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Smooth Surface Interpolation to Scattered Data Using Interpolatory Subdivision Algorithms

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Abstract—In this paper, a smooth interpolatory subdivision algorithm for the generation of interpolatory surfaces (GC^1) over arbitrary triangulations is constructed and its convergence properties over nonuniform triangulations studied. An immediate application of this algorithm to surface interpolation to scattered data in $\mathbf{R}^n$, $n \geq 3$ is also studied. For uniform data, this method is a generalization of the analyses for univariate subdivision algorithms, and for nonuniform data, an extraordinary point analysis is proposed and a local subdivision matrix analysis presented. It is proved that the subdivision algorithm produces smooth surfaces over arbitrary networks provided the shape parameters of the algorithm are kept within an appropriate range. Some error estimates for both uniform and nonuniform triangulations are also investigated. Finally, three graphical examples of surface interpolations over nonuniform data are given to show the smoothing interpolating process of the algorithm.

Keywords—Triangulation, Approximation, Surface interpolation, Scattered data interpolation, Subdivision algorithm.

1. INTRODUCTION

The scattered data interpolation problem is one of the oldest problems in applied mathematics. Although there are many ways to tackle this problem, for example, the Shepard method, the polynomial spline method and other data fitting methods, it seems there are still many difficulties in the applications of these methods. For example, one of the disadvantages using these methods is their global property. Therefore, they are not very efficient for the purpose of interactive computer aided design. In order to overcome the drawbacks of these algorithms, a different approach, by using a recursive subdivision algorithm, is proposed here to provide a solution to the scattered data interpolation problem.

Recursive subdivision algorithms, which are also referred to as binary or stationary subdivision algorithms, have been studied intensively for many years in the fields of approximation and computer aided geometric design. Typical examples of such algorithms are the de Rham's *trisection algorithm* (1947), the de Casteljau's algorithm (1959) for the Bernstein-Bézier curves, the Chaikin algorithm (1974) for curves, the Catmull-Clarke algorithm (1978) for both curves and surfaces, etc. Recently, a lot of work has been done in this area to study subdivision algorithms systematically. This includes the works by Dyn, Gregory and Levin [1–3], Cavaretta, Dahmen and Micchelli [4–7], Daubechies, Deslauriers and Dubuc [8–11] and others [12–22]. And interpolatory subdivision algorithms for both curves and surfaces play a very important role in these

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX applications. In [19,22], an interpolatory subdivision algorithm is investigated and some explicit conditions are obtained under which a subdivision algorithm could produce smooth surfaces with certain prescribed properties such as interpolatory and convexity. Thus the idea of Deslauriers-Dubuc and Dyn-Gregory-Levin's uniform analysis [1,2,9,11] for univariate subdivision algorithms is generalized to surfaces. In the papers [17–19,22] a general stepwise interpolatory subdivision algorithm for surfaces over uniform triangulations is constructed, and then the 10-point interpolatory subdivision scheme is studied in detail (cf. [19,22]). This paper is a continuation of our previous work [19,22].

The outline of the paper is as follows. In Section 2, the 10-point subdivision algorithm is introduced briefly. In Section 3, the algorithm is generalized to arbitrary triangulations first and then to arbitrary networks. A convergence analysis of the scheme over arbitrary triangulations is presented in Section 4. And some error estimates for surface interpolation using this algorithm are given in Section 5. Three examples are given at the end to show the smoothing process of the algorithm.

It should be emphasized that in the analysis presented in Sections 3 and 4, a local subdivision matrix analysis approach is employed that is different from the *Cross Difference of Directional Divided Difference* analysis used in the analysis of the algorithm over uniform data (cf. [22]). More detailed description of the analysis of the algorithm over uniform data can be found in [19,22]. Other analyses of uniform subdivision algorithms can also be found in [1–5,14,20].

The generalization of the subdivision algorithm to surface interpolation over arbitrary data has wide applications. First, it can be used to solve interpolatory-type surface fitting over arbitrary data problems. Second, it can be employed to generate smooth closed surface interpolants over scattered data, as shown in Example 2. It can also be used to generate smooth fillings over some curved polygonal holes (cf. [21]). Furthermore, it can be utilized to simplify and then solve problems like surface normal estimations and even other surface characteristic estimates. It is also hoped that nonuniform subdivision algorithms could be applied successfully in some optimization problems such as optimized data-transmission and wavelets processing, etc.

2. THE ALGORITHM OVER UNIFORM DATA AND ITS BASIC PROPERTIES

The construction of the algorithm is, originally, motivated by the ideas described in papers by Deslauriers and Dubuc [9,11], Dyn, Gregory and Levin [1–3]. The scheme was formulated in order to solve such problems as high accuracy surface fitting and fast surface representation (cf. [14,16,19]). The main properties of the scheme, in addition to the properties of general uniform subdivision schemes, are its generation of smooth interpolatory surfaces and the reproductivity of cubic parametric polynomial surfaces provided that the shape parameters are chosen appropriately.

A mathematical description of a uniform subdivision scheme over uniform triangulations, which is also called *binary subdivision algorithm*, is as follows. Suppose that the initial *control points* of a uniform triangular network in $\mathbf{R}^3$ are denoted by $\mathbf{P}^0_\alpha, \alpha \in \mathbf{Z}^2$, then, the refined control points $\{\mathbf{P}^{k+1}_\alpha, \alpha \in \mathbf{Z}^2\}$ are obtained from $\{\mathbf{P}^k_\alpha\}$ recursively by the following formula:

$$\mathbf{P}^{k+1}_\alpha = \sum_{\beta \in \mathbf{Z}^2} a_{\alpha-2\beta} \mathbf{P}^k_\beta, \qquad \alpha \in \mathbf{Z}^2, \quad k \geq 0. \tag{2.1}$$

Here, a uniform triangulation network means that it is topologically equivalent to the type I uniform triangulation shown in Figure 1. An equivalent form of this expression is

$$\mathbf{P}^{k+1}_{\gamma+2\alpha} = \sum_{\beta \in \mathbf{Z}^2} a_{\gamma-2\beta} \mathbf{P}^k_{\alpha+\beta}, \qquad \alpha \in \mathbf{Z}^2, \tag{2.2}$$

where $\gamma =: (\gamma_1, \gamma_2)$ and $\gamma_i = 0$ or 1, $i = 1, 2$. Thus, the scheme is stepwise interpolatory if and only if the coefficients $\{a_\alpha\}$ satisfy

$$a_{2\alpha} = \delta_{\alpha,0}, \qquad \forall \alpha \in \mathbf{Z}^2. \tag{2.3}$$

Equation (2.2) shows clearly that the scheme is a 4-step subdivision scheme. The 10-point scheme is given by a special choice of the coefficients $\{a_\alpha\}$ which is given explicitly in [19]. There are three parameters w_i, $i = 1, 2, 3$ in these coefficients that are used to control the shape of the limit surfaces. This special choice of the coefficients comes from the 3-directional symmetric structure of the scheme. In fact, a simpler way to describe the scheme uses only a single formula to characterize the scheme. Since the scheme is symmetric and interpolatory, only the inserted values are to be evaluated. The formula for an inserted point, $\mathbf{P}_o$, is given by (cf. Figure 2)

$$\begin{aligned} \mathbf{P}_o = {} & \frac{1}{2}(\mathbf{P}_e + \mathbf{P}_f) + w_1(\mathbf{P}_a + \mathbf{P}_c + \mathbf{P}_h + \mathbf{P}_j - 2\mathbf{P}_e - 2\mathbf{P}_f) \\ & + w_2(\mathbf{P}_b + \mathbf{P}_i - \mathbf{P}_e - \mathbf{P}_f) + w_3(\mathbf{P}_d + \mathbf{P}_g - \mathbf{P}_e - \mathbf{P}_f) \end{aligned} \tag{2.4}$$

where o is the inserted *new control point* associated with the edge joining the vertices e and f. From this construction, it is obvious that the scheme can be used (with some modifications at those so-called *extraordinary points*) to produce surfaces over arbitrary triangulations.

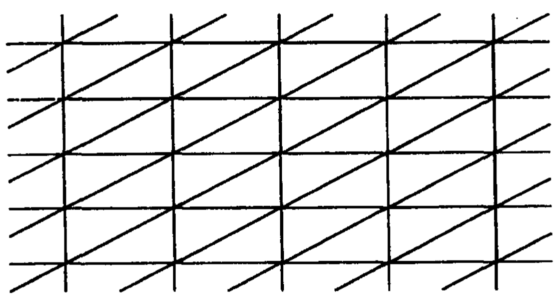


Figure 1. The type I uniform triangulation.

