On the max version of the generalized spectral radius theorem

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Abstract

Let $\Psi$ be a bounded set of $n \times n$ non-negative matrices. Recently, the max algebra version $\mu(\Psi)$ of the generalized spectral radius of $\Psi$ was introduced. We show that

$$
\mu(\Psi) = \lim_{t \to \infty} \rho(\Psi^{(t)})^{1/t},
$$

where $\rho$ denotes the generalized spectral radius and $\Psi^{(t)}$ the Hadamard power of $\Psi$. This provides a description of $\mu(\Psi)$ that uses no max terminology. As an application, we give a short proof of the max version of the generalized spectral radius theorem.

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1. Introduction

In the recent years, the algebraic system called max algebra inspired a lot of interest (see e.g. [1,3,8] and the references cited there). It has been used to describe certain conventionally non-linear systems in a linear fashion.
Following the notation from [1,8,15,16], the max algebra consists of the set of non-negative numbers with sum \( a \oplus b = \max(a, b) \) and the standard product \( ab \), where \( a, b \geq 0 \). Let \( A = [a_{ij}] \) be a \( n \times n \) non-negative matrix, i.e. \( a_{ij} \geq 0 \) for all \( i, j = 1, \ldots, n \). We may denote \( a_{ij} \) also by \( [A]_{ij} \). Given \( A \) and \( B \) two non-negative matrices, we say that \( A \preceq B \) if \( B - A \) is non-negative. Let \( \mathbb{R}^{n \times n}_+ \) be the set of all \( n \times n \) real matrices and \( \mathbb{R}^{n \times n}_+^2 \) the set of all \( n \times n \) non-negative matrices. The product of \( A, B \in \mathbb{R}^{n \times n}_+ \) in the max algebra is denoted by \( A \otimes B \), where \( [A \otimes B]_{ij} = \max_{k=1}^n a_{ik} b_{kj} \). The notation \( A_0^2 \) means \( A \otimes A \), and \( A_k^k \) denotes the \( k \)-th max power of \( A \). If \( x = [x_i] \in \mathbb{R}^n \) is a non-negative vector, then the notation \( A \otimes x \) means \( [A \otimes x]_i = \max_{j=1}^n a_{ij} x_j \). The usual associative and distributive laws hold in this algebra. Note that the standard products are denoted by \( AB \) and \( Ax \).

The weighted directed graph \( \mathcal{D}(A) \) associated with \( A \) has a vertex set \( \{1, 2, \ldots, n\} \) and edges \( (i, j) \) from a vertex \( i \) to a vertex \( j \) with weight \( a_{ij} \) if and only if \( a_{ij} > 0 \). A path of length \( k \) is a sequence of edges \( (i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1}) \). A circuit of length \( k \) is a path with \( i_{k+1} = i_1 \), where \( i_1, i_2, \ldots, i_k \) are distinct. Associated with this circuit is the circuit geometric mean known as \( (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})^{1/k} \). The maximum circuit geometric mean in \( \mathcal{D}(A) \) is denoted by \( \mu(A) \). Note that circuits \( (i_1, i_1) \) of length 1 (loops) are included here and that we also consider empty circuits, i.e., circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero.

The maximum circuit geometric mean \( \mu(A) \) was utilized in [11] and has been studied extensively since. It was proved in [12] that given \( A \in \mathbb{R}^{n \times n}_+ \)

\[
\mu(A) = \lim_{t \to \infty} \rho(A^{(t)})^{1/t},
\]

(1)

where \( A^{(t)} = [a_{ij}^t] \) is a Hadamard (or also Schur) power of \( A \) and \( \rho \) the spectral radius. The simplified proof of (1) and also some other characterizations of \( \mu(A) \) can be found in [7;14, p. 366;2, p. 130]. We also have

\[
\mu(A) \leq \rho(A) \leq n \mu(A)
\]

(2)

(see e.g. [7;14, p. 366]).

