \dots$$

Moreover, $T^{[p]}$ is non-negative so that $\nu_{1,p}(\Omega) \geq 0$. Actually, it follows easily from the definition that $\text{Ker } T^{[p]}$ is isomorphic to $H^p(\Omega)$, the p -th absolute de Rham cohomology space of Ω with real coefficients. Therefore:

- a positive lower bound of $\nu_{1,p}(\Omega)$ will imply in particular that $H^p(\Omega) = 0$;
- a positive upper bound of $\nu_{1,p}(\Omega)$ will be significant only when $H^p(\Omega) = 0$.

As Ω is connected, we see that $H^0(\Omega)$ is 1-dimensional. Therefore, in our notation, $\nu_{1,0}(\Omega) = 0$ and $\nu_{2,0}(\Omega) = \nu_2$ is the first positive eigenvalue of the classical problem (1).

Finally, using the Hodge star operator, we define a dual operator $T_D^{[p]}$, also acting on $\Lambda^p(\Sigma)$; in particular, the dual of $T^{[n]}$ defines an operator $T_D^{[0]}$ acting on $C^\infty(\Sigma)$ and different from the classical Dirichlet-to-Neumann operator T (see Section 2.1 for details).

The operator $T^{[p]}$ belongs to a family of operators depending on a complex parameter z , introduced by G. Carron in [2] and called *relative to absolute operator* (see the proof of Theorem 11). Other Dirichlet to Neumann operators acting on differential forms, but different from $T^{[p]}$, were introduced by Joshi and Lionheart in [10], and Belishev and Sharafutdinov in [1]. In the preprint [17], the operator $\star_\Sigma T^{[p]} : \Lambda^p(\Sigma) \rightarrow \Lambda^{n-p}(\Sigma)$ appears in a certain matrix decomposition of the Joshi and Lionheart operator. None of these works, however, discuss eigenvalue estimates.

1.2. Lower bounds by the extrinsic geometry

First, some notations. Fix a point $x \in \Sigma$ and let $\eta_1(x), \dots, \eta_n(x)$ be the principal curvatures of Σ^n at x (our sign convention is that the principal curvatures of the unit ball in \mathbf{R}^{n+1} are positive). The *p-curvatures* of Σ are, by definition, all possible sums $\eta_{j_1}(x) + \dots + \eta_{j_p}(x)$ for $j_1, \dots, j_p \in \{1, \dots, n\}$. Arrange the sequence of principal curvatures so that it is non-decreasing: $\eta_1(x) \leq \dots \leq \eta_n(x)$, and call

$$\sigma_p(x) \doteq \eta_1(x) + \dots + \eta_p(x)$$

the *lowest p-curvature at x*. We say that Σ is *p-convex* if $\sigma_p(x) \geq 0$ for all $x \in \Sigma$, and let

$$\sigma_p(\Sigma) = \inf_{x \in \Sigma} \sigma_p(x).$$

Note that 1-convex means, simply, *convex* (all principal curvatures are non-negative) and *n-convex* means that Σ has non-negative mean curvature because, by definition, $\sigma_n(\Sigma) = nH$, where H is a lower bound of the mean curvature of Σ . Finally, it is clear from the definition that, if Σ is *p-convex*, then it is *q-convex* for all $q \geq p$.

Recall that, if ω is a *p-form* on Ω^{n+1} , the Bochner formula gives

$$\Delta\omega = \nabla^* \nabla \omega + W^{[p]},$$

where $W^{[p]}$ is a symmetric endomorphism acting on $\Lambda^p(\Omega)$, called the *Bochner curvature term*. One knows that $W^{[1]} = \text{Ric}$, the Ricci tensor, hence $W^{[1]} \geq 0$ provided that Ω has non-negative Ricci curvature.

From the work of Gallot and Meyer (see [8]) we also know that, if γ is a lower bound of the eigenvalues of the Riemann curvature operator (seen as a symmetric endomorphism of $\Lambda^2(\Omega)$), then $W^{[p]} \geq p(n + 1 - p)\gamma$. Hence

- if the curvature operator of Ω is non-negative then $W^{[p]} \geq 0$ for all degrees p .

However, the condition $W^{[p]} \geq 0$ is sometimes much weaker than assuming the positivity of the curvature operator.

Theorem 1. Let $p = 1, \dots, n$. Assume that Ω^{n+1} satisfies $W^{[p]} \geq 0$ and that Σ is strictly p -convex, that is $\sigma_p(\Sigma) > 0$.

- (a) If $p < \frac{n+1}{2}$ then $v_{1,p}(\Omega) > \frac{n-p+2}{n-p+1}\sigma_p(\Sigma)$. The equality never holds.
- (b) If $p \geq \frac{n+1}{2}$ then

$$v_{1,p}(\Omega) \geq \frac{p+1}{p}\sigma_p(\Sigma), \tag{2}$$

which is an equality when Ω is a ball in the Euclidean space \mathbf{R}^{n+1} .

Remark. Note that under the given curvature assumptions we have in particular $H^q(\Omega) = 0$ for all $q \geq p$; so, the p -convexity has interesting topological consequences. This is not new: in [20] it was proved by other methods that, if $\sigma_p(\Sigma) > 0$ and the sectional curvatures of Ω are non-negative, then Ω has the homotopy type of a CW-complex with cells only in dimensions $\leq p - 1$. For a result in negative curvature we refer to [14]: in particular, if Ω is a p -convex domain in \mathbf{H}^n then $H^p(\Omega) = 0$ for all $q \geq p$, provided that $p > (n + 1)/2$.

The proof of Theorem 1 uses a Reilly-type formula for differential forms, proved in [12]. We characterize the equality in (2) in the following two cases: when $p = n$ and when $p > (n + 1)/2$ and Ω is a Euclidean domain. Precisely:

Theorem 2. Assume that Ω has non-negative Ricci curvature and mean-convex boundary. Then

$$v_{1,n}(\Omega) \geq (n + 1)H,$$

where H is a lower bound of the mean curvature. If $n \geq 2$, equality holds if and only if Ω is a Euclidean ball.

Theorem 3. If $p > \frac{n+1}{2}$ and Ω is a Euclidean domain, then we have equality in (2) if and only if Ω is a ball.

For Euclidean domains we also prove an inequality relating the first eigenvalues for consecutive degrees.

Theorem 4. Let Ω be any compact domain in \mathbf{R}^{n+1} , and let $\sigma_p(\Sigma)$ be a lower bound of the p -curvatures of Σ (which we do not assume to be positive).

- (i) For all $p = 1, \dots, n$ one has $v_{1,p}(\Omega) \geq v_{1,p-1}(\Omega) + \sigma_p(\Sigma)/p$.
- (ii) If Ω is convex, then $v_{1,p} > 0$ for all $p \geq 1$ and

$$v_{1,1}(\Omega) \leq v_{1,2}(\Omega) \leq \dots \leq v_{1,n}(\Omega).$$

The inequality (i) is sharp for $p > \frac{n+1}{2}$ since equality is achieved by the unit Euclidean ball. The monotonicity property in (ii) is an immediate consequence of (i), because if Ω is convex then $\sigma_p(\Sigma) \geq 0$ for all p .

We remark that the property (ii) holds also for the first eigenvalues of the Laplacian acting on p -forms of a convex Euclidean domain Ω , for the absolute boundary conditions (see [9]).

1.3. Upper bounds by the isoperimetric ratio

It turns out that the existence of parallel forms implies that, for suitable degrees, the Dirichlet-to-Neumann eigenvalues can be bounded above by the isoperimetric ratio $\text{Vol}(\Sigma)/\text{Vol}(\Omega)$. Precisely, if Ω supports a non-trivial parallel p -form, and $H^p(\Omega) = H_R^p(\Omega) = 0$, then

$$v_{1,p-1}(\Omega) + v_{1,n-p}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}. \quad (3)$$

In some cases the estimate is sharp and we can characterize equality. Either one of the two cohomology assumptions can be removed if the given parallel form is known to be exact (resp., co-exact): so, for example, (3) holds in all degrees for all domains in Euclidean space, since the parallel p -form $dx_1 \wedge \cdots \wedge dx_p$ is exact and co-exact.

The inequality (3) follows from the estimates in Section 4, which apply more generally to the ratio $\int_{\Sigma} \|\xi\|^2 / \int_{\Omega} \|\xi\|^2$, where ξ is a *harmonic field*, that is, a differential form which is closed and co-closed (we remark that on a manifold with nonempty boundary the vector space of harmonic fields of a given degree is infinite dimensional, and is properly contained in the space of harmonic forms).

As the volume form of Ω is parallel we have, for all compact manifolds with boundary, the estimate:

$$v_{1,n}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}, \quad (4)$$

which reduces to an equality when Ω is a Euclidean ball.

Then, we examine the equality case in (4). To that end, consider the mean-exit time function E , solution of the problem:

$$\begin{cases} \Delta E = 1 & \text{on } \Omega, \\ E = 0 & \text{on } \Sigma. \end{cases}$$

Any domain for which the normal derivative $\partial E/\partial N$ is constant on Σ will be called a *harmonic domain*. The reason for this terminology is given by Proposition 18, in which we observe the following simple fact: $\partial E/\partial N$ is constant on Σ if and only if the mean value of any harmonic function on Ω equals its mean value on the boundary.