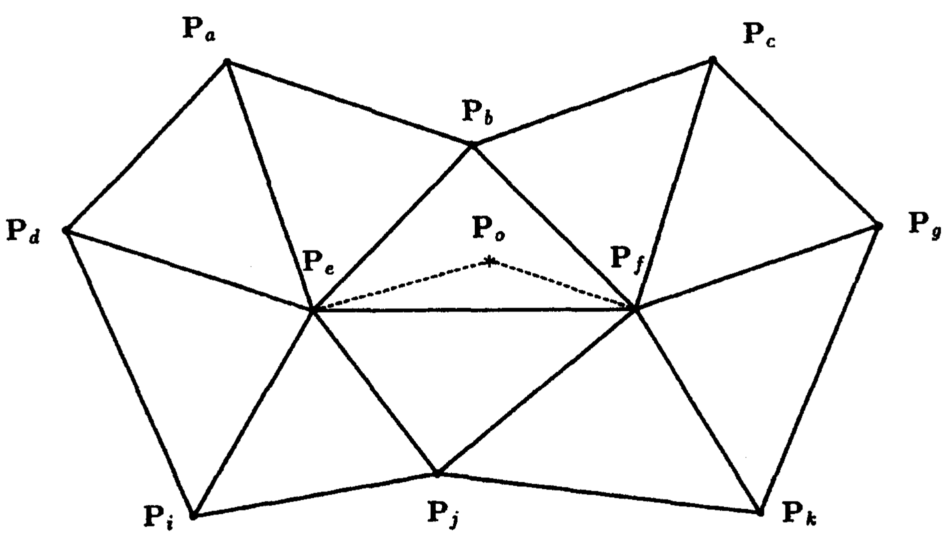


Figure 2. The geometric construction of the 10-point algorithm.

Explicitly, if the uniform control points at level k are denoted by $\{P^k_{i,j}\}$, and the type I triangulation is assumed, the 10-point scheme can then be written in the following compact form:

$$\begin{aligned}\mathbf{P}_{2i,2j}^{k+1} &= \mathbf{P}_{i,j}^{k},\\ \mathbf{P}_{2i+1,2j}^{k+1} &= \left(\frac{1}{2} - 2w_1 - w_2 - w_3\right)\left(\mathbf{P}_{i,j}^{k} + \mathbf{P}_{i+1,j}^{k}\right)\\ &\quad + w_1\left(\mathbf{P}_{i-1,j-1}^{k} + \mathbf{P}_{i+1,j-1}^{k} + \mathbf{P}_{i,j+1}^{k} + \mathbf{P}_{i+2,j+1}^{k}\right)\\ &\quad + w_2\left(\mathbf{P}_{i,j-1}^{k} + \mathbf{P}_{i+1,j+1}^{k}\right) + w_3\left(\mathbf{P}_{i-1,j}^{k} + \mathbf{P}_{i+2,j}^{k}\right)\end{aligned} \tag{2.4a}$$

with $P_{2i,2j+1}^{k+1}$ and $P_{2i+1,2j+1}^{k+1}$ being duals (rotation symmetries) of the second equation. So, in the uniform subdivision process, formula (2.4) or (2.4a) is used to evaluate all *new control points* to produce a refined uniform triangular control net in which the triangulation of the refined control net is formed by the standard type I meshing shown in Figure 1. Repeated applications of this process will therefore result in finer and finer control nets. Studies show that if the shape parameters $\{w_i\}$ are chosen appropriately, the scheme will produce smooth interpolatory surfaces. In fact, the scheme has the following properties (cf. [16,19]).

(1) The scheme is local, interpolatory and coordinate-free.
(2) The parameters $\{w_i\}$ work as tension controls along the three mesh directions, respectively.
(3) The scheme reproduces linear surfaces for all $\{w_i\}$.
(4) The scheme reproduces bivariate cubic parametric surfaces if $\{w_i\}$ satisfy

$$w_1 = t - \frac{9}{16}, \qquad w_2 = -2t + \frac{9}{8} = -2w_1, \qquad w_3 = \frac{1}{2} - t \tag{2.5}$$

where t is any real number.
(5) The scheme reduces to the butterfly scheme (cf. [2,3,14,16]) if the parameters satisfy

$$w_1 := w, \qquad w_2 := -2w, \qquad w_3 := 0. \tag{2.6}$$

(6) The scheme has certain data-dependent shape preserving properties.

The scheme produces smooth surfaces if the shape control parameters are chosen properly. More precisely, we have the following results (cf. [16,19]).

PROPOSITION 2.1. *The scheme produces C^0 surfaces if the parameters $\{w_i\}$ satisfy*

$$\begin{aligned}\left|\frac{1}{2} - 2w_1 - w_2\right| + 2|w_2| + 2|w_3| + |w_1 - w_3| &< 1,\\ 4|w_1| + 2|w_2| + 2|w_3| &< \frac{1}{2}.\end{aligned} \tag{2.7}$$

A simple symmetric solution to (2.7) is given by

$$5|w_1| + 3|w_2| + 3|w_3| < \frac{1}{2}. \tag{2.8}$$ ∎

REMARK 2.2. For the cubic precision scheme, (2.7) becomes

$$\frac{1}{2} < t < \frac{37}{64}. \tag{2.9}$$

PROPOSITION 2.3. *If $w_2 = -2w_1$, then there exists a piecewise quadratic function such that*

$$C_d^{k+2} \le B(w_1, w_3)\, C_d^k, \qquad \forall k \ge 0, \tag{2.10}$$

where

$$C_d^k := \max_{\substack{i,j,m,n,\\ m\neq n}} \left|C_{i,j,m,n}^k\right| \tag{2.11}$$

and

$$C_{i,j,m,n}^k := 2^k \Delta_m \Delta_n \mathbf{P}_{i,j}^k, \qquad \forall i,j \in \mathbf{Z}, \tag{2.12}$$

and $B(w_1, w_3) < 1$ *provided that the shape parameters* w_1, w_2 *and* w_3 *satisfy*

$$\begin{aligned} w_2 + 2w_1 &= 0,\\ w_1 &\neq 0,\\ w_1 + 7w_3 &\leq 0,\\ 8(w_1 + 0.07) - 3(w_3 - 0.01) &\geq 0,\\ (w_1 + 0.10) + (w_3 + 0.07) &\geq 0,\\ 10w_1 - 7w_3 &\leq 0. \end{aligned} \tag{2.13}$$

∎

In the above proposition, the forward difference operators $\{\Delta_i\}$, $i = 1,2,3$, along the mesh directions are defined as

$$\begin{aligned} \Delta_1 &:= P_{i+1,j}^k - P_{i,j}^k,\\ \Delta_2 &:= P_{i,j+1}^k - P_{i,j}^k,\\ \Delta_3 &:= P_{i+1,j+1}^k - P_{i,j}^k. \end{aligned} \tag{2.14}$$

PROPOSITION 2.4. *The 10-point scheme produces* C^1 *surfaces if the shape parameters satisfy (2.13). Especially, the cubic precision scheme produces smooth surfaces if the tension parameter* t *satisfies*

$$\frac{49}{100} \leq t \leq \frac{54}{100}. \tag{2.15}$$

∎

3. THE ALGORITHM OVER ARBITRARY NETWORKS

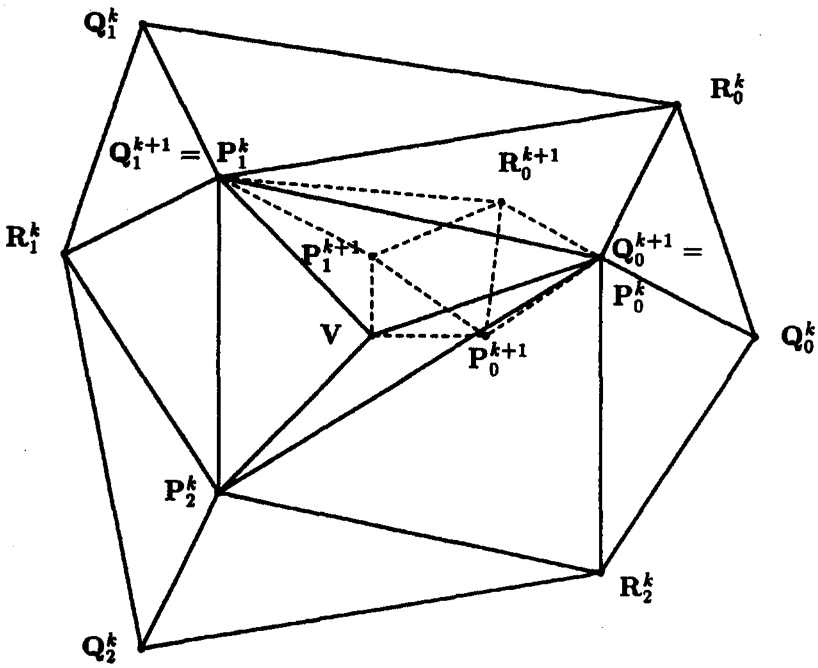
In this section, we generalize the 10-point scheme to nonuniform triangulations first and then to arbitrary networks. The main requirement of the construction is that for general initial data, the limit surface should be smooth even at the extraordinary points, provided that the parameters are chosen properly at these points. Here an extraordinary point is defined as a vertex whose valency is not equal to 6.

