Positive solutions to singular system with four-point coupled boundary conditions

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Abstract

Existence of positive solution to a nonlinear singular system with four-point coupled boundary conditions of the type

\begin{align*}
-x''(t) &= f(t, x(t), y(t)), & t \in (0, 1), \\
y''(t) &= g(t, x(t), y(t)), & t \in (0, 1), \\
x(0) &= 0, & x(1) &= \alpha y(\xi), \\
y(0) &= 0, & y(1) &= \beta x(\eta),
\end{align*}

is established. The nonlinearities \( f, g : (0, 1) \times [0, \infty) \times [0, \infty) \to [0, \infty) \) are continuous and singular at \( t = 0, t = 1 \), while the parameters \( \alpha, \beta, \xi, \eta \) satisfy \( \xi, \eta \in (0, 1), 0 < \alpha \beta \xi \eta < 1 \).

An example is included to show the applicability of our result.

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1. Introduction

Coupled boundary conditions (BCs) arise in the study of reaction–diffusion equations and Sturm–Liouville problems, see [3,4,16] and [28, Chapter 13]. The study of coupled BCs of the following type

\begin{equation}
D_1 f |_{\partial \Omega} + D_2 \frac{\partial f}{\partial \nu} = 0,
\end{equation}

where \( D_1 \) and \( D_2 \) are differential operators from \( L^2(\Omega; W) \) to \( L^2(\partial \Omega; W) \), \( \Omega \subset \mathbb{R}^n \) and \( W \) is a separable Hilbert space, for the elliptic system has been initiated by Agmon and coauthors [2]. In the study of interaction problems and elliptic operators on polygonal domains, Mehmeti [21], Mehmeti and Nicaise [22] and Nicaise [23] have studied coupled BCs. In [15, Section 8.3], Krstic and coauthors presented the Timoshenko beam model with free-end BCs

\begin{align*}
\varepsilon u_{tt}(t, x) &= (1 + d \partial_t)(u_{xx} - \theta_k), & t \geq 0, & x \in (0, 1), \\
\mu \varepsilon \theta_{tt}(t, x) &= (1 + d \partial_t)(\varepsilon \theta_{xx} + \alpha(u_x - \theta)), & t \geq 0, & x \in (0, 1), \\
u_x(t, 0) &= \theta(t, 0), & \theta_x(t, 0) &= 0,
\end{align*}

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where $u(t, x)$ denotes the displacement and $\theta(t, x)$ denotes the angle of rotation due to the bending. The positive constants $a$ and $\mu$ are proportional to the nondimensional cross-sectional area and the nondimensional moment of inertia of the beam, respectively. The parameter $\varepsilon$ is inversely proportional to the nondimensional shear modulus of the beam. The coefficient $d$ denotes the possible presence of Kelvin–Voigt damping. The meaning of the first boundary condition is that zero force is being applied at the tip, while the meaning of the second boundary condition is that zero moment is being applied at the tip. The model is controlled at $x = 1$ through the conditions on $u(t, 1)$ and $\theta(t, 1)$.

Coupled BCs have also some applications in mathematical biology. For example, Leung [16] studied the following reaction–diffusion system for prey–predator interaction:

$$
\begin{align*}
    u_t(t, x) &= \sigma_1 \Delta u + u(a + f(u, v)), \quad t > 0, \ x \in \Omega \subset \mathbb{R}^n, \\
    v_t(t, x) &= \sigma_2 \Delta v + v(-r + g(u, v)), \quad t > 0, \ x \in \Omega \subset \mathbb{R}^n,
\end{align*}
$$

subject to the coupled BCs

$$
\frac{\partial u}{\partial \eta} = 0, \quad \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \quad \text{on } \partial \Omega, 
$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $a$, $r$, $\sigma_1$, $\sigma_2$ are positive constants, $f, g : \mathbb{R}^2 \to \mathbb{R}$ have Hölder continuous partial derivatives up to second order in compact sets, $\eta$ is a unit outward normal at $\partial \Omega$ and $p$ and $q$ have Hölder continuous first derivatives in compact subsets of $[0, \infty)$. The functions $u(t, x)$, $v(t, x)$ respectively represent the density of prey and predator at time $t \geq 0$ and at position $x = (x_1, \ldots, x_n)$. Similar coupled BCs are also studied in [5] for biochemical system.

Existence theory for boundary value problems (BVPs) of ordinary differential equations is well studied; we refer the readers to [1,6–10,20,25,26] and the reference therein for two-point BVPs and [14,17–19,29,31] for three-point BVPs, while for multi-point BVPs, we refer to [12,13,24,30].

Inspired by the above mentioned work and wide applications of coupled BCs in various fields of sciences and engineering, we study existence result to a coupled singular system subject to four-point coupled BCs of the type

$$
\begin{align*}
    x''(t) &= f(t, x(t), y(t)), \quad t \in (0, 1), \\
    y''(t) &= g(t, x(t), y(t)), \quad t \in (0, 1), \\
    x(0) = 0, \quad x(1) &= \alpha y(\xi), \\
    y(0) = 0, \quad y(1) &= \beta x(\eta),
\end{align*}
$$

where the parameters $\alpha$, $\beta$, $\xi$, $\eta$ satisfy $\xi, \eta \in (0, 1)$, $0 < \alpha \beta \xi \eta < 1$. We assume that the nonlinearities $f, g : (0, 1) \times [0, \infty) \to [0, \infty)$ are continuous and singular at $t = 0$, $t = 1$. By singularity we mean that the functions $f(t, x, y)$, $g(t, x, y)$ are unbounded at $t = 0$ and $t = 1$. Here, we study existence of at least one positive solution for the system of BVPs (1.5). By a positive solution of the system (1.5), we mean that $(x, y) \in (C[0, 1] \cap C^2(0, 1)) \times (C[0, 1] \cap C^2(0, 1))$, $(x, y)$ satisfies (1.5), $x > 0$ and $y > 0$ on $(0, 1)$. Further we remark that, to the best of our knowledge in literature there is no result for a system of ordinary differential equations with four-point coupled BCs.