It is known that \( \mu(A) \) is the largest max eigenvalue of \( A. \) Moreover, if \( A \) is irreducible, then \( \mu(A) \) is the unique max eigenvalue and every max eigenvector is positive (see [1, Theorem 2] and [15, Theorem 1]). As noted in [8] this implies that

\[
\mu(A_k^k) = \mu(A)^k.
\]

(3)

As stated in [8, Lemma 4.1] one can use (3) and a diagonal scaling argument to prove that also the equality

\[
\mu(A) = \lim_{k \to \infty} \| A_k^k \|^{1/k}
\]

(4)

holds for an arbitrary matrix norm \( \| \cdot \| \). We also have

\[
\rho(A) = \lim_{k \to \infty} \mu(A_k^k)^{1/k}
\]

(5)

by (2) and \( \rho(A^k) = \rho(A)^k \) (see [1, Theorem 16]).

Let \( \| \cdot \| \) be a vector norm on \( \mathbb{R}^n \). In [15,16], the notation

\[
\eta_{\| \cdot \|}(A) = \sup_{x \neq 0, x \geq 0} \frac{\| A \otimes x \|}{\| x \|}
\]

was introduced. Here \( |A| \) is simply the matrix \([a_{ij}]\). As proved in [16, Lemma 1], \( \eta_{\| \cdot \|}(\cdot) \) is a vector norm on \( \mathbb{R}^{n \times n}_+ \), if \( \| \cdot \| \) is a monotone norm (i.e. if \( |x| \leq |y| \) implies \( \| x \| \leq \| y \| \) for all
x, y \in \mathbb{R}^n). However, \eta_{\|\cdot\|}(\cdot) is not a matrix norm in general. But on the other hand, one can easily verify that the inequality
\[
\eta_{\|\cdot\|}(A \otimes B) \leq \eta_{\|\cdot\|}(A)\eta_{\|\cdot\|}(B)
\]
holds for all \(A, B \in \mathbb{R}^{n \times n}_+\) [15, Lemma 1(ii)].

If \(A \in \mathbb{R}^{n \times n}_+\) and \(\|\cdot\|\) an arbitrary vector norm on \(\mathbb{R}^n\), then we have by (4)
\[
\mu(A) = \lim_{k \to \infty} \eta_{\|\cdot\|}(A^k)\frac{1}{k}
\]
(see also [15, Theorem 3]), since all vector norms on a finite dimensional real vector space are equivalent (see e.g. [13, p. 272]).

Given an irreducible non-negative matrix \(A\), algorithms for computing \(\mu(A)\) and the max-eigenvector \(x\) were established in [8–10]. On the other hand, the infinite-dimensional generalization of \(\mu\) was provided in [17].

Let \(\Sigma\) be a bounded set of \(n \times n\) complex matrices. For \(m \geq 1\), let
\[
\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.
\]
The generalized spectral radius of \(\Sigma\) is defined by
\[
\rho(\Sigma) = \limsup_{m \to \infty} \left[ \sup_{A \in \Sigma^m} \rho(A) \right]^{1/m}.
\]
(8)
It was shown in [4] that \(\rho(\Sigma)\) is equal to the joint spectral radius of \(\Sigma\), i.e.,
\[
\rho(\Sigma) = \lim_{m \to \infty} \left[ \sup_{A \in \Sigma^m} \|A\| \right]^{1/m},
\]
(9)
where \(\|\cdot\|\) is a matrix norm on \(\mathbb{C}^{n \times n}\). This equality is called Berger–Wang formula or also the generalized spectral radius theorem. Since then many different type of proofs of (9) were obtained (for references see e.g. [16]).

If \(\Sigma\) is a bounded set of \(n \times n\) real matrices and \(\|\cdot\|\) an arbitrary vector norm on \(\mathbb{R}^n\), then we have
\[
\rho(\Sigma) = \lim_{m \to \infty} \left[ \sup_{A \in \Sigma^m} \eta_{\|\cdot\|}(A) \right]^{1/m}.
\]
This follows from (9) since all vector norms on a finite dimensional real vector space are equivalent.

If \(\Psi\) is a bounded set of \(n \times n\) non-negative matrices, then the inequalities in (2) together with (8) imply that
\[
\rho(\Psi) = \limsup_{m \to \infty} \left[ \sup_{A \in \Psi^m} \mu(A) \right]^{1/m}.
\]
(10)
This generalizes (5).