3.1. The Algorithm over Arbitrary Triangulations

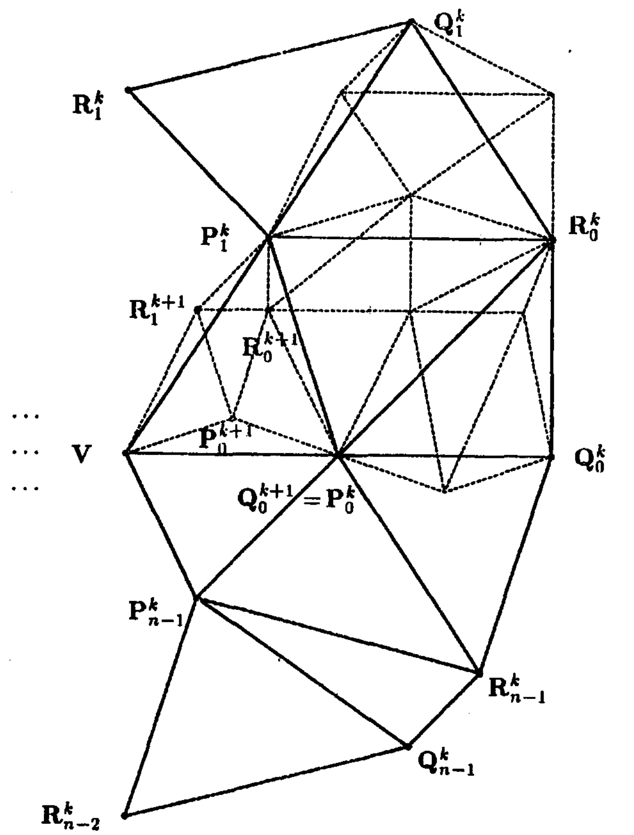
From construction of the algorithm over uniform data, it can be easily modified to refine arbitrary triangular networks which leads to the generation of surfaces on arbitrary triangular networks. Depending on the local topology (more explicitly, the valency of the vertex), the 10-point scheme can be modified to refine arbitrary triangular networks.

For simplicity, we assume also, without loss of generality, that the initial data is locally uniform except one extraordinary point V. In fact, this situation can be achieved locally after the first subdivision. However, the smooth properties of the limit surface do not depend on the first subdivision of the initial data although the limit surface depends on the first subdivision.

In the following formulation, the index i is a cyclic integer in the range: $i = 0,1,2,\dots,n-1$ where n is the valency of the vertex. For simplicity, it is also assumed in scheme that the cubic precision parameters are used, i.e., $\{w_i\}$ satisfy (2.5). The symbols $\mathbf{P}_i^k$, $\mathbf{Q}_i^k$, $\mathbf{R}_i^k$ denote the corresponding initial or refined control points in $\mathbf{R}^3$ (or in $\mathbf{R}^d$, $d \geq 3$) near the extraordinary vertex V at level k in the way shown in Figures 3a and 3b. In the figures, the solid triangulation is at level k and the dashed triangulation is the refined one at level $k+1$. For clarity, only part of the refined triangulation is shown. Using the above notation and configuration at the extraordinary points, the local schemes can be described as follows.



(a) The geometric construction of the algorithm for $n = 3$.



(b) The geometric construction of the algorithm for $n \geq 4$.

Figure 3.

CASE I. $n = 3$.

In this case, there are several choices that can be used. One of them is given by the following (cf. Figure 3a):

$$\begin{aligned}
\mathbf{P}_i^{k+1} &= t\mathbf{V} + (w_1 + t)\,\mathbf{P}_i^k + (w_1 + w_3)\,\mathbf{Q}_i^k + w_2\mathbf{R}_i^k + w_1\mathbf{P}_{i+1}^k + w_3\mathbf{R}_{i+1}^k + w_1\mathbf{P}_{i-1}^k + w_2\mathbf{R}_{i-1}^k,\\
\mathbf{Q}_i^{k+1} &= \mathbf{P}_i^k, \qquad (3.1)\\
\mathbf{R}_i^{k+1} &= w_2\mathbf{V} + t\mathbf{P}_i^k + w_1\mathbf{Q}_i^k + w_2\mathbf{R}_i^k + t\mathbf{P}_{i+1}^k + w_1\mathbf{Q}_{i+1}^k + w_3\mathbf{R}_{i+1}^k + 2w_1\mathbf{P}_{i-1}^k + w_3\mathbf{R}_{i-1}^k
\end{aligned}$$

where $\{w_i\}$ are recommended to be chosen by the cubic precision conditions (2.5) and t is a local tension control parameter. However, other choices are also acceptable if they produce smooth surfaces.

CASE II. $n \geq 4$.

In this case, the scheme is just the 10-point scheme, that is, formula (2.4) or (2.4a) is applied for the evaluation of every new point. Since in this case the scheme also produces surfaces (to be shown later in Section 4), it is not necessary to construct more complicated schemes at this vertex although other choices may also be used. In fact, a cubic precision scheme can be constructed but the the coefficients of the formulae are quite complicated. For simplicity, (2.6) is assumed in the following analysis. However, from the proof in the next section we know that the method is also valid for $w_3 \neq 0$. Applying the scheme near the extraordinary point, we obtain the following subdivision formulae (Figure 3b):

$$\begin{aligned}
\mathbf{P}_i^{k+1} &= \frac{1}{2}\mathbf{V} + \frac{1}{2}\mathbf{P}_i^k + w\mathbf{R}_i^k - 2w\mathbf{P}_{i+1}^k + w\mathbf{P}_{i+2}^k - 2w\mathbf{P}_{i-1}^k + w\mathbf{R}_{i-1}^k + w\mathbf{P}_{i-2}^k,\\
\mathbf{Q}_i^{k+1} &= \mathbf{P}_i^k, \qquad (3.2)\\
\mathbf{R}_i^{k+1} &= -2w\mathbf{V} + \frac{1}{2}\mathbf{P}_i^k + w\mathbf{Q}_i^k - 2w\mathbf{R}_i^k + \frac{1}{2}\mathbf{P}_{i+1}^k + w\mathbf{Q}_{i+1}^k + w\mathbf{P}_{i+2}^k + w\mathbf{P}_{i-1}^k
\end{aligned}$$

where w is the local tension control parameter.

REMARK 3.1. It will be seen later in our analysis that the key role in the construction of the local algorithms at the extraordinary points is the reproductivity of constants and the spectrum (contraction) conditions (4.1) and (4.3).