Throughout the paper, we assume that the following conditions hold:

$$(\mathcal{H}_1) \quad f(0, 1, 1), g(0, 1, 1) \in C((0, 1), (0, \infty)) \text{ and satisfy}$$

$$a := \int_0^1 t(1 - t)f(t, 1, 1) \, dt < +\infty, \quad b := \int_0^1 t(1 - t)g(t, 1, 1) \, dt < +\infty.$$  

$$(\mathcal{H}_2) \quad \text{There exist real constants } \alpha_i, \beta_i \text{ with } 0 \leq \alpha_i \leq \beta_i < 1, i = 1, 2; \beta_1 + \beta_2 < 1, \text{ such that for all } t \in (0, 1), x, y \in [0, \infty),$$

$$c^{\beta_1} f(t, x, y) \leq f(t, c x, y) \leq c^{\alpha_1} f(t, x, y), \quad 0 < c \leq 1,$$

$$c^{\beta_2} f(t, x, y) \leq f(t, x, c y) \leq c^{\alpha_2} f(t, x, y), \quad 0 < c \leq 1,$$

$$c^{\beta_2} f(t, x, y) \leq f(t, x, c y) \leq c^{\alpha_2} f(t, x, y), \quad 0 < c \leq 1.$$

$$(\mathcal{H}_3) \quad \text{There exist real constants } \rho_i, \rho_i \text{ with } 0 \leq \gamma_i \leq \rho_i < 1, i = 1, 2; \rho_1 + \rho_2 < 1, \text{ such that for all } t \in (0, 1), x, y \in [0, \infty),$$

$$c^{\gamma_1} g(t, x, y) \leq g(t, c x, y) \leq c^{\beta_1} g(t, x, y), \quad 0 < c \leq 1,$$

$$c^{\gamma_1} g(t, x, y) \leq g(t, x, c y) \leq c^{\beta_1} g(t, x, y), \quad 0 < c \leq 1,$$

$$c^{\gamma_1} g(t, x, y) \leq g(t, x, c y) \leq c^{\beta_1} g(t, x, y), \quad 0 < c \leq 1.$$
Here we remark that, in [27] an existence result for a coupled system of second order and fourth order BVPs has been established under hypotheses (\(H_1\))–(\(H_3\)) together with some additional assumptions on the nonlinearities. But in this paper, we prove existence results for the system (1.5) under the hypotheses (\(H_1\))–(\(H_3\)) only. This paper is organized as follows. In Section 2, we present a positive cone, a fixed point result which will be used to prove existence of positive solution, Green’s function for the system of BVPs (1.5) and some related lemmas. In Section 3, we present main result of the paper and finally an example is provided to show the applicability of our theory.

2. Preliminaries

For each \(u \in E := C[0, 1]\) we write \(\|u\| = \max \{|u(t)|: t \in [0, 1]\}\). Define

\[ P = \left\{ u \in E: \min_{t \in [\max \{\xi, \eta\}, 1]} u(t) \geq \gamma \|u\| \right\}, \]

where

\[ 0 < \gamma := \frac{\min \{1, \alpha \xi, \alpha \beta \xi, \beta \eta, \alpha \beta \eta\} \min \{\xi, \eta, 1 - \xi, 1 - \eta\}}{\max \{1, \alpha, \beta, \alpha \beta \xi, \alpha \beta \eta\}} < 1. \]

Clearly, \((E, \| \cdot \|)\) is a Banach space and \(P\) is a cone of \(E\). Similarly, for each \((x, y) \in E \times E\) we write \(\|(x, y)\|_1 = \|x\| + \|y\|\).

We need the following results in the sequel.

Lemma 2.1 (Guo–Krasnosel’skii fixed-point theorem). (See [11].) Let \(P\) be a cone of a real Banach space \(E\), and let \(\Omega_1, \Omega_2\) be bounded open neighborhoods of \(0 \in E\), and assume \(\Omega_2 \subset \Omega_1\). Suppose that \(T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P\) is completely continuous such that one of the following conditions holds:

(i) \(\|Tx\| \leq \|x\|\) for \(x \in \partial \Omega_1 \cap P\); \(\|Tx\| \geq \|x\|\) for \(x \in \partial \Omega_2 \cap P\);
(ii) \(\|Tx\| \leq \|x\|\) for \(x \in \partial \Omega_2 \cap P\); \(\|Tx\| \geq \|x\|\) for \(x \in \partial \Omega_1 \cap P\).

Then, \(T\) has a fixed point in \(P \cap (\overline{\Omega_2} \setminus \Omega_1)\).

We need the following results in the sequel.

Lemma 2.2. Let \(u, v \in C[0, 1]\), then the system of BVPs

\[-x''(t) = u(t), \quad -y''(t) = v(t), \quad t \in [0, 1],
\]

\[x(0) = y(0) = 0, \quad x(1) = \alpha y(\xi), \quad y(1) = \beta x(\eta), \]

has integral representation

\[x(t) = \int_0^1 F_{\xi \eta}(t, s)u(s) \, ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s)v(s) \, ds, \]

\[y(t) = \int_0^1 F_{\eta \xi}(t, s)v(s) \, ds + \int_0^1 G_{\alpha \beta \eta \xi}(t, s)u(s) \, ds, \]

where

\[F_{\xi \eta}(t, s) = \begin{cases}
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi \eta(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq s \leq t \leq 1, \ s \leq \eta, \\
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi \eta(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq t \leq s \leq 1, \ s \leq \eta, \\
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi \eta(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq s \leq t \leq 1, \ s \geq \eta, \\
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi \eta(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq t \leq s \leq 1, \ s \geq \eta,
\end{cases} \]

\[G_{\alpha \beta \xi \eta}(t, s) = \begin{cases}
\frac{\alpha \xi t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi \eta(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi \eta \xi(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq s, t \leq 1, \ s \leq \xi, \\
\frac{\alpha \xi t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi \eta(t-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi \eta \xi(t-s)}{1-\alpha \beta \xi \eta}, & 0 \leq s, t \leq 1, \ s \geq \xi.
\end{cases} \]
Proof. Integrating the system (2.1), we have

\[ x(t) = c_1 + c_3 t - \int_0^t (t-s)u(s) \, ds, \]
\[ y(t) = c_2 + c_4 t - \int_0^t (t-s)v(s) \, ds, \]

where \( c_i \), \( i = 1, \ldots, 4 \), are constants. Now, using the BCs, we obtain

\[ c_1 = 0, \quad c_2 = 0, \]
\[ c_3 - c_4 \alpha \xi = \int_0^1 (1-s)u(s) \, ds - \alpha \int_0^\xi (\xi-s)v(s) \, ds, \]
\[ c_3 \beta \eta - c_4 = \beta \int_0^\eta (\eta-s)u(s) \, ds - \int_0^1 (1-s)v(s) \, ds. \]