Let
\[
\Psi^m_{\otimes} = \{A_1 \otimes A_2 \otimes \cdots \otimes A_m : A_i \in \Psi\}
\]
and \(\|\cdot\|\) an arbitrary vector norm on \(\mathbb{R}^n\). In [16], the max algebra version of the generalized spectral radius \(\mu(\Psi)\) of \(\Psi\) was introduced by
\[
\mu(\Psi) = \limsup_{m \to \infty} \left[ \sup_{A \in \Psi^m_{\otimes}} \mu(A) \right]^{1/m}.
\]
(11)
The max algebra version of the joint spectral radius $\eta(\Psi)$ of $\Psi$ is defined by

$$\eta(\Psi) = \lim_{m \to \infty} \left[ \sup_{A \in \Psi_m} \eta_{\|\cdot\|}(A) \right]^{1/m}$$

(see [16,5] and the references cited there). Using (6) it was shown in [16, Lemma 6] that

$$\eta(\Psi) = \lim_{m \to \infty} \left[ \sup_{A \in \Psi_m} \eta_{\|\cdot\|}(A) \right]^{1/m}. \quad (12)$$

Since any two vector norms on a finite dimensional real vector space are equivalent, we obtain from (12) that

$$\eta(\Psi) = \lim_{m \to \infty} \left[ \sup_{A \in \Psi_m} \|A\| \right]^{1/m} \quad (13)$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n \times n}$ (see also the proof of [16, Theorem 3(i)]). Of course, the equality

$$\mu(\Psi) = \lim_{m \to \infty} \left[ \sup_{A \in \Psi_m} \rho(A) \right]^{1/m}$$

also holds by (2). The equalities (9) and (13) imply that

$$\eta(\Psi) \leq \rho(\Psi) \leq n\eta(\Psi) \quad (14)$$

(see the proof of [16, Theorem 3(ii)]). The main result of [16] was the max algebra version of the generalized spectral radius theorem [16, Theorem 2], i.e.,

$$\mu(\Psi) = \eta(\Psi). \quad (15)$$

A natural question arises, whether we can characterize $\mu(\Psi)$ in terms of (1). In the next section we give a positive answer. Using this we give a short proof of (15).

2. The main results

Let $\Psi$ be a bounded set of $n \times n$ non-negative matrices and $t > 0$. Let $\Psi^{(t)}$ be a Hadamard power of $\Psi$, i.e.,

$$\Psi^{(t)} = \{A^{(t)} : A \in \Psi\}.$$ 

Then $\Psi^{(t)}$ is also a bounded set of $n \times n$ non-negative matrices. We will show (Theorem 2.4) that

$$\mu(\Psi) = \lim_{t \to \infty} \rho(\Psi^{(t)})^{1/t} = \inf_{t \in (0, \infty)} \rho(\Psi^{(t)})^{1/t}. \quad (16)$$

Note that this provides a description of $\mu(\Psi)$ which uses no max terminology.

The following result is a special case of [6, Proposition 3.1]. The alternative proof was given in [17, Lemma 4.2].

**Proposition 2.1.** Let $A_1, \ldots, A_m \in \mathbb{R}_{+}^{n \times n}$ and $t \geq 1$. Then we have

$$A_1^{(t)} \cdots A_m^{(t)} \leq (A_1 \cdots A_m)^{(t)}. \quad (17)$$
We define
\[ \mu_1(\Psi) = \limsup_{t \to \infty} \rho(\Psi(t))^{1/t}. \]

**Proposition 2.2.** Let \( \Psi \) be a bounded set of \( n \times n \) non-negative matrices. Then the following properties hold:

(i) If \( t \geq 1 \), then \( \rho(\Psi(t)) \leq \rho(\Psi)^t \) and so \( \mu_1(\Psi) \leq \rho(\Psi) \);

(ii) \( \rho(\Psi(t))^{1/t} \) is decreasing in \( t \in (0, \infty) \) and so
\[ \mu_1(\Psi) = \inf_{t \in (0, \infty)} \rho(\Psi(t))^{1/t} = \lim_{t \to \infty} \rho(\Psi(t))^{1/t}. \]

**Proof.** (i) If \( A \in (\Psi(t))^m \), then we have \( A = A_1 \otimes \cdots \otimes A_m \) for \( A_1, \ldots, A_m \in \Psi \). By (17) we have \( \mu(A) \leq \mu((A_1 \cdots A_m)^t) = \mu(A_1 \cdots A_m)^t = \mu(B)^t \), where \( B = A_1 \cdots A_m \in \Psi^m \). This, together with (10), implies
\[ \rho(\Psi(t)) \leq \limsup_{m \to \infty} \left( \sup_{B \in \Psi^m} \mu(B)^t \right)^{1/m} = \rho(\Psi)^t, \]
which proves (i).