Theorem 5. *Let Ω be any compact domain. Then $v_{1,n}(\Omega) \leq \text{Vol}(\Sigma)/\text{Vol}(\Omega)$.*

- a) *If equality holds, then Ω is a harmonic domain.*
- b) *Conversely, if Ω is a harmonic domain, then $\text{Vol}(\Sigma)/\text{Vol}(\Omega)$ belongs to the spectrum of $T^{[n]}$ (an associated eigenform being $\star dE$).*

It remains to see how rigid the harmonicity condition is, and what conditions it imposes on the geometry of the boundary. For Euclidean domains the question was settled in a famous paper by Serrin [16] which states in particular that any harmonic domain in \mathbf{R}^{n+1} is a ball. This

rigidity result was extended by Kumaresan and Prajapat (see [11]) to domains in the hyperbolic space \mathbf{H}^{n+1} and in the hemisphere $\mathbf{R}_+^{n+1}\mathbf{S}_+^{n+1}$. To our knowledge, the classification of harmonic domains in \mathbf{S}^{n+1} is still an open (and interesting) question. Then, we have the following:

Corollary 6.

- a) For Euclidean domains the equality holds in (4) iff Ω is a ball.
- b) Let Ω be a domain in \mathbf{H}^{n+1} or in \mathbf{S}_+^{n+1} . If the equality holds in (4), then Ω is a geodesic ball.

Finally, using the estimate (4) and the inequalities of Theorems 1 and 4, one gets the following fact.

Proposition 7. For the unit Euclidean ball \mathbf{B}^{n+1} in \mathbf{R}^{n+1} one has $v_{1,p}(\mathbf{B}^{n+1}) = p + 1$ for all $p \geq (n + 1)/2$.

This calculation shows that the estimates of Theorems 1 and 4 are indeed sharp.

Remark. In a forthcoming paper, we will compute the whole spectrum of the Dirichlet-to-Neumann operator acting on p -forms of the unit Euclidean ball. In particular it turns out that, if $1 \leq p < (n + 1)/2$, then $p + 1$ is still an eigenvalue of $T^{[p]}$, however it is no longer the first. In that range one has in fact $v_{1,p}(\mathbf{B}^{n+1}) = \frac{n+3}{n+1}p$.

1.4. Upper bounds by the Hodge–Laplace eigenvalues

The Hodge Laplacian acting on p -forms of a closed manifold Σ is the operator defined by $\Delta^\Sigma = d^\Sigma \delta^\Sigma + \delta^\Sigma d^\Sigma$, where d^Σ and δ^Σ denote respectively the differential and the co-differential acting on forms of Σ . We let $\lambda'_{1,p}(\Sigma)$ (resp. $\lambda''_{1,p}(\Sigma)$) be the first eigenvalue of Δ^Σ restricted to the subspace of exact (resp. co-exact) forms (these subspaces are preserved by Δ^Σ because it commutes with d^Σ and δ^Σ). Differentiating eigenforms, one sees that, if $\lambda_{1,p}(\Sigma)$ is the first positive eigenvalue of Δ^Σ , then $\lambda_{1,p}(\Sigma) = \min\{\lambda'_{1,p}(\Sigma), \lambda'_{1,p+1}(\Sigma)\}$.

We then have the following lower bound.

Theorem 8. Assume that $H_R^p(\Omega) = 0$, $\min(\sigma_p(\Sigma), \sigma_{n-p+1}(\Sigma)) \geq 0$ and $W^{[p]} \geq 0$. Then, for all $p = 1, \dots, n$:

$$\lambda'_{1,p}(\Sigma) \geq \frac{1}{2}(\sigma_p(\Sigma)v_{1,n-p}(\Omega) + \sigma_{n-p+1}(\Sigma)v_{1,p-1}(\Omega)).$$

Observe that $\lambda'_{1,1}(\Sigma) = \lambda_1(\Sigma)$, the first positive eigenvalue of the Laplacian acting on functions of Σ . Taking $p = 1$ in the previous theorem we obtain the following sharp lower bound.

Theorem 9. Assume that Ω has non-negative Ricci curvature and that Σ is strictly convex, with principal curvatures bounded below by $\sigma_1(\Sigma) > 0$. Then:

$$\lambda_1(\Sigma) \geq \frac{1}{2}(\sigma_1(\Sigma)v_{1,n-1}(\Omega) + nHv_{2,0}(\Omega)),$$

where H is a lower bound of the mean curvature of Σ , and $v_{2,0}(\Omega)$ is the first positive eigenvalue of the Dirichlet-to-Neumann operator on functions. Moreover, if $n = \dim(\Sigma) \geq 3$, the equality holds if and only if Ω is a Euclidean ball.

The motivation for looking at such a bound was given by the following estimate of Escobar [6], which holds under the same assumptions of Theorem 9:

$$\lambda_1(\Sigma) > \frac{nH}{2} v_{2,0}(\Omega). \quad (5)$$

We observe that the defect $\lambda_1(\Sigma) - \frac{nH}{2} v_{2,0}(\Omega)$ in (5) is bounded below by the first Dirichlet-to-Neumann eigenvalue in the degree $n - 1$, thus obtaining a sharp bound.

1.5. An upper bound by the first biharmonic Steklov eigenvalue

The following problem on functions is classical, and is known as the fourth order (or *biharmonic*) Steklov eigenvalue problem:

$$\begin{cases} \Delta^2 f = 0 & \text{on } \Omega, \\ f = 0, \quad \Delta f = \mu \frac{\partial f}{\partial N} & \text{on } \Sigma. \end{cases} \quad (6)$$

For recent results on the problem, we refer to [7] and [19]. An immediate application of the min–max principle associated to the Dirichlet-to-Neumann operator on n -forms gives:

Theorem 10. *One has always $\mu_1(\Omega) \geq v_{1,n}(\Omega)$, where $\mu_1(\Omega)$ is the first eigenvalue of (6). If the equality holds, then Ω is a harmonic domain.*

In [19] Wang and Xia prove that, if the Ricci curvature of Ω is non-negative and the mean curvature of Σ is bounded below by $H > 0$, then $\mu_1(\Omega) \geq (n + 1)H$. Moreover equality occurs if and only if Ω is isometric to a ball of \mathbf{R}^{n+1} . Combining Theorem 10 and our estimate of Theorem 2 we see that, under the given assumptions:

$$\mu_1(\Omega) \geq v_{1,n}(\Omega) \geq (n + 1)H$$

which implies the result of Wang and Xia. On the other hand, it is easy to observe that $\mu_1(\Omega) \leq \text{Vol}(\Sigma) / \text{Vol}(\Omega)$ (see for example [19]). Then the estimate (4) is a direct consequence of this fact and Theorem 10.

The paper is organized as follows. In Section 2 we state the main properties of the operator $T^{[p]}$. In Section 3 we prove the lower bounds and in Section 4 we give the proof of the upper bounds. Finally, in Appendix A, we prove a rigidity result needed for the equality case of Theorem 3.

2. Generalities on the Dirichlet-to-Neumann operator

Before stating the main properties of $T^{[p]}$, let us recall the following well-known facts. The Hodge–de Rham theorem for manifolds with boundary asserts that $H_{\text{dR}}^p(\Omega, \mathbf{R})$, the absolute

de Rham cohomology space in degree p with real coefficients, is isomorphic to the (finite dimensional) vector space of harmonic p -forms ϕ satisfying the absolute boundary conditions ($i_N\phi = i_Nd\phi = 0$ on Σ), which we denote by $H^p(\Omega)$. Equivalently, one has:

$$H^p(\Omega) = \{\phi \in \Lambda^p(\Omega): d\phi = \delta\phi = 0 \text{ on } \Omega, i_N\phi = 0 \text{ on } \Sigma\}.$$

By duality, the relative de Rham cohomology space in degree p is isomorphic to the vector space

$$H^p_R(\Omega) = \{\phi \in \Lambda^p(\Omega): d\phi = \delta\phi = 0 \text{ on } \Omega, J^*\phi = 0 \text{ on } \Sigma\}.$$

Theorem 11. *Let Ω^{n+1} be a compact domain with smooth boundary Σ^n . Let $T^{[p]}$ be the Dirichlet-to-Neumann operator acting on p -forms of Σ , as defined in Section 1.1. Then:*

- (a) $T^{[p]}$ is non-negative and self-adjoint.
- (b) The kernel of $T^{[p]}$ consists of the boundary values of absolute cohomology classes, and the restriction J^* induces an isomorphism between $H^p(\Omega)$ and $\text{Ker}(T^{[p]})$.
- (c) $T^{[p]}$ is an elliptic pseudo-differential operator of order one. Hence it admits an increasing sequence of eigenvalues with finite multiplicities

$$v_{1,p}(\Omega) \leq v_{2,p}(\Omega) \leq \dots$$

with $v_{1,p}(\Omega) = 0$ repeated $b_p(\Omega) = \dim H^p(\Omega)$ times. In particular, $v_{1,p}(\Omega) > 0$ if and only if $H^p(\Omega) = 0$.

- (d) The first eigenvalue of $T^{[p]}$ satisfies the min–max principle

$$v_{1,p}(\Omega) = \inf \left\{ \frac{\int_{\Omega} \|d\hat{\phi}\|^2 + \|\delta\hat{\phi}\|^2}{\int_{\Sigma} \|\hat{\phi}\|^2} \right\} \tag{7}$$

where the infimum is taken over all p -forms $\hat{\phi}$ on Ω such that $i_N\hat{\phi} = 0$ on Σ .