Solving for \( c_3 \) and \( c_4 \), we get

\[ c_3 = \frac{1}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)u(s) \, ds - \frac{\alpha \beta \xi}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds \]
\[ + \frac{\alpha \xi}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)v(s) \, ds - \frac{\alpha}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds, \]
\[ c_4 = \frac{\beta \eta}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)u(s) \, ds - \frac{\beta}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds \]
\[ + \frac{1}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)v(s) \, ds - \frac{\alpha \beta \eta}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds. \]

Thus system (2.5) becomes

\[ x(t) = \frac{1}{1 - \alpha \beta \xi \eta} \int_0^t (1-s)u(s) \, ds - \frac{\alpha \beta \xi}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds - \int_0^t (t-s)u(s) \, ds \]
\[ + \frac{\alpha \xi}{1 - \alpha \beta \xi \eta} \int_0^t (1-s)v(s) \, ds - \frac{\alpha}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds, \]
\[ y(t) = \frac{1}{1 - \alpha \beta \xi \eta} \int_0^t (1-s)v(s) \, ds - \frac{\alpha \beta \eta}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds - \int_0^t (t-s)v(s) \, ds \]
\[ + \frac{\beta \eta}{1 - \alpha \beta \xi \eta} \int_0^t (1-s)u(s) \, ds - \frac{\beta}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds, \]

which is equivalent to the system (2.2). \( \square \)

**Lemma 2.3.** The functions \( F_{\xi \eta} \) and \( G_{\alpha \beta \xi \eta} \) satisfy

(i) \( F_{\xi \eta}(t, s) \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1-s), \) \( t, s \in [0, 1], \)

(ii) \( G_{\alpha \beta \xi \eta}(t, s) \leq \frac{\alpha \beta \eta}{1 - \alpha \beta \xi \eta} s(1-s), \) \( t, s \in [0, 1]. \)
Proof. For \((t, s) \in [0, 1] \times [0, 1]\), we discuss various cases.

**Case 1.** \(s \leq \eta, t \geq s\); from (2.3), we obtain

\[
F_{\xi \eta}(t, s) = s + (\alpha \beta \xi - 1) \frac{ts}{1 - \alpha \beta \xi \eta}.
\]

If \(\alpha \beta \xi > 1\), the maximum occurs at \(t = 1\), hence

\[
F_{\xi \eta}(t, s) \leq F_{\xi \eta}(1, s) = \frac{\alpha \beta \xi s(1 - \eta)}{1 - \alpha \beta \xi \eta} \leq \frac{\alpha \beta \xi s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

And if \(\alpha \leq 1\), the maximum occurs at \(t = s\), hence

\[
F_{\xi \eta}(t, s) \leq F_{\xi \eta}(s, s) = \frac{s(1 - s) + \alpha \beta \xi (s - \eta)}{1 - \alpha \beta \xi \eta} \leq \frac{s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

**Case 2.** \(s \leq \eta, t \leq s\); using (2.3), we have

\[
F_{\xi \eta}(t, s) = t(1 - s) - \frac{\alpha \beta \xi t(\eta - s)}{1 - \alpha \beta \xi \eta} \leq t(1 - s) - \frac{s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

**Case 3.** \(s \geq \eta, t \geq s\); using (2.3), we get

\[
F_{\xi \eta}(t, s) = s + \frac{t(\alpha \beta \xi \eta - s)}{1 - \alpha \beta \xi \eta}.
\]

If \(\alpha \beta \xi \eta > s\), the maximum occurs at \(t = 1\), hence

\[
F_{\xi \eta}(t, s) \leq F_{\xi \eta}(1, s) = \frac{\alpha \beta \xi \eta(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\alpha \beta \xi s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s),
\]

And if \(\alpha \beta \xi \eta \leq s\), the maximum occurs at \(t = s\), so

\[
F_{\xi \eta}(t, s) \leq F_{\xi \eta}(s, s) = \frac{s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

**Case 4.** \(s \geq \eta, t \leq s\); using (2.3), we get

\[
F_{\xi \eta}(t, s) = \frac{t(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{s(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\max(1, \alpha \beta \xi)}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

Now we prove (ii). For \((t, s) \in [0, 1] \times [0, 1]\), we discuss two cases.

**Case 1.** \(s \leq \xi\); using (2.4), we obtain

\[
G_{\alpha \beta \xi \eta}(t, s) = \frac{\alpha ts(1 - \xi)}{1 - \alpha \beta \xi \eta} \leq \frac{\alpha}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

**Case 2.** \(s \geq \xi\); using (2.4), we have

\[
G_{\alpha \beta \xi \eta}(t, s) = \frac{\alpha \xi t(1 - s)}{1 - \alpha \beta \xi \eta} \leq \frac{\alpha}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

**Remark 2.4.** In view of Lemma 2.3, we have

\[
F_{\eta \xi}(t, s) \leq \frac{\max(1, \alpha \beta \eta)}{1 - \alpha \beta \xi \eta} s(1 - s), \quad t, s \in [0, 1],
\]

\[
G_{\beta \alpha \eta \xi}(t, s) \leq \frac{\beta}{1 - \alpha \beta \xi \eta} s(1 - s), \quad t, s \in [0, 1].
\]

**Lemma 2.5.** The functions \(F_{\xi \eta}\) and \(G_{\alpha \beta \xi \eta}\) satisfy

(i) \(F_{\xi \eta}(t, s) \geq \frac{\min(\alpha \beta \xi \eta \max(1 - \eta, 1 - s)}{1 - \alpha \beta \xi \eta} s(1 - s), \quad t, s \in [\eta, 1] \times [0, 1].\)

(ii) \(G_{\alpha \beta \xi \eta}(t, s) \geq \frac{\max(1 - s, 1 - \xi)}{1 - \alpha \beta \xi \eta} s(1 - s), \quad t, s \in [\xi, 1] \times [0, 1].\)
Proof. Here for \((t, s) \in [\eta, 1] \times [0, 1] , \) we discuss various cases.