Let \( s > t > 0 \). Since \( \frac{s}{t} \geq 1 \) we have by (i)
\[ \rho(\Psi(s))^{1/s} = \rho(\Psi(t)^{\frac{s}{t}})^{1/s} = \rho(\Psi(t))^{1/s} \leq \rho(\Psi(t))^{1/t}, \]
which proves (ii). \( \square \)

The following proposition is an analogue of (14) for \( \mu(\Psi) \).

**Proposition 2.3.** If \( \Psi \) is a bounded set of \( n \times n \) non-negative matrices, then we have
\[ \mu(\Psi) \leq \rho(\Psi) \leq n \mu(\Psi). \] (18)

**Proof.** Let \( A_1, \ldots, A_m \in \Psi \). It is easy to see that
\[ A_1 \otimes \cdots \otimes A_m \leq A_1 \cdots A_m \leq n^{m-1}(A_1 \otimes \cdots \otimes A_m). \]
Since \( \mu \) is monotone we have
\[ \sup_{A \in \Psi^m} \mu(A) \leq \sup_{A \in \Psi^m} \mu(A) \leq n^{m-1} \sup_{A \in \Psi^m} \mu(A). \]
Taking the \( m \)-th root and letting \( m \to \infty \) we obtain (18) by (11) and (10). \( \square \)

**Theorem 2.4.** If \( \Psi \) is a bounded set of \( n \times n \) non-negative matrices, then
\[ \mu(\Psi) = \mu_1(\Psi) = \eta(\Psi). \]

**Proof.** Let \( t > 0 \). We obviously have \( \mu_1(\Psi(t)) = \mu_1(\Psi)^t \). Next we show that
\[ \mu(\Psi(t)) = \mu(\Psi)^t \quad \text{and} \quad \eta(\Psi(t)) = \eta(\Psi)^t. \] (19)
If \( A \in (\Psi(t))^m \), then we have \( A = A_1 \otimes \cdots \otimes A_m \) for \( A_1, \ldots, A_m \in \Psi \). One can easily verify that \( A = (A_1 \otimes \cdots \otimes A_m)^t \) (in other words \((\Psi(t))^m)^{\otimes} = (\Psi^m)^{(t^t)}\)). This implies that
\[
\mu(\Psi(t)) = \lim_{m \to \infty} \left( \sup_{B \in \Psi_m^m} \mu(B^{(t)}) \right)^{1/m}.
\]
Since \( \mu(B^{(t)}) = \mu(B)^t \) we obtain \( \mu(\Psi^{(t)}) = \mu(\Psi)^t \). Since we also have \( \|B^{(t)}\|_{\infty} = \|B\|^t_{\infty} \) (where \( \|B\|_{\infty} := \max_{i,j=1,...,n} b_{ij} \)), we have by (13)

\[
\eta(\Psi^{(t)}) = \lim_{m \to \infty} \left( \sup_{B \in \Psi_m^m} \|B\|^t_{\infty} \right)^{1/m} = \eta(\Psi)^t,
\]
which proves (19).

Now we have by (18)

\[
\mu(\Psi) = \mu(\Psi)^{1/t} \leq \rho(\Psi^{(t)})^{1/t} \leq n^{1/t} \mu(\Psi^{(t)})^{1/t} = n^{1/t} \mu(\Psi).
\]

Letting \( t \to \infty \) in (20) we obtain that \( \mu(\Psi) = \mu_1(\Psi) \). Using (14) instead of (18) in the argument above one obtains \( \eta(\Psi) = \mu_1(\Psi) \), which completes the proof. \( \square \)

Note that by taking \( \Psi = \{A\} \) in the proof above, we obtain an alternative proof of (4) and consequently (3).

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