3.2. Matrix Representation of the Algorithm at an Extraordinary Point

Writing (3.2) in a matrix form, we obtain:

$$\begin{aligned}
\begin{pmatrix}\mathbf{P}_i^{k+1}\\ \mathbf{Q}_i^{k+1}\\ \mathbf{R}_i^{k+1}\end{pmatrix} &= \begin{pmatrix}\frac{1}{2} & 0 & w\\ 1 & 0 & 0\\ \frac{1}{2} & w & w_2\end{pmatrix}\begin{pmatrix}\mathbf{P}_i^k\\ \mathbf{Q}_i^k\\ \mathbf{R}_i^k\end{pmatrix} + \begin{pmatrix}w_2 & 0 & 0\\ 0 & 0 & 0\\ \frac{1}{2} & w & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i+1}^k\\ \mathbf{Q}_{i+1}^k\\ \mathbf{R}_{i+1}^k\end{pmatrix}\\
&+ \begin{pmatrix}w & 0 & 0\\ 0 & 0 & 0\\ w & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i+2}^k\\ \mathbf{Q}_{i+2}^k\\ \mathbf{R}_{i+2}^k\end{pmatrix} + \begin{pmatrix}w_2 & 0 & w\\ 0 & 0 & 0\\ w & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i-1}^k\\ \mathbf{Q}_{i-1}^k\\ \mathbf{R}_{i-1}^k\end{pmatrix} \qquad (3.3)\\
&+ \begin{pmatrix}w & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{P}_{i-2}^k\\ \mathbf{Q}_{i-2}^k\\ \mathbf{R}_{i-2}^k\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\ 0\\ w_2\end{pmatrix}\mathbf{V}.
\end{aligned}$$

From this expression, we introduce the following basic matrices:

$$\begin{aligned}
C_0 &:= \begin{pmatrix}\frac{1}{2} & 0 & w\\ 1 & 0 & 0\\ \frac{1}{2} & w & w_2\end{pmatrix}, \quad C_1 := \begin{pmatrix}w_2 & 0 & 0\\ 0 & 0 & 0\\ \frac{1}{2} & w & 0\end{pmatrix}, \quad C_2 := \begin{pmatrix}w & 0 & 0\\ 0 & 0 & 0\\ w & 0 & 0\end{pmatrix}, \qquad (3.4)\\
C_3 &:= \begin{pmatrix}w_2 & 0 & w\\ 0 & 0 & 0\\ w & 0 & 0\end{pmatrix}, \quad C_4 := \begin{pmatrix}w & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

and the local control point vectors:

$$\mathbf{F}^k := \left(\mathbf{V}, \mathbf{P}_0^k, \mathbf{Q}_0^k, \mathbf{R}_0^k, \mathbf{P}_1^k, \mathbf{Q}_1^k, \mathbf{R}_1^k, \dots, \mathbf{P}_{n-1}^k, \mathbf{Q}_{n-1}^k, \mathbf{R}_{n-1}^k\right)^\top, \qquad k \geq 0. \tag{3.5}$$

Here, $\mathbf{F}^k$ is a vector of length $3n+1$. Thus, the subdivision process (3.2) at V can be written in a more compact form:

$$\mathbf{F}^{k+1} = \mathbf{A} \cdot \mathbf{F}^k, \qquad k = 0, 1, 2, \dots \tag{3.6}$$

where $\mathbf{A}$ is called the local subdivision matrix. More explicitly, the matrix is given in the form

$$\mathbf{A} := \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{a} & \mathbf{A}' \end{pmatrix}. \tag{3.7}$$

Here, $\mathbf{a}$ is a vector of length $3n$ and $\mathbf{A}'$ is a *block circulant matrix* defined by

$$\begin{aligned} \mathbf{A}' &= B\mathrm{circ}\,(A_0, A_1, A_2, \dots, A_{n-1}) \\ &:= \begin{pmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} \\ A_n & A_0 & A_1 & \dots & A_{n-2} \\ A_{n-1} & A_n & A_0 & \dots & A_{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ A_1 & A_2 & A_3 & \dots & A_0 \end{pmatrix} \end{aligned} \tag{3.8}$$

and $\{A_i\}$ are some 3×3 matrices defined by $\{C_i\}$. More precisely, for $n = 4$, we have

$$\begin{aligned} A_0 &:= C_0, \\ A_1 &:= C_1, \\ A_2 &:= C_2 + C_4, \\ A_3 &:= C_3, \end{aligned} \tag{3.9}$$

and for $n \geq 5$, we have

$$\begin{aligned} A_0 &:= C_0, \\ A_1 &:= C_1, \\ A_2 &:= C_2, \\ A_i &:= \mathbf{0}, \qquad i = 3, 4, \dots, n-3, \\ A_{n-2} &:= C_3, \\ A_{n-1} &:= C_4. \end{aligned} \tag{3.10}$$

Similarly, for $n = 3$, using the same notation, the subdivision process (3.1) can also be written in the form (3.6)–(3.8), with the basic subdivision matrices given by

$$A_0 := \begin{pmatrix} w_1 + t & w_1 + w_3 & w_2 \\ 1 & 0 & 0 \\ t & w_1 & w_2 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} w_1 & 0 & w_3 \\ 0 & 0 & 0 \\ t & w_1 & w_3 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} w_1 & 0 & w_2 \\ 0 & 0 & 0 \\ 2w_1 & 0 & w_3 \end{pmatrix}. \tag{3.11}$$

3.3. The Algorithm over Arbitrary Networks

Since the algorithm works on all triangulations, for scattered data with an arbitrary topology, it suffices if a triangulation can be formed from the data. A straightforward way to form a triangulation from the given scattered data with its topology (i.e., a polygonal network) is to add more edges such that all the polygons are partitioned into triangles. However, because the interpolant depends very much upon the initial triangulation of the data, much care should be taken in forming the initial triangulation. In order to produce a proper interpolant, we propose the following strategies in the generation of the initial triangulation.

A (*Point insertion*). New points near the centres of polygons can be introduced to form a local star-like triagulation for that polygon.

B (*Edge removal*). Some edges of the given topology can be removed to form a better triagulation of the vertices.

C (*Optimum valency*). The best value for the valency of each vertex is 6. Therefore, in the process of forming the triagulation, the valency of each vertex should be made close to 6. From experiments, a good range of valency is 5–8.

D (*Symmetric preserving*). The symmetry of the data should be preserved. Thus symmetric formulae should be used in the generation of new points and edges.

E (*Shape preserving*). Both the convexity and monotonicity of the data should be preserved.

F (*Quasi-uniform*). The triangulation is "quasi-uniform," that is, the sizes (i.e., both edges and areas) of adjacent triangles do not vary too much.

G (*Boundary considerations*). For surface patches, boundary properties and conditions must be considered in the refinement process of the data so that the resulting interpolant satisfies the given boundary conditions.

H (*Iteration*). If necessary, the above strategies A–F should be applied repeatedly and interactively to obtain good triangulations.

An application of these strategies is shown in Example 3, Section 6.

REMARK 3.2. Since the interpolant is almost determined by the initial triangulations and the initial data may contain errors, many things should be taken into account in order to construct a good interpolant or approximant. In paper [23], some methods and are proposed for the preprocessing of the initial data and its associated topology so that the modified data and topology can be used to produce better results.

4. CONVERGENCE OF THE ALGORITHM OVER ARBITRARY NETWORKS

In this section, we show that the algorithm generates smooth surfaces over arbitrary triangulations if the parameters $\{w_i\}$ are well chosen. The analyses of the scheme here are different from the previous analyses of the scheme over uniform data (cf. [16,19]). In fact, the analysis presented here is an extraordinary point analysis. The block-circulant matrix theory is used here. This technique is quite suitable for the nonuniform analysis (cf. [10]).

4.1. Spectrum Properties of the Subdivision Matrix

To study the C^0 and C^1 properties of the scheme over arbitrary triangulations, it is sufficient to prove that the limit surface of the schemes is C^0 or C^1 at the extraordinary point since from [19] we know that the limit surface is C^1 everywhere else provided that the tension parameter satisfies (2.13). Since the eigenproperties of the subdivision matrix play a very important role in the convergence analysis (cf. [10,16]), we first study the eigenproperties of the subdivision matrix $\mathbf{A}$. It should be stressed that all the eigenvalues and their corresponding eigenvectors of $\mathbf{A}$ can be evaluated analytically since the matrix is a *block-circulant matrix* composed of 3×3 submatrices; therefore these eigenvalues are roots of cubic polynomials which can be found explicitly by Cardino's formulae.

Let the eigenvalues and their corresponding (generalized) eigenvectors of $\mathbf{A}$ be denoted by $\{\lambda_i, \mu_i\}$, where $|\lambda_i| \geq |\lambda_{i+1}|$ for all i. Then, by direct evaluations, we have the following results.