Case 1. \(s \leq \eta, \ t \geq s; \) using (2.3), we obtain
\[
F_{\xi \eta}(t, s) = s + (\alpha \beta \xi - 1) \frac{ts}{1 - \alpha \beta \xi \eta}.
\]
If \(\alpha \beta \xi < 1, \) the minimum occurs at \(t = 1, \) hence
\[
F_{\xi \eta}(t, s) \geq F_{\xi \eta}(1, s) = \alpha \beta \xi s(1 - \eta) = \frac{\alpha \beta \xi s(1 - \eta)}{1 - \alpha \beta \xi \eta} \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s),
\]
and if \(\alpha \beta \xi \geq 1, \) the minimum occurs at \(t = \eta, \) then
\[
F_{\xi \eta}(t, s) \geq F_{\xi \eta}(\eta, s) = \frac{s(1 - \eta)}{1 - \alpha \beta \xi \eta} \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

Case 2. \(s \geq \eta, \ t \geq s; \) using (2.3), we have
\[
F_{\xi \eta}(t, s) = s - t(s - \alpha \beta \xi \eta) \frac{s(1 - s)}{1 - \alpha \beta \xi \eta}.
\]
If \(s > \alpha \beta \xi \eta, \) the minimum occurs at \(t = 1, \) hence
\[
F_{\xi \eta}(t, s) \geq F_{\xi \eta}(1, s) = \frac{\alpha \beta \xi \eta (1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s),
\]
and if \(s \leq \alpha \beta \xi \eta, \) the minimum occurs at \(t = s, \) therefore
\[
F_{\xi \eta}(t, s) \geq F_{\xi \eta}(s, s) = \frac{s(1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\eta(1 - s) - s(1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s).
\]
Case 3. \(s \geq \eta, \ t \leq s; \) using (2.3), we have
\[
F_{\xi \eta}(t, s) = \frac{t(1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\eta(1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s).
\]
Now we prove (ii). For \((t, s) \in [\xi, 1] \times [0, 1], \) we discuss two cases.

Case 1. \(s \leq \xi; \) using (2.4), we have
\[
G_{\alpha \beta \xi}(t, s) = \frac{at s(1 - \xi)}{1 - \alpha \beta \xi \eta} \geq \frac{\alpha \xi s(1 - \xi)}{1 - \alpha \beta \xi \eta} \geq \frac{\alpha \xi \min \{\xi, 1 - \xi\}}{1 - \alpha \beta \xi \eta} s(1 - s).
\]
Case 2. \(s \geq \xi; \) using (2.4), we get
\[
G_{\alpha \beta \xi}(t, s) = \frac{\alpha \xi (1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\alpha \xi (1 - s)}{1 - \alpha \beta \xi \eta} \geq \frac{\alpha \xi \min \{\xi, 1 - \xi\}}{1 - \alpha \beta \xi \eta} s(1 - s).
\]

Remark 2.6. In view of Lemma 2.5, we have
\[
F_{\xi \eta}(t, s) \geq \frac{\min \{1, \alpha \beta \xi \} \min \{\xi, 1 - \xi\}}{1 - \alpha \beta \xi \eta} s(1 - s), \quad (t, s) \in [\xi, 1] \times [0, 1],
\]
\[
G_{\alpha \beta \xi}(t, s) \geq \frac{\beta \eta \min \{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1 - s), \quad (t, s) \in [\eta, 1] \times [0, 1].
\]

Remark 2.7. From Lemma 2.3 and Remark 2.4, for \(t, s \in [0, 1], \) we have
\[
F_{\xi \eta}(t, s) \leq \mu s(1 - s), \quad F_{\xi \eta}(t, s) \leq \mu s(1 - s), \quad G_{\alpha \beta \xi}(t, s) \leq \mu s(1 - s), \quad G_{\beta \alpha \xi}(t, s) \leq \mu s(1 - s).
\]
Also, from Lemma 2.5 and Remark 2.6, for \((t, s) \in [\max \{\xi, \eta\}, 1] \times [0, 1], \) we have
\[
F_{\xi \eta}(t, s) \geq \nu s(1 - s), \quad F_{\xi \eta}(t, s) \geq \nu s(1 - s), \quad G_{\alpha \beta \xi}(t, s) \geq \nu s(1 - s), \quad G_{\beta \alpha \xi}(t, s) \geq \nu s(1 - s),
\]
where \(\mu = \frac{\max \{1, \alpha \beta \xi, \min \{\eta, 1 - \xi\}\}}{1 - \alpha \beta \xi \eta} \) and
\[
\nu = \frac{\min \{1, \alpha \beta \xi, \beta \eta, \min \{\eta, 1 - \xi, 1 - \eta\}\}}{1 - \alpha \beta \xi \eta}.
Lemma 2.8. The Green’s functions $F_{\xi \eta}$ and $G_{\alpha \beta \xi \eta}$ can be expressed as:

$$F_{\xi \eta}(t, s) = H(t, s) + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} H(\eta, s),$$

$$G_{\alpha \beta \xi \eta}(t, s) = \frac{\alpha t}{1 - \alpha \beta \xi \eta} H(\xi, s).$$

where

$$H(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. From (2.2), consider the integral equation

$$x(t) = \int_0^1 F_{\xi \eta}(t, s) u(s) \, ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s) v(s) \, ds$$

$$= \frac{t}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds - \int_0^t (t-s)u(s) \, ds$$

$$+ \frac{\alpha \xi t}{1 - \alpha \beta \xi \eta} \int_0^1 (1-s)v(s) \, ds - \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds$$