We remark that (b) has already been observed in [17], and (c) follows from the work of Carron in [2].

Proof. (a) We prove that the operator is self-adjoint. Recall the Stokes formula:

$$\int_{\Omega} \langle d\omega_1, \omega_2 \rangle = \int_{\Omega} \langle \omega_1, \delta\omega_2 \rangle - \int_{\Sigma} \langle J^*\omega_1, i_N\omega_2 \rangle$$

for all $\omega_1 \in \Lambda^{p-1}(\Omega)$ and $\omega_2 \in \Lambda^p(\Omega)$. Now let $\phi, \psi \in \Lambda^p(\Sigma)$ and denote by $\hat{\phi}, \hat{\psi}$ their harmonic tangential extensions on Ω . The definition of $T^{[p]}$ and the Stokes formula give:

$$\int_{\Sigma} \langle T^{[p]}\phi, \psi \rangle = - \int_{\Sigma} \langle i_Nd\hat{\phi}, J^*\hat{\psi} \rangle = \int_{\Omega} \langle d\hat{\phi}, d\hat{\psi} \rangle - \langle \delta d\hat{\phi}, \hat{\psi} \rangle.$$

As $\hat{\psi}$ is harmonic and $i_N \hat{\psi} = 0$ we have

$$-\int_{\Omega} \langle \delta d\hat{\phi}, \hat{\psi} \rangle = \int_{\Omega} \langle d\delta\hat{\phi}, \hat{\psi} \rangle = \int_{\Omega} \langle \delta\hat{\phi}, \delta\hat{\psi} \rangle.$$

So $\int_{\Sigma} \langle T^{[p]}\phi, \psi \rangle = \int_{\Omega} \langle d\hat{\phi}, d\hat{\psi} \rangle + \langle \delta\hat{\phi}, \delta\hat{\psi} \rangle$ which shows that $T^{[p]}$ is self-adjoint. Taking $\psi = \phi$ yields:

$$\int_{\Sigma} \langle T^{[p]}\phi, \phi \rangle = \int_{\Omega} \|d\hat{\phi}\|^2 + \|\delta\hat{\phi}\|^2 \geq 0$$

and $T^{[p]}$ is non-negative.

(b) If $\phi \in \text{Ker}(T^{[p]})$ then its harmonic tangential extension $\hat{\phi}$ satisfies, on Σ : $i_N \hat{\phi} = i_N d\hat{\phi} = 0$. Hence ϕ is the restriction of a form (cohomology class) in $H^p(\Omega)$. Conversely, it is clear by the definition that an absolute cohomology class restricts to a form in the kernel of $T^{[p]}$. Then:

$$\text{Ker}(T^{[p]}) = J^*(H^p(\Omega)).$$

We observe that the map $J^* : H^p(\Omega) \rightarrow J^*(H^p(\Omega))$ is injective: in fact, if $J^*\hat{\phi} = 0$ for some cohomology class $\hat{\phi}$, then $\hat{\phi}$ is harmonic and zero on the boundary, which implies $\hat{\phi} = 0$. Then the dimension of $\text{Ker}(T^{[p]})$ equals $b_p(\Omega)$.

(c) The proof that $T^{[p]}$ is an elliptic pseudo-differential operator follows the lines of the proof done in Section 6.4 of [2]. There, in studying determinants, G. Carron considers the linear operator $T_z : \Lambda^p(\Sigma) \rightarrow \Lambda^p(\Sigma)$ depending on a complex parameter $z \in \mathbf{C} \setminus [0, \infty)$ and defined by

$$T_z \phi = -i_N d\hat{\phi}_z,$$

where $\hat{\phi}_z$ is the unique solution of

$$\begin{cases} \Delta \hat{\phi}_z = z\hat{\phi}_z & \text{on } \Omega, \\ i_N \hat{\phi}_z = 0, \quad J^* \hat{\phi}_z = \phi & \text{on } \Sigma. \end{cases} \tag{8}$$

Carron shows that T_z is an elliptic, pseudo-differential, invertible operator. In fact, the inverse S_z of T_z is shown to be the operator obtained by restricting to the boundary the Green kernel of the Hodge Laplacian Δ acting on p -forms of Ω , for the absolute boundary conditions; as S_z is pseudo-differential of order -1 , the operator T_z is pseudo-differential of order 1. The restriction on z is imposed precisely because then T_z will be invertible, since z avoids the spectrum of Δ (which is contained in the non-negative half-line).

Our operator is obtained by taking $z = 0$ in (8): it is no longer invertible when $H^p(\Omega) \neq \{0\}$ but it is still pseudo-differential and elliptic because, by (b), its kernel is finite dimensional, isomorphic to $H^p(\Omega)$. In fact, the operator S_0 is now invertible modulo compact operators, given by the projection onto the kernel of T_0 and its transpose. The rest of Carron’s proof carries over and so $T_0 = T^{[p]}$ is an elliptic PDO of order 1. More generally, T_z is an elliptic PDO for all z , and is invertible as long as z does not belong to the spectrum of Δ .

The rest of (c) now follows from the standard theory of elliptic PDO (see [18]).

(d) The min–max principle gives

$$v_{1,p}(\Omega) = \inf \left\{ \frac{\int_{\Omega} \|d\hat{\phi}\|^2 + \|\delta\hat{\phi}\|^2}{\int_{\Sigma} \|\hat{\phi}\|^2} : \Delta\hat{\phi} = 0, i_N\hat{\phi} = 0 \right\}.$$

We only have to show that we can remove the condition $\Delta\hat{\phi} = 0$. This follows from the fact that among all tangential extensions ξ of a given form $\phi \in \Lambda^p(\Sigma)$, the harmonic tangential extension $\hat{\phi}$ minimizes the quadratic form $\int_{\Omega} \|d\xi\|^2 + \|\delta\xi\|^2$. Indeed, assume that $J^*\xi = \phi = J^*\hat{\phi}$ and $i_N\xi = 0 = i_N\hat{\phi}$. Let $\psi = \xi - \hat{\phi}$ so that $\psi = 0$ on the boundary. Using the Stokes formula one verifies that:

$$0 \leq \int_{\Omega} \|d\psi\|^2 + \|\delta\psi\|^2 = \int_{\Omega} \|d\xi\|^2 + \|\delta\xi\|^2 - \int_{\Omega} \|d\hat{\phi}\|^2 + \|\delta\hat{\phi}\|^2,$$

and the assertion follows. \square

2.1. The dual problem

Let $p = 0, \dots, n$. Given a p -form ϕ on Σ consider the unique $(p + 1)$ -form on Ω $\tilde{\phi}$ which satisfies:

$$\begin{cases} \Delta\tilde{\phi} = 0 & \text{on } \Omega, \\ J^*\tilde{\phi} = 0, \quad i_N\tilde{\phi} = \phi & \text{on } \Sigma. \end{cases}$$

The form $\tilde{\phi}$ will be called the *harmonic normal extension* of ϕ . Its existence and uniqueness is also proved in Schwarz [15]. We set

$$T_D^{[p]}\phi = J^*(\delta\tilde{\phi})$$

and call $T_D^{[p]}$ the *dual Dirichlet-to-Neumann operator*. It defines another elliptic pseudo-differential operator of order one acting on $\Lambda^p(\Sigma)$, which is self-adjoint and non-negative. These properties can easily be derived from Theorem 11 and the fact that $T_D^{[p]}$ is related to the operator $T^{[p]}$ by the identity $T_D^{[p]} = (-1)^{p(n-p)} \star_{\Sigma} T^{[n-p]} \star_{\Sigma}$, where \star_{Σ} denotes the Hodge-star operator acting on forms on Σ . Denoting by $v_{1,p}^D(\Omega)$ the first eigenvalue of $T_D^{[p]}$, we have

$$v_{1,p}^D(\Omega) = v_{1,n-p}(\Omega).$$

Moreover, the min–max principle for the dual problem takes the form:

$$v_{1,p}^D(\Omega) = \inf \left\{ \frac{\int_{\Omega} \|d\hat{\phi}\|^2 + \|\delta\hat{\phi}\|^2}{\int_{\Sigma} \|\hat{\phi}\|^2} : \hat{\phi} \in \Lambda^{p+1}(\Omega), J^*\hat{\phi} = 0 \right\}. \tag{9}$$

Note that $T_D^{[0]}$ is an operator acting on functions, which clearly differs from the operator $T^{[0]}$.

3. Lower bounds: proofs

3.1. Reilly formula for differential forms

The main tool used in the proof of the lower bound is a Reilly-type formula for differential forms proved by the authors in [12], which we state below.