THEOREM 4.1. *The subdivision matrix* $\mathbf{A}$ *defined by (3.7) has the following properties:*

$$\begin{aligned} \lambda_1 &= 1, \qquad \mu_1 = (1,1,1,\dots,1)^t, \\ |\lambda_i| &< 1, \qquad i = 2,3,\dots,3n,3n+1 \end{aligned} \tag{4.1}$$

provided that

$$0.3125 < t < 0.6000, \qquad \text{for } n = 3,$$
$$-\frac{1}{12} < w < 0, \qquad \text{for } n \geq 4. \tag{4.2}$$

Furthermore, we have

$$0 < \lambda_2 = \lambda_3 < \lambda_1, \qquad |\lambda_i| < \lambda_2, \qquad i \geq 4, \quad \textbf{dim span}\{\mu_2, \mu_3\} = 2 \tag{4.3}$$

if

$$0.5275 < t < 0.5500, \qquad \text{for } n = 3,$$
$$-\frac{1}{12} < w < 0, \qquad \text{for } n \geq 4. \tag{4.4}$$ ∎

REMARK 4.2. The eigenvalue λ_2 is a double eigenvalue of $\mathbf{A}$ and has two linearly independent eigenvectors. This can be shown explicitly by using the block-circulant matrix theory or the Fourier Transform technique (cf. [10]).

4.2. Convergence Analysis

From the above results, it can be shown that the limit surface has tangent plane continuity at the extraordinary point. Thus, the surface is smooth everywhere. Since the number of extraordinary points in a triangulation is a constant and the 10-point scheme is used at all these regular (uniform) points, the C^0 and C^1 properties of the limit surface at these regular points are obvious. By using a pointwise analysis, the following C^0 and C^1 convergence results can be obtained.

THEOREM 4.3. *The limit surface is C^0 if (4.2) holds.*

PROOF. If the conditions are satisfied, all the local control points in the vector $\mathbf{F}^k$ will converge uniformly to the control point V as k approaches infinity. Hence the result follows from the corresponding C^0 convergence in [16,19]. Moreover, the control net sequence converges to the limit surface uniformly due to the fact that the total number of extraordinary points is fixed. ∎

THEOREM 4.4. *The limit surface is GC^1 (smooth) for nondegenerate data if (4.4) holds.*

PROOF. It suffices to show the limit surface is smooth at the extraordinary point $\mathbf{V}$. First, we consider any secant plane, $\mathbf{\Pi}^s$, i.e., a local linear interpolant to the limit surface at $\mathbf{V}$, of the surface at the point $\mathbf{V}$ determined by three control points $\mathbf{E}_1^s$, $\mathbf{E}_2^s$ and $\mathbf{V}$ on the control net $\mathbf{P}^s$ at level s. Therefore, $\mathbf{E}_1^s$ and $\mathbf{E}_2^s$ will be in the local control point vector $\mathbf{F}^k$ for some k provided $\mathbf{E}_1^s$ and $\mathbf{E}_2^s$ are closed to $\mathbf{V}$ enough. Without loss of generality, we may assume $k = s$. Hence the unit normal of $\mathbf{\Pi}^s$ can be written as

$$\mathbf{N}^s = \frac{(\mathbf{E}_1^s - \mathbf{V}) \times (\mathbf{E}_1^s - \mathbf{V})}{\|(\mathbf{E}_1^s - \mathbf{V}) \times (\mathbf{E}_1^s - \mathbf{V})\|_2}. \tag{4.5}$$

From (3.5), (3.6) and (4.3), we have the following expansion:

$$\mathbf{F}^s = \mathbf{V} + \mathbf{G}_2\lambda_2^s\mu_2 + \mathbf{G}_3\lambda_3^s\mu_3 + \mathbf{G}_4\lambda_4^s\mu_4 + \cdots \tag{4.6}$$

where $\{\mathbf{G}_i\} \in \mathbf{R}^3$ are determined by the initial data and the tension parameters. Thus, $\mathbf{E}_1^s - \mathbf{V}$ and $\mathbf{E}_2^s - \mathbf{V}$ can be written in the following form:

$$c_2\mathbf{G}_2\lambda_2^s + c_3\mathbf{G}_3\lambda_3^s + c_4\mathbf{G}_4\lambda_4^s + \cdots \tag{4.7}$$

where $\{c_i\}$ are some constants determined by the eigenvectors. Therefore, for general initial data, the normal vector (4.5) becomes

$$\mathbf{N}^s = \frac{\mathbf{G}_2 \times \mathbf{G}_3}{\|\mathbf{G}_2 \times \mathbf{G}_3\|_2} + \mathcal{O}\left(\frac{\lambda_4}{\lambda_2}\right)^s \tag{4.8}$$

which converges uniformly to the constant unit vector $\mathbf{N}_v$ defined by

$$\mathbf{N}_v := \frac{\mathbf{G}_2 \times \mathbf{G}_3}{\|\mathbf{G}_2 \times \mathbf{G}_3\|_2} \tag{4.9}$$

as $\mathbf{E}_1^s$ and $\mathbf{E}_2^s$ approach $\mathbf{V}$, i.e., as k goes to infinity.

For an arbitrary secant plane $\mathbf{\Pi}^k$ of the surface determined by $\mathbf{P}_1^k$, $\mathbf{P}_2^k$ and $\mathbf{V}$ on the surface close enough to $\mathbf{V}$, because the algorithm is interpolatory, there exist two points $\mathbf{E}_1^k$, $\mathbf{E}_2^k$ on the control net of level k such that

$$\mathbf{\Pi}^k = \mathbf{span}\left\{\mathbf{Q}_1^k - \mathbf{V},\, \mathbf{Q}_2^k - \mathbf{V}\right\}$$

and

$$\mathbf{Q}_i^k = \left(1-\theta_i^k\right)\mathbf{E}_i^k + \theta_i^k \mathbf{E'}_i^k, \qquad 0 \le \theta_i^k \le 1, \quad i = 1,2$$

for some control points $\{\mathbf{E}_i^k,\, \mathbf{E'}_i^k\}$ at level k since the limit surface is continuous and the interpolatory points are dense on the limit surface. Hence by using expansion (4.6) again and assuming that the initial data is a general data, we have

$$\begin{aligned}
\mathbf{N}^k &= \frac{\left(\mathbf{Q}_1^k - \mathbf{V}\right) \times \left(\mathbf{Q}_1^k - \mathbf{V}\right)}{\left\|\left(\mathbf{Q}_1^k - \mathbf{V}\right) \times \left(\mathbf{Q}_1^k - \mathbf{V}\right)\right\|_2} \\
&= \frac{\left(\mathbf{E}_1^k + \theta_1^k\left(\mathbf{E}_1^k - \mathbf{E'}_1^k\right) - \mathbf{V}\right) \times \left(\mathbf{E}_2^k + \theta_2^k\left(\mathbf{E}_2^k - \mathbf{E'}_2^k\right) - \mathbf{V}\right)}{\left\|\left(\mathbf{E}_1^k + \theta_1^k\left(\mathbf{E}_1^k - \mathbf{E'}_1^k\right) - \mathbf{V}\right) \times \left(\mathbf{E}_2^k + \theta_2^k\left(\mathbf{E}_2^k - \mathbf{E'}_2^k\right) - \mathbf{V}\right)\right\|_2} \\
&= \frac{\mathbf{G}_2 \times \mathbf{G}_3}{\|\mathbf{G}_2 \times \mathbf{G}_3\|_2} + \mathcal{O}\left(\frac{\lambda_4}{\lambda_2}\right)^k
\end{aligned}$$

which converges uniformly to the same unit vector $\mathbf{N}_v$ defined in (4.9) which is the unit normal of the surface at $\mathbf{V}$. This competes the proof. ∎

REMARK 4.5. It can be shown that a necessary condition for the limit surface to have a unique tangent plane at the extraordinary point is that the local subdivision matrix has the properties:

(i) $\lambda_1 = 1$, $\mu_1 = (1,1,\dots,1)^{\mathrm{T}}$;
(ii) there exists $N_0 \ge 3$, such that $0 < \lambda_2 = \lambda_3 = \lambda_4 = \cdots = \lambda_{N_0} < 1$,
$\mathbf{dim\ span}\{\mu_2, \mu_3, \dots, \mu_{N_0}\} = 2$; (4.10)
(iii) $|\lambda_i| < \lambda_2$, $i = N_0 + 1, \dots, 3n+1$.

REMARK 4.6. Formulae (4.5), (4.8) and (4.9) can be used to estimate the unit normal directions of surface interpolants from scattered surface data at extraordinary points. It should be noted that these estimates depend not only on the parameters $\{w_i\}$ and the intial data, but also on the triangulation of the data.