$$= \int_0^1 H(t, s)u(s) \, ds + \frac{1}{1 - \alpha \beta \xi \eta} \int_0^t (1-s)u(s) \, ds + \frac{1}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds$$

$$- \int_0^t s(1-t)u(s) \, ds + \int_0^t (t-s)u(s) \, ds + \frac{1}{1 - \alpha \beta \xi \eta} \int_0^1 t(1-s)u(s) \, ds$$

$$- \int_0^1 t(1-s)u(s) \, ds + \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^\xi (1-s)v(s) \, ds - \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds$$

$$= \int_0^1 H(t, s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^t \eta(1-s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds$$

$$+ \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta \eta(1-s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds$$

$$+ \frac{\alpha \xi t}{1 - \alpha \beta \xi \eta} \int_0^\xi (1-s)v(s) \, ds + \frac{\alpha \xi t}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds$$

$$= \int_0^1 H(t, s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta \eta(1-s)u(s) \, ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta (\eta-s)u(s) \, ds$$

$$+ \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^\eta \eta(1-s)u(s) \, ds + \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^\xi (1-s)v(s) \, ds + \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^\xi (\xi-s)v(s) \, ds$$
Clearly, if

\begin{equation*}
\rho = \int_0^1 H(t, s)u(s)\,ds + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi} \int_0^1 H(\eta, s)u(s)\,ds + \frac{\alpha t}{1 - \alpha \beta \eta} \int_0^1 H(\xi, s)v(s)\,ds
\end{equation*}

\begin{equation*}
\rho = \int_0^1 \left( H(t, s) + \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi} H(\eta, s) \right)u(s)\,ds + \frac{\alpha t}{1 - \alpha \beta \eta} H(\xi, s)v(s)\,ds.
\end{equation*}

This proves (2.6). 

Employing Lemma 2.2, the system (1.5) can be expressed as

\begin{equation*}
x(t) = \int_0^1 F_{\xi}(t, s)f(s, x(s), y(s))\,ds + \int_0^1 G_{\alpha \beta \xi}(t, s)g(s, x(s), y(s))\,ds, \quad t \in [0, 1],
\end{equation*}

\begin{equation*}
y(t) = \int_0^1 F_{\eta}(t, s)g(s, x(s), y(s))\,ds + \int_0^1 G_{\beta \alpha \eta}(t, s)f(s, x(s), y(s))\,ds, \quad t \in [0, 1].
\end{equation*}

By a solution of the system (1.5), we mean a solution of the corresponding system of integral equations (2.7). Define an operator \( T : P \times P \to P \times P \) by

\[ T(x, y) = (A(x, y), B(x, y)), \]

where operators \( A, B : P \times P \to P \) are defined by

\begin{equation*}
A(x, y)(t) = \int_0^1 F_{\xi}(t, s)f(s, x(s), y(s))\,ds + \int_0^1 G_{\alpha \beta \xi}(t, s)g(s, x(s), y(s))\,ds, \quad t \in [0, 1],
\end{equation*}

\begin{equation*}
B(x, y)(t) = \int_0^1 F_{\eta}(t, s)g(s, x(s), y(s))\,ds + \int_0^1 G_{\beta \alpha \eta}(t, s)f(s, x(s), y(s))\,ds, \quad t \in [0, 1].
\end{equation*}

Clearly, if \((x, y) \in P \times P \) is a fixed point of \( T \), then \((x, y) \) is a solution of system (1.5).

**Lemma 2.9.** Under the hypotheses \( (H_1) - (H_3) \), the map \( T : \overline{P} \subseteq (P \times P) \to P \times P \) is completely continuous.

**Proof.** First we show that \( A(P \times P) \subseteq P \). For \((x, y) \in P \times P, t \in [0, 1] \), using (2.8) and Remark 2.7, we have

\begin{equation*}
A(x, y)(t) = \int_0^1 F_{\xi}(t, s)f(s, x(s), y(s))\,ds + \int_0^1 G_{\alpha \beta \xi}(t, s)g(s, x(s), y(s))\,ds \\
\leq \mu \int_0^1 s(1 - s)f(s, x(s), y(s))\,ds + \mu \int_0^1 s(1 - s)g(s, x(s), y(s))\,ds,
\end{equation*}

which implies that

\begin{equation*}
\|A(x, y)\| \leq \mu \int_0^1 s(1 - s)f(s, x(s), y(s))\,ds + \mu \int_0^1 s(1 - s)g(s, x(s), y(s))\,ds.
\end{equation*}

Also, for \((x, y) \in P \times P, t \in [\max\{\xi, \eta\}, 1] \), using (2.8), Remark 2.7 and (2.10), we obtain

\begin{equation*}
A(x, y)(t) = \int_0^1 F_{\xi}(t, s)f(s, x(s), y(s))\,ds + \int_0^1 G_{\alpha \beta \xi}(t, s)g(s, x(s), y(s))\,ds \\
\geq \nu \int_0^1 s(1 - s)f(s, x(s), y(s))\,ds + \nu \int_0^1 s(1 - s)g(s, x(s), y(s))\,ds.
\end{equation*}
\[ A(x, y) = \gamma \mu \int_0^1 s(1-s) f(s, x(s), y(s)) \, ds + \gamma \mu \int_0^1 s(1-s) g(s, x(s), y(s)) \, ds \]
\[ \geq \gamma \| A(x, y) \|. \tag{2.11} \]