Denote by S the shape operator of the immersion of Σ in Ω ; it is defined as $S(X) = -\nabla_X N$ for all tangent vectors $X \in T\Sigma$. S admits a canonical extension acting on p -forms on Σ and denoted by $S^{[p]}$. Explicitly, if ω is a p -form on Σ one has:

$$S^{[p]}\omega(X_1, \dots, X_p) = \sum_{j=1}^p \omega(X_1, \dots, S(X_j), \dots, X_p),$$

for tangent vectors $X_1, \dots, X_p \in T\Sigma$. It is clear from the definition that the eigenvalues of $S^{[p]}$ are precisely the p -curvatures of Σ : therefore we have immediately

$$\langle S^{[p]}\omega, \omega \rangle \geq \sigma_p(\Sigma) \|\omega\|^2$$

at all points of Σ and for all p -forms ω . Now let ω be a p -form on Ω . The Reilly formula says that

$$\int_{\Omega} \|d\omega\|^2 + \|\delta\omega\|^2 = \int_{\Omega} \|\nabla\omega\|^2 + \langle W^{[p]}(\omega), \omega \rangle + 2 \int_{\Sigma} \langle i_N\omega, \delta^{\Sigma}(J^*\omega) \rangle + \int_{\Sigma} \mathcal{B}(\omega, \omega), \quad (10)$$

where the boundary term has the following expression:

$$\begin{aligned} \mathcal{B}(\omega, \omega) &= \langle S^{[p]}(J^*\omega), J^*\omega \rangle + nH \|i_N\omega\|^2 - \langle S^{[p-1]}(i_N\omega), i_N\omega \rangle \\ &= \langle S^{[p]}(J^*\omega), J^*\omega \rangle + \langle S^{[n-p+1]}(J^*\star\omega), J^*\star\omega \rangle. \end{aligned}$$

By convention, we set $S^{[0]} = S^{[n+1]} = 0$. For a detailed proof of (10) see [12].

3.2. Proof of Theorem 1

We assume that $W^{[p]} \geq 0$, and that the p -curvatures of Σ are bounded below by $\sigma_p(\Sigma) > 0$. We have to prove that, if $p < \frac{n+1}{2}$ then:

$$v_{1,p}(\Omega) > \frac{n-p+2}{n-p+1} \sigma_p(\Sigma), \quad (11)$$

and if $p \geq \frac{n+1}{2}$ then

$$v_{1,p}(\Omega) \geq \frac{p+1}{p} \sigma_p(\Sigma). \quad (12)$$

Let ω be an eigenform associated to $v_{1,p}(\Omega)$ and let $\hat{\omega}$ be its harmonic tangential extension to Ω . By the variational characterization (7):

$$\int_{\Omega} \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 = \nu_{1,p}(\Omega) \int_{\Sigma} \|\omega\|^2 \tag{13}$$

because, on the boundary, $\|\hat{\omega}\|^2 = \|\omega\|^2 + \|i_N\hat{\omega}\|^2 = \|\omega\|^2$. We apply the Reilly formula to $\hat{\omega}$. As $W^{[p]} \geq 0$ and $i_N\hat{\omega} = 0$ we get

$$\begin{aligned} \int_{\Omega} (\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2) &\geq \int_{\Omega} \|\nabla\hat{\omega}\|^2 + \int_{\Sigma} \langle S^{[p]}(\omega), \omega \rangle \\ &\geq \int_{\Omega} \|\nabla\hat{\omega}\|^2 + \sigma_p(\Sigma) \int_{\Sigma} \|\omega\|^2. \end{aligned} \tag{14}$$

We will use the following estimate of Gallot and Meyer [8], valid for any p -form $\hat{\omega}$:

$$\|\nabla\hat{\omega}\|^2 \geq \frac{\|d\hat{\omega}\|^2}{p+1} + \frac{\|\delta\hat{\omega}\|^2}{n-p+2}. \tag{15}$$

When $p < \frac{n+1}{2}$ one has $p+1 < n-p+2$ hence:

$$\|\nabla\hat{\omega}\|^2 \geq \frac{\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2}{n-p+2}, \tag{16}$$

and the equality implies $d\hat{\omega} = 0$. Inserting (16) in (14), and taking into account (13), we obtain (11). Note that then $\nu_{1,p}(\Omega) > 0$. Equality in (16) implies that $d\hat{\omega} = 0$ hence $i_N d\hat{\omega} = 0$: but this is impossible because otherwise $\nu_{1,p}(\Omega) = 0$. So the inequality is always strict.

If $p \geq \frac{n+1}{2}$ one has

$$\|\nabla\hat{\omega}\|^2 \geq \frac{\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2}{p+1} \tag{17}$$

and proceeding as before we obtain (12). The inequality (12) is sharp: for the unit Euclidean ball we have $\sigma_p(\Sigma) = p$ and $\nu_{1,p}(\mathbf{B}^{n+1}) = p+1$ (see Proposition 7). We finally remark that, if $p > \frac{n+1}{2}$ and the equality holds in (12), it holds also in (17) and then $\delta\hat{\omega} = 0$. \square

Now we study the equality case of this estimate. Recall that the p -form $\hat{\omega}$ is a *conformal Killing form* if it satisfies the differential equation

$$\nabla_X \hat{\omega} = \frac{1}{p+1} i_X d\hat{\omega} - \frac{1}{n-p+2} X^* \wedge \delta\hat{\omega}$$

for all $X \in T\Omega$. A co-closed conformal Killing form is called a *Killing form*. It is well known that the inequality (15) is an equality if and only if $\hat{\omega}$ is a conformal Killing form (see for example [8]). We then have:

Proposition 12. Assume $p \geq (n + 1)/2$. If equality holds in (12) then the harmonic tangential extension of a p -eigenform associated to $v_{1,p}(\Omega)$ is a conformal Killing p -form (a Killing form if $p > \frac{n+1}{2}$) and the p lowest principal curvatures of the boundary are constant, equal to $c = v_{1,p}(\Omega)/(p + 1)$.

Proof. Looking at the proof of (12) we see immediately that if the equality holds then $\hat{\omega}$ is a conformal Killing form and, by the last remark in the proof, it is a Killing form when $p > \frac{n+1}{2}$. It remains to show the last assertion. Now, the Gauss formula leads to the following relations (see Section 6 in [12]):

$$\begin{cases} \nabla_X^\Sigma(i_N \hat{\omega}) = i_N \nabla_X \hat{\omega} - i_{S(X)} J^* \hat{\omega}, \\ \nabla_X^\Sigma(J^* \hat{\omega}) = J^*(\nabla_X \hat{\omega}) + S(X)^* \wedge i_N \hat{\omega}, \end{cases} \tag{18}$$

for all $X \in T\Sigma$, where ∇^Σ is the Levi–Civita connection of Σ . Since $\hat{\omega}$ is the harmonic tangential extension of ω , we have $i_N \hat{\omega} = 0$ and the first equation in (18) reads:

$$i_N \nabla_X \hat{\omega} = i_{S(X)} \omega. \tag{19}$$

On the other hand, since $\hat{\omega}$ is a conformal Killing p -form we have for all $X \in \Gamma(T\Sigma)$:

$$i_N \nabla_X \hat{\omega} = -\frac{1}{p + 1} i_X(i_N d\hat{\omega}) = \frac{v_{1,p}(\Omega)}{p + 1} i_X \omega. \tag{20}$$

We used the fact that $i_N \delta \hat{\omega} = -\delta^\Sigma(i_N \hat{\omega}) = 0$, which immediately implies $i_N(X^* \wedge \delta \hat{\omega}) = 0$. Combining (19) and (20) gives:

$$i_{S(X) - \frac{v_{1,p}(\Omega)}{p+1} X} \omega = 0 \tag{21}$$

for all $X \in \Gamma(T\Sigma)$. The form ω , being an eigenform of an elliptic operator, can't vanish on an open set and therefore is non-zero a.e. on Σ . Take a point x where it does not vanish: then, at x , there exists p principal directions, say v_1, \dots, v_p , such that $\omega(v_1, \dots, v_p) \neq 0$. Choosing successively $X = v_1, \dots, v_p$ one sees from (21) that the associated principal curvatures satisfy $\lambda_1 = \dots = \lambda_p = \frac{v_{1,p}(\Omega)}{p+1}$. \square

3.3. Proof of Theorem 2

Assume that Ω^{n+1} has non-negative Ricci curvature and that Σ has mean curvature bounded below by $H > 0$. Then $\sigma_n = nH$ and applying Theorem 1 for $p = n$ we get $v_{1,n}(\Omega) \geq (n + 1)H$. It remains to show that, if the equality holds, then Ω is a Euclidean ball. Now, under the given assumptions, we have $\text{Vol}(\Sigma)/\text{Vol}(\Omega) \geq (n + 1)H$ by Theorem 1 in [13], with equality if and only if Ω is a Euclidean ball. It is then enough to show that

$$\frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)} = (n + 1)H.$$

From Proposition 12, we know that if $\omega \in \Lambda^n(\Sigma)$ is an eigenform associated with $\nu_{1,n}(\Omega) = (n + 1)H$, then its harmonic tangential extension $\hat{\omega} \in \Lambda^n(\Omega)$ is a Killing n -form on Ω ; in particular, $\delta\hat{\omega} = 0$. We can write $d\hat{\omega} = f\Psi_\Omega$, where Ψ_Ω is the volume form of Ω and f is a smooth function. As $\hat{\omega}$ is harmonic and co-closed, we have

$$0 = \delta d\hat{\omega} = \delta(f\Psi_\Omega) = -i_{\nabla f}\Psi_\Omega,$$

which immediately implies $\nabla f = 0$. By renormalization, we can assume that $f = 1$ and so $d\hat{\omega}$ is the volume form of Ω . By assumption, $J^*d\hat{\omega} = 0$ and $i_N d\hat{\omega} = -(n + 1)H\omega$. Then, on Σ

$$1 = \|i_N d\hat{\omega}\|^2 = (n + 1)^2 H^2 \|\omega\|^2.$$

On the other hand, by the Stokes formula and the fact that $d\hat{\omega}$ has constant unit norm:

$$\text{Vol}(\Omega) = \int_\Omega \|d\hat{\omega}\|^2 = - \int_\Sigma \langle \omega, i_N d\hat{\omega} \rangle = (n + 1)H \int_\Sigma \|\omega\|^2 = \frac{\text{Vol}(\Sigma)}{(n + 1)H},$$

which proves the assertion.