From the above expansion and the uniform convergence analysis, the following uniform convergence result can be concluded. This can also be obtained from the error estimates (5.6) and (5.6a) in the next section.

THEOREM 4.7. *The convegence of the control net sequence of arbitrary triangulations to the limit surface is uniform and simultaneous with the unit normal vector sequence; that is, the unit normal of the control net convergences uniformly to the unit normal of the limit surface.* ∎

5. ERROR ESTIMATES

In this section, we study the approximation properties of the surfaces generated by the algorithm. It is shown that over uniform triangulations, scheme (2.4a) with cubic precision has the power of approximation $\mathcal{O}(h^4)$ or otherwise it has a lower power of approximation $\mathcal{O}(h^2)$. It is also shown that the approximation is simultaneous with their (partial) derivatives. However, over nonuniform triangulations, scheme (2.4a) with modifications (3.1) and (3.2) at extraordinary points has only the power of approximation $\mathcal{O}(h)$. This lower order of approximation is caused by the simplicity of the modification formulae (3.1) and (3.2). If linear precision modification formulae are used, then the the power of approximation will be $\mathcal{O}(h^2)$.

First, by using the bivariate polynomial interpolation, the reproductivity of the 10-point scheme, and the Taylor expansion for bivariate functions, we can obtain the following result.

THEOREM 5.1. *Suppose $\mathbf{F}(u,v)$, $u,v \in \mathbf{R}$, is a regular and C^4 surface in $\mathbf{R}^m$, $m \geq 3$. Let $\mathbf{P}(u,v)$, $u,v \in \mathbf{R}$, be the limit surface generated by the 10-point scheme (2.4a) from the initial data (uniform) $\mathbf{P}(i,j) := \mathbf{F}(ih_1, jh_2)$, $i \in Z$, $0 < h_1$, $h_2 < 1$ with the parameters satisfying (2.5) and (2.13). Then, on any finite rectangle $[a,b] \times [c,d]$, we have the following error estimate:*

$$\left\|\mathbf{F}(h_1 u, h_2 v) - \mathbf{P}(u,v)\right\|_\infty \leq \frac{M_4(\mathbf{F})}{24} h^4 = \mathcal{O}(h^4), \tag{5.1}$$

where

$$h := \max\{h_1,\, h_2\}$$

and the number $M_4(\mathbf{F})$ depends only on the fourth-order partial derivatives of $\mathbf{F}(u,v)$. For general parameters $\{w_i\}$ satisfying (2.13) but not necessarily satisfying (2.5), we have the following lower order estimate:

$$\left\|\mathbf{F}(h_1 u, h_2 v) - \mathbf{P}(u,v)\right\|_\infty \leq \frac{M_2(\mathbf{F})}{2} h^2 = \mathcal{O}(h^2) \tag{5.2}$$

where the number $M_2(\mathbf{F})$ depends only on the second-order partial derivatives of $\mathbf{F}(u,v)$. Furthermore, for estimates (5.1) and (5.2), the corresponding approximations of the first-order partial derivatives are $\mathcal{O}(h^3)$ and $\mathcal{O}(h)$, respectively, provided the parameters $\{w_i\}$ satisfy (2.13).

PROOF. Without loss of generality, we assume $a = c = 0$, $b = d = 1$ since the algorithm is a local algorithm. From the local property of the scheme, we know that the patch $\mathbf{P}(u,v)$, $0 \leq u,v \leq 1$ is completely determined by the points $\mathbf{P}_{i,j}$, $i,j = -6,-5,\dots,7$. Let $\mathbf{Q}_3(u,v)$ be any cubic polynomial which interpolates ten of the above points, say $\mathbf{P}_{i,j}$, $i,j = 0,1,2,3$, $j \leq i$ at the corresponding nodes $(u,v) = (h_1 i, h_2 j)$. Then from the Lagrange polynomial interpolation formula and the corresponding error representation formula, we have the following estimate over the square $[0,1]^2$:

$$\left\|\mathbf{Q}_3(u,v) - \mathbf{F}(u,v)\right\|_\infty \leq \frac{M_4(\mathbf{F})}{2(4)!} h^4 = \mathcal{O}\left(h^4\right), \tag{5.3}$$

where the number $M_4(\mathbf{F})$ depends only on the fourth-order partial derivatives of $\mathbf{F}(u,v)$, and an explicit upper bound of it is given by (5.5).

Since the scheme reproduces cubic polynomial surfaces if (2.5) is satisfied, then using the error formula we have, over the domain $[0,1]^2$,

$$\begin{aligned}
&\left\|\mathbf{P}(u,v) - \mathbf{F}(h_1 u, h_2 v)\right\|_\infty \\
&\quad = \left\|\mathbf{Q}_3(h_1 u, h_2 v) - \mathbf{F}(h_1 u, h_2 v) + \mathbf{P}(u,v) - \mathbf{Q}_3(h_1 u, h_2 v)\right\|_\infty \\
&\quad \leq \left\|\mathbf{Q}_3(h_1 u, h_2 v) - \mathbf{F}(h_1 u, h_2 v)\right\|_\infty + \left\|\mathbf{P}(u,v) - \mathbf{Q}_3(h_1 u, h_2 v)\right\|_\infty \\
&\quad = \left\|\mathbf{Q}_3(h_1 u, h_2 v) - \mathbf{F}(h_1 u, h_2 v)\right\|_\infty + \max_{-6 \leq i,j \leq 7} \left\|\mathbf{P}(i,j) - \mathbf{Q}_3(h_1 i, h_2 j)\right\|_\infty \\
&\quad = \left\|\mathbf{Q}_3(h_1 u, h_2 v) - \mathbf{F}(h_1 u, h_2 v)\right\|_\infty + \max_{-6 \leq i,j \leq 7} \left\|\mathbf{F}(h_1 i, h_2 j) - \mathbf{Q}_3(h_1 i, h_2 j)\right\|_\infty \\
&\quad = 2\left\|\mathbf{Q}_3(u,v) - \mathbf{F}(h_1 u, h_2 v)\right\|_\infty \\
&\quad \leq \frac{M_4(\mathbf{F})}{4!} h^4 \\
&\quad = \mathcal{O}\left(h^4\right).
\end{aligned} \tag{5.4}$$

For general parameters, the scheme is only exact for linear surfaces, and therefore, by a similar argument, estimate (5.2) will follow.

For the error estimates of the partial derivatives, the results follow from the classical error estimates. This completes the proof. ∎

REMARK 5.2. The finite conditions on the domain in Theorem 5.1 can be removed since the concerned schemes are all local schemes. Therefore both the approximation and convergence are local and uniform.

REMARK 5.3. The approximation order of an interpolatory subdivision algorithm with a finite mask depends on its precision for polynomials. By Taylor's expansion, it can be shown that the approximation order of an interpolatory subdivision algorithm, either univariate or multivariate, is at least $N+1$ if it reproduces polynomials of degree N.

REMARK 5.4. Explicitly, the number $M_4(\mathbf{F})$ can be bounded by

$$M_4(\mathbf{F}) \leq 2 \cdot 16 \cdot 7^4 \left\| \mathbf{F}^{(4)} \right\|_\infty \tag{5.5}$$

where $\|\mathbf{F}^{(4)}\|_\infty$ denotes the maximum value of all the fourth-order directional and mixed partial derivatives of $\mathbf{F}(u,v)$. Similarly we have the following estimates for the bounds:

$$M_3(\mathbf{G}) \leq 2\cdot 8\cdot 7^3 \left\| \mathbf{G}^{(3)} \right\|_\infty, \qquad M_2(\mathbf{G}) \leq 2\cdot 4\cdot 7^2 \left\| \mathbf{G}^{(2)} \right\|_\infty, \qquad M_1(\mathbf{G}) \leq 2\cdot 7 \left\| \mathbf{G}^{(1)} \right\|_\infty \tag{5.5a}$$

where it is assumed that $\mathbf{G}(u,v)$ has the required smoothness, and the number 7 comes from the local property of the algorithm.

For nonuniform triangulations, using a similar proof, we can obtain the following weaker estimates.