Consequently, \( A(x, y) \in \mathcal{P} \). Similarly, we can show that \( B(P \times P) \subset \mathcal{P} \). Hence, \( T(P \times P) \subset (P \times P) \). Now, we show that the operator \( A : \mathcal{D} \cap (P \times P) \to \mathcal{P} \) is uniformly bounded. Choose a real constant \( c \in (0, 1) \) such that \( cr \leq 1 \). For \( (x, y) \in \mathcal{D} \cap (P \times P), t \in [0, 1] \), using (2.8), Remark 2.7 and \((\mathcal{H}_1)-(\mathcal{H}_3)\), we have

\[
A(x, y)(t) = \int_0^1 F(t, s) f(s, x(s), y(s)) \, ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s) g(s, x(s), y(s)) \, ds
\]
\[
\leq \mu \int_0^1 s(1-s) f(s, x(s), y(s)) \, ds + \mu \int_0^1 s(1-s) g(s, x(s), y(s)) \, ds
\]
\[
= \mu \int_0^1 s(1-s) \left( \frac{c x(s)}{c}, \frac{c y(s)}{c} \right) \, ds + \mu \int_0^1 s(1-s) g(s, c x(s), c y(s)) \, ds
\]
\[
\leq \mu c^{-\beta_1} \int_0^1 s(1-s) f(s, c x(s), c y(s)) \, ds + \mu c^{-\beta_1} \int_0^1 s(1-s) g(s, c x(s), c y(s)) \, ds
\]
\[
\leq \mu c^{\alpha_1-\beta_1-\beta_2} \int_0^1 s(1-s) (x(s))^{\alpha_1} f(s, 1, c y(s)) \, ds
\]
\[
+ \mu c^{\gamma_1-\beta_1-\beta_2} \int_0^1 s(1-s) (x(s))^{\gamma_1} g(s, 1, c y(s)) \, ds
\]
\[
\leq \mu c^{\alpha_2-\beta_1-\beta_2} \int_0^1 s(1-s) (x(s))^{\alpha_2} f(s, 1, 1) \, ds
\]
\[
+ \mu c^{\gamma_2-\beta_1-\beta_2} \int_0^1 s(1-s) (x(s))^{\gamma_2} g(s, 1, 1) \, ds
\]
\[
\leq \mu c^{\alpha_1+\alpha_2-\beta_1-\beta_2} \mu^{\alpha_1+\alpha_2} + \mu c^{\gamma_1+\gamma_2-\beta_1-\beta_2} \mu^{\gamma_1+\gamma_2},
\]

which implies that \( A(\mathcal{D} \cap (P \times P)) \) is uniformly bounded. Similarly, using (2.8), Remark 2.7 and \((\mathcal{H}_1)-(\mathcal{H}_3)\), we can show that \( B(\mathcal{D} \cap (P \times P)) \) is also uniformly bounded. Thus, \( T(\mathcal{D} \cap (P \times P)) \) is uniformly bounded. Now we show that \( A(\mathcal{D} \cap (P \times P)) \) is equicontinuous. For \( (x, y) \in \mathcal{D} \cap (P \times P), t \in [0, 1] \), using (2.8) and Lemma 2.8, we have

\[
A(x, y)(t) = \int_0^1 F(t, s) f(s, x(s), y(s)) \, ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s) g(s, x(s), y(s)) \, ds
\]
\[
= \int_0^1 H(t, s) f(s, x(s), y(s)) \, ds + \frac{\alpha \beta \xi \eta}{1 - \alpha \beta \xi \eta} \int_0^1 H(\eta, s) f(s, x(s), y(s)) \, ds
\]
\[
+ \frac{\alpha \eta}{1 - \alpha \beta \xi \eta} \int_0^1 H(\xi, s) g(s, x(s), y(s)) \, ds
\]
\[
\begin{aligned}
&= \int_0^t s(1-t)f(s, x(s), y(s)) \, ds + \int_0^t t(1-s)f(s, x(s), y(s)) \, ds \\
&+ \frac{\alpha \beta \xi t}{1 - \alpha \beta \xi \eta} \int_0^1 H(\eta, s)f(s, x(s), y(s)) \, ds + \frac{\alpha t}{1 - \alpha \beta \xi \eta} \int_0^1 H(\xi, s)g(s, x(s), y(s)) \, ds.
\end{aligned}
\]

Differentiating with respect to \( t \), we obtain
\[
A(x, y)'(t) = -\int_0^t s f(s, x(s), y(s)) \, ds + \int_0^t (1-s) f(s, x(s), y(s)) \, ds \\
+ \frac{\alpha \beta \xi}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)f(s, x(s), y(s)) \, ds + \frac{\alpha}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)g(s, x(s), y(s)) \, ds.
\]

which implies that
\[
|A(x, y)'(t)| \leq \int_0^t s f(s, x(s), y(s)) \, ds + \int_0^t (1-s) f(s, x(s), y(s)) \, ds \\
+ \frac{\alpha \beta \xi}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)f(s, x(s), y(s)) \, ds + \frac{\alpha}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)g(s, x(s), y(s)) \, ds.
\]

Now using (H1)–(H5), we have
\[
|A(x, y)'(t)| \leq \int_0^t s f(s, x(s), y(s)) \, ds + \int_0^t (1-s) f(s, x(s), y(s)) \, ds \\
+ \frac{\alpha \beta \xi}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)f(s, x(s), y(s)) \, ds + \frac{\alpha}{1 - \alpha \beta \xi \eta} \int_0^1 s(1-s)g(s, x(s), y(s)) \, ds \\
\leq c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} \left( \int_0^1 s f(s, 1, 1) \, ds + \int_0^1 (1-s) f(s, 1, 1) \, ds \right) \\
+ \frac{\alpha}{1 - \alpha \beta \xi \eta} \left( \beta \xi \alpha c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} + b \psi_1 + \psi_2 - \psi_1 - \psi_2 \right).
\]

Let
\[
h(t) = c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} \left( \int_0^1 s f(s, 1, 1) \, ds + \int_0^1 (1-s) f(s, 1, 1) \, ds \right) \\
+ \frac{\alpha}{1 - \alpha \beta \xi \eta} \left( \beta \xi \alpha c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} + b \psi_1 + \psi_2 - \psi_1 - \psi_2 \right).
\]