3.4. The equality case for Euclidean domains: proof of Theorem 3

We fix $c > 0$ and let $\mathcal{F}_p(c)$ denote the set of p -forms $\hat{\omega}$ on Ω , $p = 0, \dots, n$, with the following properties:

- a) $\hat{\omega}$ is harmonic and tangential (that is $i_N \hat{\omega} = 0$ on Σ).
- b) $\hat{\omega}$ is Killing and $d\hat{\omega}$ is parallel.
- c) $i_N d\hat{\omega} = -(p + 1)c\omega$, where $\omega = J^* \hat{\omega}$ is the restriction to Σ .

Note that $\mathcal{F}_0(c)$ consists of all harmonic functions \hat{f} with parallel gradient and such that $\frac{\partial \hat{f}}{\partial N} = -c\hat{f}$: if \hat{f} is not trivial, its restriction to the boundary is a Dirichlet-to-Neumann eigenfunction associated to the eigenvalue c .

Lemma 13. *Let $p \geq 1$. If $\hat{\omega} \in \mathcal{F}_p(c)$ and V is a parallel vector field on \mathbf{R}^{n+1} , then $i_V \hat{\omega} \in \mathcal{F}_{p-1}(c)$.*

Proof. The Cartan formula gives $di_V \hat{\omega} + i_V d\hat{\omega} = \mathcal{L}_V \hat{\omega}$, where \mathcal{L}_V is the Lie derivative along V . If V is parallel and $\hat{\omega}$ is Killing, we have $\mathcal{L}_V \hat{\omega} = \nabla_V \hat{\omega} = \frac{1}{p+1} i_V d\hat{\omega}$ and then:

$$di_V \hat{\omega} = -\frac{p}{p + 1} i_V d\hat{\omega}. \tag{22}$$

Now $\nabla_V d\hat{\omega} = \mathcal{L}_V d\hat{\omega} = di_V d\hat{\omega} = 0$ by Cartan formula and (22). This holds for all parallel vector fields: in particular, any Killing form of degree $p \geq 1$ in Euclidean space has parallel exterior derivative.

Fix $\hat{\omega} \in \mathcal{F}_p(c)$. As V is parallel, i_V commutes with Δ and anticommutes with i_N . Then $i_V \hat{\omega}$ satisfies a).

As i_V anticommutes with δ , we see that $i_V \hat{\omega}$ is co-closed. On the other hand, since V is parallel:

$$\nabla_X i_V \hat{\omega} = i_V \nabla_X \hat{\omega} = \frac{1}{p+1} i_V i_X d\hat{\omega} = -\frac{1}{p+1} i_X i_V d\hat{\omega} = \frac{1}{p} i_X di_V \hat{\omega}$$

where we used (22) in the last equality. Hence $i_V \hat{\omega}$ is a Killing $(p-1)$ -form. A similar calculation shows that $\nabla_X di_V \hat{\omega} = 0$, hence $di_V \hat{\omega}$ is parallel and b) follows.

Finally, again using (22):

$$i_N di_V \hat{\omega} = -\frac{P}{p+1} i_N i_V d\hat{\omega} = \frac{P}{p+1} i_V i_N d\hat{\omega} = -pci_V \hat{\omega},$$

and c) follows as well. \square

Now assume that Ω is an extremal domain for our inequality, and let $\hat{\omega}$ be the tangential harmonic extension of an eigenform ω associated to $\nu_{1,p}(\Omega)$. Set $c = \nu_{1,p}(\Omega)/(p+1)$. By Proposition 12, $\hat{\omega}$ is a Killing p -form: in particular, as observed in the proof of Lemma 13, $d\hat{\omega}$ is parallel. Moreover, $i_N d\hat{\omega} = -(p+1)c\omega$ by definition. This means that $\hat{\omega}$ is a form in $\mathcal{F}_p(c)$. As $\hat{\omega}$ is non-trivial, we can find p parallel vector fields V_1, \dots, V_p such that the function $\hat{f} = \hat{\omega}(V_1, \dots, V_p)$ is non-trivial. Applying the lemma successively to the parallel fields V_1, \dots, V_p , we see that $\hat{f} \in \mathcal{F}_0(c)$, that is, \hat{f} satisfies

$$\begin{cases} \nabla d\hat{f} = 0 & \text{on } \Omega, \\ \frac{\partial \hat{f}}{\partial N} = -c\hat{f} & \text{on } \Sigma. \end{cases}$$

By Proposition 12, the lowest p principal curvatures are constant, equal to c , and then $S \geq c$. We now apply Theorem 19 in Appendix A, to conclude that Ω is a Euclidean ball. The proof of Theorem 3 is now complete.

3.5. An inequality for consecutive degrees: proof of Theorem 4

We have to show that if Ω is a domain in \mathbf{R}^{n+1} , then for all $p = 1, \dots, n$:

$$\nu_{1,p}(\Omega) \geq \nu_{1,p-1}(\Omega) + \frac{\sigma_p(\Sigma)}{p}. \tag{23}$$

For the proof, we consider the family of unit length parallel vector fields on \mathbf{R}^{n+1} , which is naturally identified with \mathbf{S}^n .

Let $\omega \in \Lambda^p(\Sigma)$ be an eigenform associated to the eigenvalue $\nu_{1,p}(\Omega)$ and denote by $\hat{\omega}$ its harmonic tangential extension. Let V be a unit length parallel vector field. Since Δ commutes with the contraction i_V , the $(p-1)$ -form $i_V \hat{\omega}$ is harmonic. Moreover we clearly have $i_N i_V \hat{\omega} = 0$. Hence we can use $i_V \hat{\omega}$ as test form for the eigenvalue $\nu_{1,p-1}(\Omega)$, and by the min–max principle we have

$$\nu_{1,p-1}(\Omega) \int_{\Sigma} \|i_V \hat{\omega}\|^2 \leq \int_{\Omega} \|di_V \hat{\omega}\|^2 + \|\delta i_V \hat{\omega}\|^2 \tag{24}$$

for all $V \in \mathbf{S}^n$. Now we want to integrate this inequality with respect to $V \in \mathbf{S}^n$. In order to simplify the formulae, we use the renormalized measure

$$d\mu = \frac{n+1}{\text{Vol}(\mathbf{S}^n)} \text{dvol}_{\mathbf{S}^n},$$

where $\text{dvol}_{\mathbf{S}^n}$ is the canonical measure of \mathbf{S}^n . Then, we have the following identities, which are valid pointwise and are proved in [9] (Lemma 4.8, p. 336):

$$\begin{aligned} \int_{\mathbf{S}^n} \|i_V \hat{\omega}\|^2 d\mu(V) &= p \|\hat{\omega}\|^2, \\ \int_{\mathbf{S}^n} \|di_V \hat{\omega}\|^2 d\mu(V) &= \|\nabla \hat{\omega}\|^2 + (p-1) \|d\hat{\omega}\|^2, \\ \int_{\mathbf{S}^n} \|\delta i_V \hat{\omega}\|^2 d\mu(V) &= \int_{\mathbf{S}^n} \|i_V \delta \hat{\omega}\|^2 d\mu(V) = (p-1) \|\delta \hat{\omega}\|^2. \end{aligned}$$

Integrating (24) with respect to $V \in \mathbf{S}^n$ and using the previous identities, we then have, by the Fubini theorem:

$$p\nu_{1,p-1}(\Omega) \int_{\Sigma} \|\hat{\omega}\|^2 \leq \int_{\Omega} \|\nabla \hat{\omega}\|^2 + (p-1) \int_{\Omega} \|d\hat{\omega}\|^2 + \|\delta \hat{\omega}\|^2.$$

On the other hand, the Reilly formula (10) applied to $\hat{\omega}$ gives:

$$\int_{\Omega} \|d\hat{\omega}\|^2 + \|\delta \hat{\omega}\|^2 = \int_{\Omega} \|\nabla \hat{\omega}\|^2 + \int_{\Sigma} \langle S^{[p]}(J^* \hat{\omega}, \cdot) J^* \hat{\omega} \rangle \geq \int_{\Omega} \|\nabla \hat{\omega}\|^2 + \sigma_p(\Sigma) \int_{\Sigma} \|\hat{\omega}\|^2.$$

Eliminating $\int_{\Omega} \|\nabla \hat{\omega}\|^2$ in the previous two inequalities leads to:

$$\begin{aligned} p\nu_{1,p-1}(\Omega) \int_{\Sigma} \|\hat{\omega}\|^2 &\leq p \int_{\Omega} (\|d\hat{\omega}\|^2 + \|\delta \hat{\omega}\|^2) - \sigma_p(\Sigma) \int_{\Sigma} \|\hat{\omega}\|^2 \\ &= p\nu_{1,p}(\Omega) \int_{\Sigma} \|\hat{\omega}\|^2 - \sigma_p(\Sigma) \int_{\Sigma} \|\hat{\omega}\|^2. \end{aligned}$$

Dividing both sides by $p \int_{\Sigma} \|\hat{\omega}\|^2$ proves (23).