THEOREM 5.5. *Suppose $\mathbf{F}(u,v)$, $u,v \in \mathbf{R}$, is a regular and C^2 surface in $\mathbf{R}^m$, $m \geq 3$. Let $\mathbf{P}(u,v)$, $u,v \in \mathbf{R}$, be the limit surface generated by the 10-point scheme (2.4a) with modifications (3.1) and (3.2) at the extraordinary points, from a nonuniform triagulation of the initial data $\mathbf{F}(u_i,v_i)$, $i \in Z$ on the surface with the parameters satisfying (2.13) and (4.4) at the regular points and extraordinary points, respectively. Then, on any finite rectangle $[a,b] \times [c,d]$, we have the following error estimates:*

$$\begin{aligned} \mathbf{dist}\,(\mathbf{F}-\mathbf{P}) &\leq \frac{M_2(\mathbf{F})}{24}\,h^2 + M_1(\mathbf{P})h = \mathcal{O}(h), \\ \mathbf{dist}\,(\mathbf{N_F}-\mathbf{N_P}) &\leq o(h), \end{aligned} \tag{5.6}$$

where the distant norm is defined as

$$\mathbf{dist}\,(\mathbf{F}-\mathbf{P}) := \sup_{u,v}\left\{\inf_{r,s} |\mathbf{F}(u,v)-\mathbf{P}(r,s)|\right\},$$

and $\mathbf{N_F}$, $\mathbf{N_P}$ are the unit normal vectors of the surfaces $\mathbf{F}$ and $\mathbf{P}$, respectively, and h is the size of the triangulation (mesh size) and the number $M_2(\mathbf{F})$ depends only on the second-order partial derivatives of $\mathbf{F}(u,v)$, and $M_1(\mathbf{P})$ depends only on the first-order partial derivatives of $\mathbf{P}(u,v)$.

PROOF. The proof is similar to the proof of Theorem 5.1 but we can only introduce a linear local interpolant $\mathbf{Q}_1(u,v)$ instead of the cubic local interpolant $\mathbf{Q}_3(u,v)$. The first term $(M_2(\mathbf{F})/24)\,h^2$ in estimate (5.6) comes from the fact $\mathbf{F}(u,v)$ and $\mathbf{Q}_1(u,v)$ are C^2 surfaces, and the second term $M_1(\mathbf{P})h$ in the estimate comes from the fact $\mathbf{P}(u,v)$ is only C^1 surface.

The estimate for the unit normal vector comes from the Taylor expansion of the surfaces with local parameterizations, interpolation property of $\mathbf{P}$ and the convergence result of Theorem 4.4.

The distant norm $\mathbf{dist}\,(\mathbf{F}-\mathbf{P})$ is used here since for nonuniform triangulations, the parameterizations of the surfaces $\mathbf{F}(u,v)$ and $\mathbf{P}(u,v)$ are quite different. ∎

The above weaker estimates for nonuniform triangulations can be improved which requires the following lemma.

LEMMA 5.6. *The 10-point scheme (2.4a) can be modified properly at the extraordinary points, so that it reproduces linear or even cubic surfaces from any triagulations of uniform data.*

PROOF. In fact, the refinement formulae at the extraordinary points can be such derived that they produce the same points as a local cubic or linear interpolant to the data points if the same parameterization is used. Therefore, using such a modification at the extraordinary points, the algorithm will reproduce cubic or linear surfaces from any triangulations of uniform data. It should be noted that the cubic precision local algorithm is quite complicated, and thus it is seldom used in practice. ∎

From this lemma and a similar argument to the proof of Theorem 5.1, we can obtain better error estimates for well modified algorithms over arbitrary triangulations.

THEOREM 5.7. *Suppose* $\mathbf{F}(u,v)$, $u,v \in \mathbf{R}$, *is a regular and* C^4 *surface in* $\mathbf{R}^m$, $m \geq 3$. *Let* $\mathbf{P}(u,v)$, $u,v \in \mathbf{R}$, *be the limit surface generated by the 10-point scheme (2.4a) with cubic or linear modifications at the extraordinary points, from a nonuniform triangulation of the initial data* $\mathbf{F}(u_i,v_i)$, $i \in Z$ *on the surface with the parameters satisfying (2.13) and (4.4) at the regular points and extraordinary points, respectively. Then, on any finite rectangle* $[a,b] \times [c,d]$, *for the cubic precision modification we have the following error estimate:*

$$\mathbf{dist}\,(\mathbf{F}-\mathbf{P}) \leq \frac{M_4(\mathbf{F})}{24}\,h^4 = \mathcal{O}(h^4), \tag{5.7}$$

and for the linear precision modification we have the following error estimate:

$$\mathbf{dist}\,(\mathbf{F}-\mathbf{P}) \leq \frac{M_2(\mathbf{F})}{2}\,h^2 = \mathcal{O}(h^2), \tag{5.8}$$

where h *is again the size of the triangulation (mesh size), and the numbers* $M_4(\mathbf{F})$ *and* $M_2(\mathbf{F})$ *have similar meanings as in Theorems 5.1 and 5.5. Furthermore, for estimates (5.7) and (5.8) the corresponding approximations of the unit normal vector are* $\mathcal{O}(h^3)$ *and* $\mathcal{O}(h)$, *respectively, provided the parameters* $\{w_i\}$ *satisfy (2.13) and (4.4).* ∎

6. CONCLUSIONS AND EXAMPLES

The 10-point interpolatory subdivision algorithm for surfaces is generalized for smooth surface interpolation over arbitrary networks, and its convergence properties over nonuniform triangulations is studied. In the convergence analysis, the local subdivision matrix analysis is used and it is proved that the algorithm produces smooth surfaces over arbitrary triangular networks if the shape parameters are chosen properly. It is also shown that the power of approximation using the algorithms is $\mathcal{O}(h^2)$, and it can be made much smaller if the data is accurate and the shape control parameters are well chosen. Numerical examples have shown that generally the subdivision method produces quite satisfactory interpolants.

The analyses of the scheme here are different from the uniform analyses discussed in our previous work about the scheme over uniform data. In fact, the analysis presented here is a pointwise analysis and matrix approach is applied here. This technique is quite suitable for the nonuniform analysis.

One problem that still remains unsolved is the optimum choice of the parameters and the initial triangulation of the scattered data such that the generated surface has some desired properties such as convexity and minimum energy.

The smoothing process of the algorithm for surface interpolation over nonuniform data is shown by the following graphical examples.

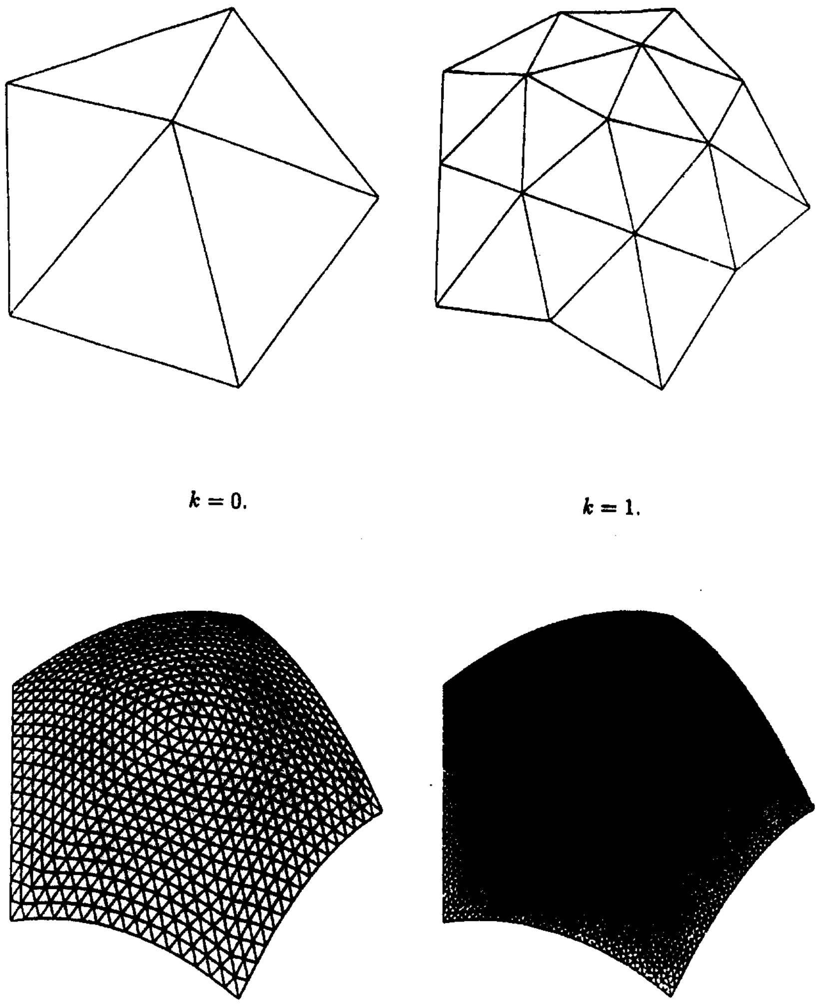


Figure 4. The control nets for $k = 0, 1, 4, 5$ generated by the algorithm for $n = 5$.