Then using (H1), we have
\[
\int_0^1 h(t) \, dt = c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} \left( \int_0^1 \int_0^1 s f(s, 1, 1) \, ds \, dt + \int_0^1 \int_0^1 (1-s) f(s, 1, 1) \, ds \, dt \right) \\
+ \frac{\alpha}{1 - \alpha \beta \xi \eta} \left( \beta \xi \alpha c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} + b \psi_1 + \psi_2 - \psi_1 - \psi_2 \right) \\
= c^{\alpha_1+\alpha_2-\beta_1-\beta_2,\xi_1+\alpha_2} \left( \int_0^1 s(1-s) f(s, 1, 1) \, ds + \int_0^1 s(1-s) f(s, 1, 1) \, ds \right)
\]
Then by using (2.8) and Remark 2.7, we have

$$
\frac{2+\alpha \beta \xi -2\alpha \beta \xi \eta}{1-\alpha \beta \xi \eta}c^{\alpha_1+\alpha_2} - \beta_1 - \beta_2 \gamma_2^{\alpha_1+\alpha_2} + b c^{\gamma_1+\gamma_2} - \rho_1 - \rho_2 \gamma_1^{\alpha_1+\alpha_2} + \alpha bc^{\gamma_1+\gamma_2} - \rho_1 - \rho_2 \gamma_1^{\alpha_1+\alpha_2}.
$$

(2.12)

Thus for any given $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$ and $(x, y) \in \overline{B}_r \cap (P \times P)$, we have

$$
|A(x, y)(t_1) - A(x, y)(t_2)| = \left| \int_{t_1}^{t_2} A(x, y)'(t) \, dt \right| \leq \int_{t_1}^{t_2} h(t) \, dt.
$$

(2.13)

which together with (2.12) yields that $A(\overline{B}_r \cap (P \times P))$ is equicontinuous on $[0, 1]$. Similarly, we can also show that $B(\overline{B}_r \cap (P \times P))$ is also equicontinuous. Thus, $T(\overline{B}_r \cap (P \times P))$ is equicontinuous. From this together with uniform boundedness of $T(\overline{B}_r \cap (P \times P))$ and the Arzelà–Ascoli theorem, it follows that $T(\overline{B}_r \cap (P \times P))$ is relatively compact. Hence, $T$ is a compact operator.

Now we show that $T$ is continuous. Let $(x_m, y_m), (x, y) \in \overline{B}_r \cap (P \times P)$ such that $\| (x_m, y_m) - (x, y) \|_1 \to 0$ as $m \to +\infty$. Then by using (2.8) and Remark 2.7, we have

$$
A(x_m, y_m)(t) - A(x, y)(t) = \left| \int_{0}^{1} F_{\xi \eta}(t, s) \left( f(s, x_m(s), y_m(s)) - f(s, x(s), y(s)) \right) \, ds 
+ \int_{0}^{1} G_{\xi \eta}(t, s) \left( g(s, x_m(s), y_m(s)) - g(s, x(s), y(s)) \right) \, ds \right|
\leq \int_{0}^{1} \left| f(s, x_m(s), y_m(s)) - f(s, x(s), y(s)) \right| \, ds
+ \int_{0}^{1} \left| g(s, x_m(s), y_m(s)) - g(s, x(s), y(s)) \right| \, ds
\leq \mu \int_{0}^{1} s(1-s) \left| f(s, x_m(s), y_m(s)) - f(s, x(s), y(s)) \right| \, ds
+ \mu \int_{0}^{1} s(1-s) \left| g(s, x_m(s), y_m(s)) - g(s, x(s), y(s)) \right| \, ds.
$$

Consequently,

$$
\| A(x_m, y_m) - A(x, y) \| \leq \mu \int_{0}^{1} s(1-s) \left| f(s, x_m(s), y_m(s)) - f(s, x(s), y(s)) \right| \, ds
+ \mu \int_{0}^{1} s(1-s) \left| g(s, x_m(s), y_m(s)) - g(s, x(s), y(s)) \right| \, ds.
$$

By the Lebesgue dominated convergence theorem, it follows that

$$
\| A(x_m, y_m) - A(x, y) \| \to 0 \quad \text{as } m \to +\infty.
$$

(2.14)

Similarly, by using (2.8) and Remark 2.7, we have

$$
\| B(x_m, y_m) - B(x, y) \| \to 0 \quad \text{as } m \to +\infty.
$$

(2.15)

From (2.14) and (2.15), it follows that

$$
\| T(x_m, y_m) - T(x, y) \|_1 \to 0 \quad \text{as } m \to +\infty,
$$

that is, $T : \overline{B}_r \cap (P \times P) \to P \times P$ is continuous. Hence, $T : \overline{B}_r \cap (P \times P) \to P \times P$ is completely continuous. □
3. Main result: Existence of at least one positive solution

**Theorem 3.1.** Under the hypotheses \((\mathcal{H}_1)\)–\((\mathcal{H}_3)\), the system \((1.5)\) has at least one positive solution.

**Proof.** Choose a constant \(R > 0\) such that

\[
R \geq \max \left\{ 1, (4a\mu)^{1-\beta_1}, (4b\mu)^{1-\beta_2} \right\}.
\]

Let \(cR = 1\), for some real constant \(c\). Then, for any \((x, y) \in \partial \Omega_R \cap (P \times P)\), \(t \in [0, 1]\), using \((2.8)\), Remark 2.7, \((\mathcal{H}_1)\)–\((\mathcal{H}_3)\), we have

\[
A(x, y)(t) = \int_0^1 F_{\xi\eta}(t, s)f(s, x(s), y(s))\, ds + \int_0^1 G_{a\beta\eta}(t, s)\, g(s, x(s), y(s))\, ds \\
\leq \mu \int_0^1 s(1-s)f(s, x(s), y(s))\, ds + \mu \int_0^1 s(1-s)g(s, x(s), y(s))\, ds \\
= \mu \int_0^1 s(1-s)f(s, \frac{cx(s)}{c}, \frac{cy(s)}{c})\, ds + \mu \int_0^1 s(1-s)g(s, \frac{cx(s)}{c}, \frac{cy(s)}{c})\, ds \\
\leq \mu c\alpha_1 - \beta_1 R^\alpha_1 \int_0^1 s(1-s)f(s, 1, 1)\, ds + \mu c\alpha_1 - \beta_2 R^\beta_2 \int_0^1 s(1-s)g(s, 1, 1)\, ds \\
= a\mu R^{\beta_1 + \beta_2} + b\mu R^{\alpha_1 + \alpha_2}.
\]