4. Upper bounds: proofs

4.1. Upper bounds by the isoperimetric ratio

A p -form ξ is said to be a *harmonic field* if $d\xi = \delta\xi = 0$. We start from the following:

Proposition 14. *Let ξ be a harmonic field of degree p on Ω .*

- (a) *If ξ is exact and $p = 2, \dots, n + 1$ then $\nu_{1,p-1}(\Omega) \int_{\Omega} \|\xi\|^2 \leq \int_{\Sigma} \|i_N \xi\|^2$.*
- (b) *If ξ is exact and $p = 1$ then $\nu_{2,0}(\Omega) \int_{\Omega} \|\xi\|^2 \leq \int_{\Sigma} \|i_N \xi\|^2$.*
- (c) *If ξ is co-exact and $p = 1, \dots, n$ then $\nu_{1,n-p}(\Omega) \int_{\Omega} \|\xi\|^2 \leq \int_{\Sigma} \|J^* \xi\|^2$.*

Proof. (a) By the Hodge–Morrey decomposition (see [15]) if ξ is an exact p -form, there is a unique co-exact (hence co-closed) $(p - 1)$ -form ω , called the *canonical primitive of ξ* , which satisfies:

$$\begin{cases} d\omega = \xi, \\ i_N \omega = 0 \quad \text{on } \Sigma. \end{cases}$$

We use ω as a test-form for the eigenvalue $\nu_{1,p-1}(\Omega)$ and then

$$\nu_{1,p-1}(\Omega) \int_{\Sigma} \|\omega\|^2 \leq \int_{\Omega} \|d\omega\|^2.$$

By the Stokes formula $\int_{\Omega} \|d\omega\|^2 = - \int_{\Sigma} \langle i_N d\omega, J^* \omega \rangle$; by the Schwarz inequality

$$\left(\int_{\Omega} \|d\omega\|^2 \right)^2 \leq \int_{\Sigma} \|i_N d\omega\|^2 \cdot \int_{\Sigma} \|\omega\|^2.$$

Eliminating $\int_{\Sigma} \|\omega\|^2$ from the previous two inequalities we get

$$\nu_{1,p-1}(\Omega) \int_{\Omega} \|d\omega\|^2 \leq \int_{\Sigma} \|i_N d\omega\|^2,$$

which is the assertion. We remark that the equality holds if and only if the canonical primitive of ξ is an eigenform of $T^{[p-1]}$ associated to $\nu_{1,p-1}(\Omega)$.

(b) If ξ is an exact harmonic field of degree 1, then $\xi = df$ for a harmonic function f . We can assume that f integrates to zero on Σ , and so we can use f as a test function for the eigenvalue $\nu_{2,0}(\Omega)$. The rest of the proof is as in (a).

(c) Let ξ be a co-exact p -harmonic field. Then $\star\xi$ is an exact $(n - p + 1)$ -harmonic field and we can apply (a) to it. The inequality follows because $\|i_N \star\xi\|^2 = \|\star_{\Sigma} J^* \xi\|^2 = \|J^* \xi\|^2$. If the equality holds, then the canonical primitive of $\star\xi$ is an eigenform of $T^{[n-p]}$ associated to $\nu_{1,n-p}(\Omega)$.

We can also characterize the equality by duality, as follows. If ξ is co-exact, it has a unique *canonical co-primitive*, that is, a unique exact $(p + 1)$ -form α such that:

$$\begin{cases} \delta\alpha = \xi, \\ J^*\alpha = 0 \quad \text{on } \Sigma. \end{cases}$$

It is clear that if we have equality then α is an eigenform of the dual operator $T_D^{[p]}$ associated to $v_{1,p}^D(\Omega) = v_{1,n-p}(\Omega)$. That is,

$$J^*(\delta\alpha) = v_{1,n-p}(\Omega)i_N\alpha. \quad \square$$

We remark that if $H^p(\Omega) = 0$ (resp. $H_R^p(\Omega) = 0$) then any p -harmonic field is automatically exact (resp. co-exact). Therefore, as at any point of the boundary one has $\|\xi\|^2 = \|J^*\xi\|^2 + \|i_N\xi\|^2$, we have, summing the two inequalities of the proposition:

Corollary 15. *Assume that $H^p(\Omega) = H_R^p(\Omega) = 0$. Let ξ be a harmonic field of degree p .*

(a) *If $p = 2, \dots, n$ then $v_{1,p-1}(\Omega) + v_{1,n-p}(\Omega) \leq \int_\Sigma \|\xi\|^2 / \int_\Omega \|\xi\|^2$. If ξ is parallel then it has constant norm and*

$$v_{1,p-1}(\Omega) + v_{1,n-p}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}.$$

(b) *If $p = 1$ then $v_{2,0}(\Omega) + v_{1,n-1}(\Omega) \leq \int_\Sigma \|\xi\|^2 / \int_\Omega \|\xi\|^2$.*

(c) *In particular, if $H_R^1(\Omega) = 0$ and f is any harmonic function then*

$$v_{2,0}(\Omega) + v_{1,n-1}(\Omega) \leq \frac{\int_\Sigma \|df\|^2}{\int_\Omega \|df\|^2}.$$

On the other hand, the volume form of Ω is parallel, exact and has degree $n + 1$. Then it follows directly from the first point of Proposition 14 that, for all compact manifolds with boundary, one has the sharp bound:

$$v_{1,n}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}. \tag{25}$$

We have equality in (25) when $\Omega = \mathbf{B}^{n+1}$ is the unit Euclidean ball: in fact $\text{Vol}(\Sigma) / \text{Vol}(\Omega) = n + 1$ and by the main lower bound (Theorem 2) we have $v_{1,n}(\mathbf{B}^{n+1}) \geq n + 1$. So $v_{1,n}(\mathbf{B}^{n+1}) = n + 1$. We will reprove (25) and discuss its equality case in Section 4.2.

We end this section with the following calculation.

Proposition 16. *We have $v_{1,p}(\mathbf{B}^{n+1}) = p + 1$ for all $p \geq \frac{n+1}{2}$.*

Proof. Let $\Omega = \mathbf{B}^{n+1}$ and let $v_{1,p} = v_{1,p}(\mathbf{B}^{n+1})$. We just observed that $v_{1,n} = n + 1$. We now use Theorem 4; as $\sigma_p(\Sigma) / p = 1$ for all p , we see that $v_{1,p} \geq v_{1,p-1} + 1$. Then $v_{1,n-1} \leq n$ and, by induction, $v_{1,p} \leq p + 1$ for all p . However, when $p \geq (n + 1) / 2$, Theorem 1 applied to Ω gives $v_{1,p} \geq p + 1$ and so $v_{1,p} = p + 1$. \square

For later use, we observe the following:

Proposition 17. Assume that Ω supports a non-constant linear function, that is, a smooth function f with df non-trivial and parallel. If $H_R^1(\Omega) = 0$, then:

$$v_{2,0}(\Omega) + v_{1,n-1}(\Omega) \leq \frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)}.$$

- (a) If the equality holds, then Σ has constant positive mean curvature $H = v_{1,n-1}(\Omega)/n$, and the restriction of f to Σ is an eigenfunction of Δ^Σ associated to the eigenvalue $\lambda \doteq v_{2,0}(\Omega)v_{1,n-1}(\Omega)$.
- (b) If $n = \dim(\Sigma) \geq 3$ and $\Omega \subset \mathbf{R}^{n+1}$, then the equality holds if and only if Ω is a ball.

Proof. The inequality follows immediately from (c) of Corollary 15 applied to df (which has constant norm by our assumptions). We can assume that f integrates to zero on Σ .

(a) If the equality holds, then f has to be a Dirichlet-to-Neumann eigenfunction associated to $v_{2,0}(\Omega)$:

$$\frac{\partial f}{\partial N} = -v_{2,0}(\Omega)f,$$

and (see the proof of Proposition 14) the canonical co-primitive α of df , solution of the problem

$$\begin{cases} \delta\alpha = df, & d\alpha = 0, \\ J^*\alpha = 0 & \text{on } \Sigma \end{cases}$$

must be a dual eigenform associated to $v_{1,1}^D(\Omega) = v_{1,n-1}(\Omega)$:

$$J^*(\delta\alpha) = v_{1,n-1}(\Omega)i_N\alpha.$$

As $J^*(\delta\alpha) = d^\Sigma f$ we have $d^\Sigma f = v_{1,n-1}(\Omega)i_N\alpha$. It follows that

$$\begin{aligned} \Delta^\Sigma f &= \delta^\Sigma d^\Sigma f = v_{1,n-1}(\Omega)\delta^\Sigma i_N\alpha = -v_{1,n-1}(\Omega)i_N\delta\alpha \\ &= -v_{1,n-1}(\Omega)\frac{\partial f}{\partial N} = v_{2,0}(\Omega)v_{1,n-1}(\Omega)f, \end{aligned}$$

that is, f is an eigenfunction of Δ^Σ associated to $v_{2,0}(\Omega)v_{1,n-1}(\Omega)$, as asserted. Observe that then $v_{1,n-1}(\Omega) > 0$ otherwise f would be constant. To prove the first assertion, recall that, for any smooth function on Ω one has, at all points of Σ :

$$\Delta f = \Delta^\Sigma f - \frac{\partial^2 f}{\partial N^2} + nH \frac{\partial f}{\partial N}.$$

As $\nabla^2 f = 0$, we have $\Delta f = 0$ and $\frac{\partial^2 f}{\partial N^2} = 0$, and we easily obtain $nH = v_{1,n-1}(\Omega)$.