EXAMPLE 1. In this example, 6 points in $\mathbf{R}^3$ are given and the initial triangulation is formed by 5 triangles in a star-like configuration shown in Figure 4 for $k = 0$. The initial triangulation is nonuniform since the center point is of valency 5, i.e., $n = 5$. The subsequent control nets for $k = 1, 4, 5$ are also shown in Figure 4. For the computation of the refined points, some boundary scheme is used to provide near-the-boundary points, and the control parameters are given by the following special cubic precision choice:

$$w_1 = -\frac{1}{16}, \qquad w_2 = -2w_1, \qquad w_3 = 0. \tag{6.1}$$

EXAMPLE 2. In this example, 8 points in $\mathbf{R}^3$, which in fact are the vertices of the unit cube, are given and the initial triangulation is formed by 12 triangles that form a configuration shown in Figure 5 for $k = 0$. The triangulation is nonuniform since all the 8 points are not of valency 6, i.e., either $n = 4$ or $n = 5$. The subsequent control nets for $k = 2, 4, 5$ are also shown in Figure 5. For the computation of the refined points, the control parameters are given by the following:

$$w_1 = -\frac{1}{20}, \qquad w_2 = -2w_1, \qquad w_3 = -\frac{1}{30}. \tag{6.2}$$

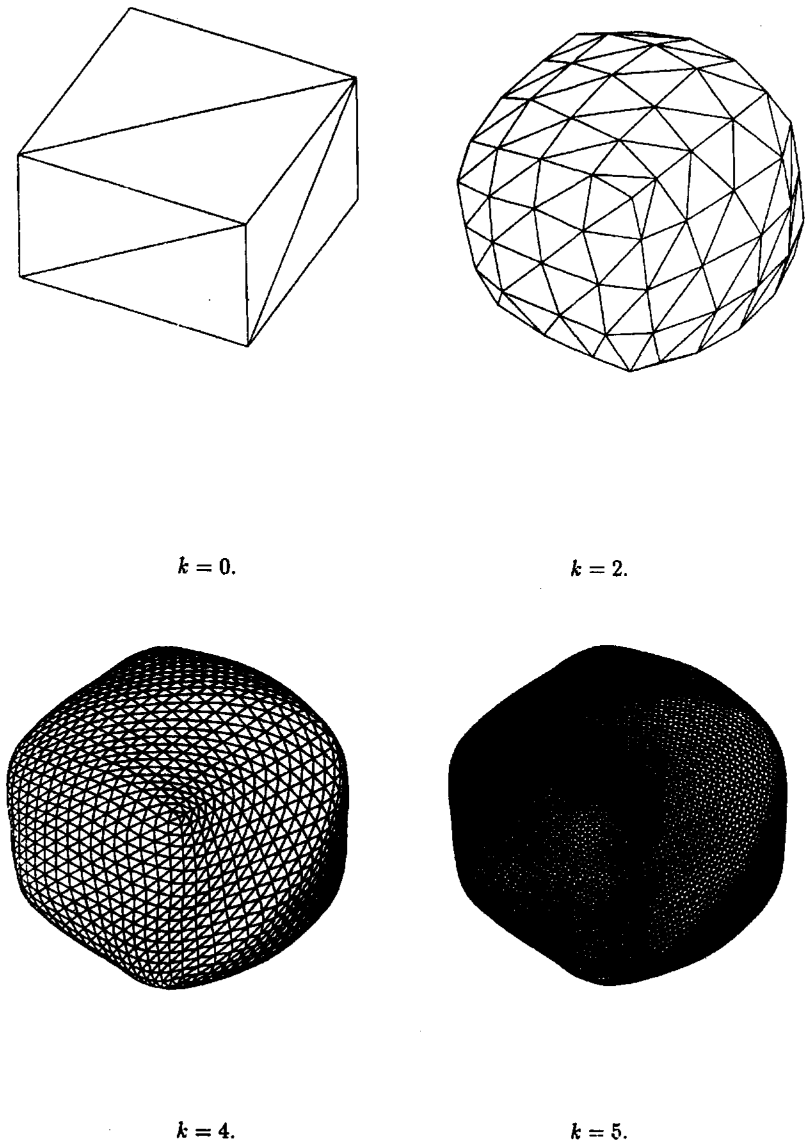


Figure 5. The interpolants to the cube produced by the algorithm at different levels.

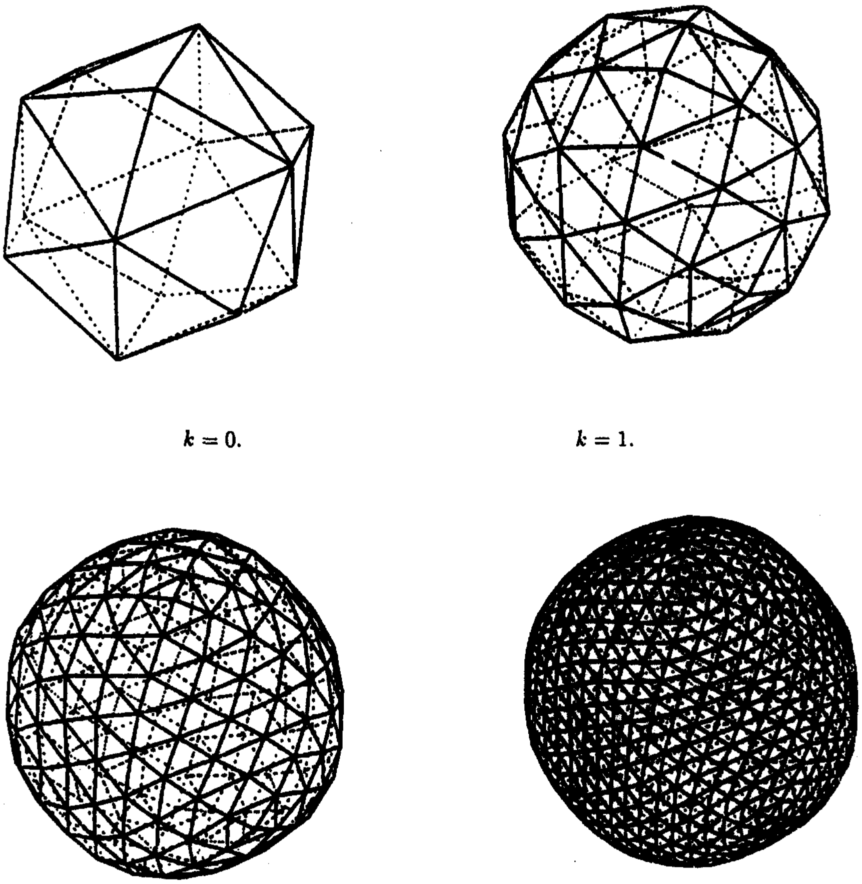
EXAMPLE 3. Similar to Example 2, the given data are the 8 vertices of the unit cube in $\mathbf{R}^3$ with all the 12 edges of the cube. The initial triangulation of the data is formed by introducing 6 more "face-points" which are obtained by using the formula (2.4) with a proper choice of the parameters. The 6 points are the symmetries of the point

$$\left(0, 0, \frac{\sqrt{3}}{2}\right). \tag{6.3}$$

Thus, an initial triangulation with 14 vertices, 24 triangles form a configuration shown in Figure 6 for $k = 0$. The triangulation is nonuniform since 6 of the 14 points are not of valency 6, in fact, $n = 4$. The subsequent control nets for $k = 1, 2, 3$ are also shown in the figure. The following control parameters are used in the computations of the refined points:

$$w_1 = -\frac{1}{25}, \qquad w_2 = -2w_1, \qquad w_3 = -\frac{1}{50}. \tag{6.4}$$

It is obvious that this interpolant is better than that obtained in Example 2.



$k = 0.$ $k = 1.$

$k = 2.$ $k = 3.$

Figure 6. The interpolants to the cube data at different levels in Example 3.

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