Thus, in view of \((3.1)\), we have

\[
\|A(x, y)\| \leq \frac{\|x, y\|_1}{2}, \quad \text{for all } (x, y) \in \partial \Omega_R \cap (P \times P).
\]

Similarly, using \((2.8)\), Remark 2.7, \((\mathcal{H}_4)\)–\((\mathcal{H}_6)\), we have

\[
\|B(x, y)\| \leq \frac{\|x, y\|_1}{2}, \quad \text{for all } (x, y) \in \partial \Omega_R \cap (P \times P).
\]

From \((3.2)\) and \((3.3)\), it follows that

\[
\|T(x, y)\|_1 \leq \frac{\|x, y\|_1}{2}, \quad \text{for all } (x, y) \in \partial \Omega_R \cap (P \times P).
\]

Choose a real constant \(r \in (0, R)\) such that

\[
r \leq \min \left\{ 1, \left( 4\nu \gamma_1^{\rho_1 + \rho_2} \int_{\max\{\xi, \eta\}}^1 s(1-s)f(s, 1, 1)\, ds \right)^{1-\beta_1 - \beta_2}, \left( 4\nu \gamma_1^{\rho_1 + \rho_2} \int_{\max\{\xi, \eta\}}^1 s(1-s)g(s, 1, 1)\, ds \right)^{1-\beta_1 - \beta_2} \right\}.
\]

Then, for any \((x, y) \in \partial \Omega_r \cap (P \times P), t \in [0, 1]\), using \((2.8)\), Remark 2.7, \((\mathcal{H}_2)\)–\((\mathcal{H}_3)\), we have

\[
A(x, y)(t) = \int_0^1 F_{\xi\eta}(t, s)f(s, x(s), y(s))\, ds + \int_0^1 G_{a\beta\eta}(t, s)\, g(s, x(s), y(s))\, ds \\
\geq \nu \int_0^1 s(1-s)f(s, x(s), y(s))\, ds + \nu \int_0^1 s(1-s)g(s, x(s), y(s))\, ds \\
\geq \nu \int_0^1 s(1-s)(x(s))^\beta_1 f(s, 1, y(s))\, ds + \nu \int_0^1 s(1-s)g(s, x(s), y(s))\, ds.
\]
\[ \frac{1}{v} \int_{0}^{1} s(1-s)(x(s))^\beta_1 (y(s))^\beta_2 f(s,1,1) \, ds + \frac{1}{v} \int_{0}^{1} s(1-s)g(s,x(s),y(s)) \, ds \]
\[ \geq \frac{1}{v} \int_{0}^{1} s(1-s)(x(s))^\beta_1 (y(s))^\beta_2 f(s,1,1) \, ds + \frac{1}{v} \int_{0}^{1} s(1-s)g(s,1,y(s)) \, ds \]
\[ \geq \frac{1}{v} \int_{\max\{\ell,\eta\}}^{1} s(1-s)(x(s))^\beta_1 (y(s))^\beta_2 f(s,1,1) \, ds + \frac{1}{v} \int_{\max\{\ell,\eta\}}^{1} s(1-s)g(s,1,1) \, ds \]
\[ + v \int_{\max\{\ell,\eta\}}^{1} s(1-s)(x(s))^\beta_1 (y(s))^\beta_2 g(s,1,1) \, ds \]
\[ \geq v^{\beta_1 + \beta_2} s(1-s) f(s,1,1) \, ds + v^{\beta_1 + \beta_2} s(1-s) g(s,1,1) \, ds \]

Thus in view of (3.5), we have
\[ \|A(x,y)\| \geq \frac{\|(x,y)\|}{2}, \quad \text{for all } (x,y) \in \partial\Omega_r \cap (P \times P). \quad (3.6) \]
Similarly, using (2.8), Remark 2.7, (H2)--(H3), in view of (3.5), we have
\[ \|B(x,y)\| \geq \frac{\|(x,y)\|}{2}, \quad \text{for all } (x,y) \in \partial\Omega_r \cap (P \times P). \quad (3.7) \]
From (3.6) and (3.7), it follows that
\[ \|T(x,y)\| \geq \|\partial(x,y)\|, \quad \text{for all } (x,y) \in \partial\Omega_r \cap (P \times P). \quad (3.8) \]

Hence in view of (3.4) and (3.8), by Lemma 2.1, T has a fixed point \((x,y) \in (\overline{\Omega}_R \setminus \Omega_r) \cap (P \times P)\). That is, \(x = A(x,y)\) and \(y = B(x,y)\). Moreover, \((x,y)\) is positive. In fact, by concavity of \(x\) and by construction of the cone \(P\), we have
\[ x(1) \geq \min_{t \in [\max\{\ell,\eta\},1]} x(t) \geq y \|x\| > 0, \]
which implies that \(x(t) > 0\) for all \(t \in (0,1]\). Similarly, \(y(t) > 0\) for all \(t \in (0,1]\). Hence, \((x,y)\) is a positive solution of (1.5). \(\square\)

**Example 3.2.** Let
\[ f(t,x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} t^{p_i}(1-t)^{q_j} x^i y^j, \]
\[ g(t,x,y) = \sum_{k=1}^{m'} \sum_{l=1}^{n'} t^{p_i}(1-t)^{q_j} x^i y^j, \]
where the real constants \(p_i, q_j, r_i, s_j\) satisfy \(p_i, q_j > -2, 0 \leq r_i, s_j < 1, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n,\) with \(\max_{1 \leq i \leq m} r_i + \max_{1 \leq j \leq n} s_j < 1,\) and the real constants \(p_i', q_j', r_i', s_j'\) satisfy \(p_i', q_j' > -2, 0 \leq r_i', s_j' < 1, k = 1, 2, \ldots, m'; l = 1, 2, \ldots, n',\) with \(\max_{1 \leq k \leq m'} r_i + \max_{1 \leq l \leq n'} s_j' < 1.\) Clearly, \(f\) and \(g\) satisfy hypotheses (H1)--(H2). Hence, by Theorem 3.1, the system of BVPs (1.5) has a positive solution.
References