(b) The equality holds for the Euclidean unit ball, by Proposition 7 (it is known that $v_{2,0}(\mathbf{B}^{n+1}) = 1$). Now, if the equality holds, then Σ has constant mean curvature by (a), hence Σ is a sphere by a well-known result of Alexandrov. \square

4.2. Harmonic domains

Recall that the domain Ω is called *harmonic* if $\partial E/\partial N$ is constant on Σ , where E is the mean-exit time function, solution of the problem $\Delta E = 1$ on Ω , $E = 0$ on Σ . Any ball in a constant curvature space form is harmonic, simply because the mean-exit time function is radially symmetric. We observe the following equivalent condition.

Proposition 18. Ω is harmonic if and only if, for all harmonic functions f on Ω , one has:

$$\frac{1}{\text{Vol}(\Omega)} \int_{\Omega} f = \frac{1}{\text{Vol}(\Sigma)} \int_{\Sigma} f$$

(that is, the mean value of any harmonic function on the domain equals its mean value on the boundary).

Proof. Assume that Ω is harmonic and let f be any harmonic function on Ω . By the definition of E and the Green formula, we have:

$$\int_{\Omega} f = \int_{\Omega} f \Delta E = \int_{\Sigma} f \frac{\partial E}{\partial N}.$$

As $\partial E/\partial N$ is constant, say equal to c , we have $\int_{\Omega} f = c \int_{\Sigma} f$. Taking $f = 1$ we see that $c = \text{Vol}(\Omega)/\text{Vol}(\Sigma)$ and the first half is proved.

Conversely, assume that the above mean-value property is true for all harmonic functions on Ω . Fix a point $x \in \Sigma$ and let $f_k \in C^{\infty}(\Sigma)$ be a sequence of functions converging to the Dirac measure of Σ at x as $k \rightarrow \infty$. Let \hat{f}_k be the harmonic extension of f_k . Then $\int_{\Omega} \hat{f}_k = \int_{\Sigma} f_k \partial E/\partial N$ and the assumption gives

$$\frac{\text{Vol}(\Omega)}{\text{Vol}(\Sigma)} \int_{\Sigma} f_k = \int_{\Sigma} f_k \frac{\partial E}{\partial N}$$

for all k . Letting $k \rightarrow \infty$ we obtain

$$\frac{\text{Vol}(\Omega)}{\text{Vol}(\Sigma)} = \frac{\partial E}{\partial N}(x).$$

As x is arbitrary, we see that $\partial E/\partial N$ is indeed constant on Σ . \square

4.3. Proof of Theorem 5

It is perhaps simpler to reprove the inequality using the dual operator $T_D^{[0]}$, with first eigenvalue $\nu_{1,0}^D(\Omega) = \nu_{1,n}(\Omega)$. So, we need to show that $\nu_{1,0}^D(\Omega) \leq \text{Vol}(\Sigma)/\text{Vol}(\Omega)$. Consider the 1-form $\alpha = dE$. Then $J^*\alpha = 0$ and we can use α as a test-form for $\nu_{1,0}^D(\Omega)$. Since $i_N\alpha = \partial E/\partial N$, by the variational characterization (9) we get

$$v_{1,0}^D(\Omega) \int_{\Sigma} \left(\frac{\partial E}{\partial N} \right)^2 \leq \int_{\Omega} \|\delta\alpha\|^2 = \text{Vol}(\Omega).$$

By the Schwarz inequality:

$$\int_{\Sigma} \left(\frac{\partial E}{\partial N} \right)^2 \geq \frac{1}{\text{Vol}(\Sigma)} \left(\int_{\Sigma} \frac{\partial E}{\partial N} \right)^2 = \frac{\text{Vol}(\Omega)^2}{\text{Vol}(\Sigma)}$$

and the inequality follows immediately.

If the equality holds then $\partial E/\partial N$ must be constant and then Ω is a harmonic domain. Conversely, assume that Ω is harmonic. Then the normal derivative of E is constant along Σ , and equals $c = \text{Vol}(\Omega)/\text{Vol}(\Sigma)$. Let $\alpha = dE$. Then

$$\begin{cases} \Delta\alpha = 0, \\ J^*\alpha = 0, \end{cases} \quad i_N\alpha = \frac{\partial E}{\partial N} = c.$$

By the definition of $T_D^{[0]}$:

$$T_D^{[0]}(c) = J^*(\delta\alpha) = 1$$

because $\delta\alpha = \Delta E = 1$. This shows that $1/c$ is an eigenvalue of $T_D^{[0]}$ as asserted, and the associated eigenfunction is constant.

4.4. Hodge–Laplace eigenvalues: proof of Theorem 8

Fix a degree $p = 1, \dots, n$. We assume that $H_R^p(\Omega) = 0$, $\min(\sigma_p(\Sigma), \sigma_{n-p+1}(\Sigma)) \geq 0$ and $W^{[p]} \geq 0$. We have to show:

$$\lambda'_{1,p}(\Sigma) \geq \frac{1}{2}(\sigma_p(\Sigma)v_{1,n-p}(\Omega) + \sigma_{n-p+1}(\Sigma)v_{1,p-1}(\Omega)). \tag{26}$$

Let ϕ be a co-exact eigenform associated to $\lambda = \lambda''_{1,p-1}(\Sigma) = \lambda'_{1,p}(\Sigma)$ and consider the exact p -eigenform $\omega = d^\Sigma\phi$ also associated to λ . Let $\hat{\phi}$ be a solution of

$$\begin{cases} \Delta\hat{\phi} = 0 & \text{on } \Omega, \\ J^*\hat{\phi} = \phi, \quad J^*(\delta\hat{\phi}) = 0 & \text{on } \Sigma, \end{cases}$$

which exists by Lemma 3.4.7 in [15]. Then, using the Stokes formula one checks that $\delta d\hat{\phi} = 0$ on Ω (the extension $\hat{\phi}$ first appeared in the paper of Duff and Spencer [4]).

If we let $\hat{\omega} = d\hat{\phi}$, then $\hat{\omega}$ is an exact p -harmonic field satisfying:

$$\begin{cases} d\hat{\omega} = \delta\hat{\omega} = 0 & \text{on } \Omega, \\ J^*\hat{\omega} = \omega & \text{on } \Sigma. \end{cases}$$

We apply the Reilly formula (10) to $\hat{\omega}$; as $W^{[p]} \geq 0$ and $\delta^\Sigma(J^*\hat{\omega}) = \delta^\Sigma d^\Sigma \phi = \lambda\phi$ we obtain

$$-2\lambda \int_\Sigma \langle i_N \hat{\omega}, \phi \rangle \geq \int_\Sigma \langle S^{[p]}(J^*\hat{\omega}), J^*\hat{\omega} \rangle + \langle S^{[n-p+1]}(J^*\hat{\omega}), J^*\hat{\omega} \rangle.$$

The Stokes formula gives:

$$\int_\Sigma \langle i_N \hat{\omega}, \phi \rangle = \int_\Sigma \langle i_N d\hat{\phi}, J^*\hat{\phi} \rangle = \int_\Omega \langle \hat{\phi}, \delta d\hat{\phi} \rangle - \int_\Omega \|d\hat{\phi}\|^2 = - \int_\Omega \|\hat{\omega}\|^2.$$

By our curvature assumptions, we end-up with

$$2\lambda \int_\Sigma \|\hat{\omega}\|^2 \geq \sigma_p(\Sigma) \int_\Sigma \|J^*\hat{\omega}\|^2 + \sigma_{n-p+1}(\Sigma) \int_\Sigma \|i_N \hat{\omega}\|^2.$$

The p -harmonic field $\hat{\omega}$ is exact, and also co-exact because $H_R^p(\Omega) = 0$. We can then apply Proposition 14 to estimate the boundary integrals in the right-hand side, and the estimate (26) follows.

4.5. Proof of Theorem 9

Let $\lambda_1(\Sigma)$ be the first positive eigenvalue of the Laplacian on functions of Σ . We assume that Ω has non-negative Ricci curvature and that Σ is strictly convex, with principal curvatures bounded below by $\sigma_1(\Sigma) > 0$. We have to show that

$$\lambda_1(\Sigma) \geq \frac{1}{2}(\sigma_1(\Sigma)v_{1,n-1}(\Omega) + nHv_{2,0}(\Omega)). \tag{27}$$

Moreover, if $n = \dim(\Sigma) \geq 3$, the equality holds if and only if Ω is a Euclidean ball.

Proof. Let ϕ be an eigenfunction associated to $\lambda_1(\Sigma)$, $\hat{\phi}$ its harmonic extension to Ω and $\hat{\omega} = d\hat{\phi}$. Then $\hat{\omega}$ is a harmonic field of degree 1. We apply the Reilly formula to $\hat{\omega}$; as $\|\nabla \hat{\omega}\|^2 \geq 0$ and $\sigma_n(\Sigma) = nH$, we obtain:

$$2\lambda \int_\Sigma \|\hat{\omega}\|^2 \geq \sigma_1(\Sigma) \int_\Sigma \|J^*\hat{\omega}\|^2 + nH \int_\Sigma \|i_N \hat{\omega}\|^2.$$

Note that, if the equality holds, then $\hat{\omega}$ must be parallel. Our curvature assumptions imply in particular that $H_R^1(\Omega) = 0$. Therefore we can apply Proposition 14 and obtain

$$\int_\Sigma \|J^*\hat{\omega}\|^2 \geq v_{1,n-1}(\Omega) \int_\Omega \|\hat{\omega}\|^2 \quad \text{and} \quad \int_\Sigma \|i_N \hat{\omega}\|^2 \geq v_{2,0}(\Omega) \int_\Omega \|\hat{\omega}\|^2. \tag{28}$$

The lower bound (27) follows. The estimate is sharp because, for the Euclidean unit ball, we have $\lambda_1(\Sigma) = \lambda_1(\mathbf{S}^n) = n$, $v_{2,0}(\mathbf{B}^{n+1}) = 1$ and, for $n \geq 3$, $v_{1,n-1}(\mathbf{B}^{n+1}) = n$.

Now assume that (27) is an equality. Then $\hat{\omega} = d\hat{\phi}$ is parallel, and we can apply Proposition 17 to $f = \hat{\phi}$. However, as we must have equalities in (28), we conclude that

$$\frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)} = v_{2,0}(\Omega) + v_{1,n-1}(\Omega),$$

and we are in the equality case of Proposition 17. So the mean curvature is constant: $nH = v_{1,n-1}(\Omega)$ and $\lambda_1(\Sigma) = v_{2,0}(\Omega)v_{n-1}(\Omega)$. By assumption

$$2\lambda_1(\Sigma) = \sigma_1(\Sigma)v_{1,n-1}(\Omega) + nHv_{2,0}(\Omega)$$

and we easily obtain $v_{2,0}(\Omega) = \sigma_1(\Sigma)$. Now, at each point of Σ , the mean curvature is always no less than the lowest principal curvature, which implies that $H \geq \sigma_1(\Sigma) = v_{2,0}(\Omega)$. We arrive at the inequality

$$\frac{\text{Vol}(\Sigma)}{\text{Vol}(\Omega)} \leq (n + 1)H.$$

By the result of Ros already cited ([13]) we know that $\text{Vol}(\Sigma)/\text{Vol}(\Omega) \geq (n + 1)H$ with equality if and if Ω is a Euclidean ball. Then Ω must be a Euclidean ball, and the proof is complete. \square

4.6. Biharmonic operator: proof of Theorem 10

We now consider the fourth order Steklov problem (6) and its first eigenvalue $\mu_1(\Omega)$. As $v_{1,n}(\Omega) = v_{1,0}^D(\Omega)$ it is enough to show that

$$\mu_1(\Omega) \geq v_{1,0}^D(\Omega).$$

Let f be a first eigenfunction associated to $\mu_1(\Omega)$. As $J^*(df) = 0$ we can use df as a test-form in (9). Then

$$v_{1,0}^D(\Omega) \leq \frac{\int_{\Omega} (\Delta f)^2}{\int_{\Sigma} \left(\frac{\partial f}{\partial N}\right)^2} = \mu_1(\Omega)$$

where the equality follows from the Rayleigh–Ritz characterization of $\mu_1(\Omega)$ (see [7]). If equality holds, then df must be an eigenform of $T_D^{[0]}$ associated to $v_{1,0}^D(\Omega)$, hence $\Delta df = 0$. But then Δf is a constant, and we can assume $\Delta f = 1$. As $f = 0$ on Σ we see that $f = E$, the mean-exit time function, and the boundary conditions satisfied by f imply that the normal derivative of E is constant. Hence Ω is harmonic.

Appendix A

Here we state a general result which gives sufficient conditions on a manifold to be isometric with a Euclidean ball. This result is used in the proof of Theorem 3 but it is perhaps of independent interest.

Theorem 19. *Let (Ω^{n+1}, g) a compact, connected Riemannian manifold with smooth boundary Σ . Assume that there exist a non-trivial function $f \in C^\infty(\Omega)$ and a number $c > 0$ such that:*

$$\begin{cases} \nabla df = 0 & \text{on } \Omega, \\ \frac{\partial f}{\partial N} = -cf & \text{on } \Sigma. \end{cases}$$

If Ω has non-negative sectional curvature and the second fundamental form of Σ satisfies $S \geq c$, then Ω is isometric with a Euclidean ball.

Proof. It is enough to prove that the boundary is isometric to a round sphere. Then, by Theorem 1 in [21], we conclude that (Ω^{n+1}, g) is isometric with a Euclidean ball.

Here are the main steps. We prove that:

- a) Σ is connected.
- b) $\text{Ric}^\Sigma \geq (n - 1)c^2$.
- c) Σ has diameter greater than or equal to $\frac{\pi}{c}$.

The proof of the theorem will follow by observing that, by Myers’ theorem and a), b), one has $\text{diam}(\Sigma) \leq \frac{\pi}{c}$; hence, by c), the diameter is equal to $\frac{\pi}{c}$. By the rigidity theorem of Cheng [3], Σ is isometric to a sphere of radius $1/c$, as asserted.

We prove a). Looking at the long exact sequence of the pair (Ω, Σ) , it is enough to show that $H_R^1(\Omega) = 0$: in fact, in that case $H^0(\Sigma) \sim H^0(\Omega) \sim \mathbf{R}$. Now the Ricci curvature of Ω is non-negative and the mean curvature of Σ is bounded below by $c > 0$: by Theorem 2 we have $\nu_{1,n}(\Omega) > 0$ and then $H^n(\Omega) = H_R^1(\Omega) = 0$.

We prove b). It is enough to prove that, for any unit length tangent vector $X \in T\Sigma$, one has $\text{Ric}^\Sigma(X, X) \geq (n - 1)c^2$. The Gauss lemma and the non-negativity of the sectional curvatures of Ω give:

$$\text{Ric}^\Sigma(X, X) \geq nH\langle S(X), X \rangle - |S(X)|^2.$$

Fix an orthonormal frame (e_1, \dots, e_n) of principal directions, so that $S(e_j) = \eta_j e_j$ for all j . Then:

$$\text{Ric}^\Sigma(X, X) \geq \sum_{j=1}^n (\eta_j(nH - \eta_j))\langle X, e_j \rangle^2;$$

as $\eta_j \geq c$ for all j one sees that $\eta_j(nH - \eta_j) \geq (n - 1)c^2$ for all j and the assertion follows.

We finally prove c). Since ∇f is parallel we have that $|\nabla f|$ is constant on Ω , and we can assume that it is equal to 1. The restriction of f is continuous on Σ , which is compact: then let $p_+ \in \Sigma$ (resp. $p_- \in \Sigma$) be a point where the restriction of f is maximum (resp. minimum). We prove $d(p_-, p_+) \geq \frac{\pi}{c}$. Now:

$$1 = |\nabla f|^2(p_\pm) = |\nabla^\Sigma f|^2(p_\pm) + \left(\frac{\partial f}{\partial N}\right)^2(p_\pm) = c^2 f(p_\pm)^2.$$

The function f is not constant on Σ (because it is harmonic on Ω and $c > 0$) therefore:

$$f(p_+) = \frac{1}{c}, \quad f(p_-) = -\frac{1}{c}.$$

As Σ is connected, there exists a minimizing geodesic $\gamma : [0, l] \rightarrow \Sigma$ parametrized by arc length and joining p_- with p_+ . So we have $\gamma(0) = p_-$, $\gamma(l) = p_+$ and the distance from p_- to p_+ is l . It is now enough to prove that $l \geq \frac{\pi}{c}$.

Let $\alpha(t) := f \circ \gamma(t)$ for $t \in [0, l]$, so that $\alpha'(t) \leq |\nabla^\Sigma f(\gamma(t))|$. Since ∇f has unit length we have: $1 = |\nabla^\Sigma f|^2(\gamma(t)) + c^2\alpha(t)^2$ and therefore

$$|\alpha'(t)|^2 \leq 1 - c^2\alpha(t)^2.$$

Fix $\varepsilon > 0$ and let $A = \{t \in [0, l]: \alpha'(t) > 0\}$. Then:

$$l \geq \int_A dt \geq \int_A \frac{\alpha'(t) dt}{\sqrt{1 - c^2\alpha(t)^2} + \varepsilon} \geq \int_0^l \frac{\alpha'(t) dt}{\sqrt{1 - c^2\alpha(t)^2} + \varepsilon}.$$

Changing variables and observing that $\alpha(0) = -\frac{1}{c}$ and $\alpha(l) = \frac{1}{c}$ we have

$$l \geq \frac{1}{c} \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2} + \varepsilon}.$$

Letting $\varepsilon \rightarrow 0^+$ gives $l \geq \frac{\pi}{c}$, as asserted. \square

Finally, we remark that the conclusion of the theorem holds also if the assumption on the non-negativity of the sectional curvature is replaced by the following assumptions: the Ricci curvature of Ω is non-negative, and the mean curvature of Σ is constant. We omit the details.

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