Spline approximation for Cauchy principal value integrals *

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Abstract: In this paper we prove the uniform convergence of some quadrature formulas based on spline approximation for Cauchy principal value integrals of the type \( \int_\alpha^\beta w(x) f^{(k)}(x)/(x - y) \, dx \) (\( k = 0, 1, \ldots \)) and we present some numerical applications. In particular we apply our rules to the well-known Prandtl’s integral equation.

Keywords: Cauchy principal value integrals, quadrature rules, splines.

1. Introduction

The numerical evaluation of Cauchy principal value integrals of the form

\[
I(f^{(k)}; y) = \int_a^b w(x) \frac{d^k}{dx^k} f(x) \frac{1}{x-y} \, dx, \quad k = 0, 1, \ldots,
\]

is a problem of interest, especially in connection with the numerical solution of singular integral equations.

Note that the integrals of the form

\[
\frac{d^k}{dy^k} \int_a^b w^*(x) f(x) \frac{1}{x-y} \, dx, \quad a < y < b, \quad k = 0, 1, \ldots,
\]

where the weight function \( w^*(x) \) is such that all its derivatives of order \( i = 0, 1, \ldots, k - 1 \) vanish at the endpoints, can be easily reduced to a linear combination of integrals of type (1) since, in this case, integration by parts leads to the identity

\[
\frac{d^k}{dy^k} \int_a^b w^*(x) f(x) \frac{1}{x-y} \, dx = \int_a^b \frac{d^k}{dx^k} \left[ w^*(x) f(x) \right] \frac{1}{x-y} \, dx.
\]

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Interpolatory quadrature rules for integrals of type (2) based on polynomial approximation of \( f \) have been studied by many authors and uniform convergence theorems have been established for particular choices of the weight function \( w^*(x) \) and of the nodes (see, for example, [4-6,20,22]); in particular, in [6], the uniform and mean convergence results in the whole open interval \((-1, 1) - \{\text{singularities of } w^*\} \) are given in the case \( k = 0, w^*(x) \) generalized smooth Jacobi weight function and nodes coinciding with the zeros of the \( n \)-th degree orthogonal polynomial on \((-1, 1)\) with respect to \( w^*(x) \).

Piecewise polynomial and spline functions have also been used to approximate integrals of type (2) in the case \( k = 0 \) and to solve numerically singular integral equations (see [11-14,17,19,25,27]); however, no uniform convergence results have been proved for the quadrature formulas based on spline approximation.

In this paper, we examine quadrature rules for integrals of type (1) of the form

\[
I(f^{(k)}; y) = I_n(f^{(k)}; y) = \sum_{i=0}^{n} w^{(k)}_i(y) f(x_i), \quad k = 0, 1, \ldots, (3)
\]

where the nodes \( \{x_i\}_{i=0}^{n} \) are fixed on a mesh defined by

\[
a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, (4)
\]

with \( h = \max_i (x_{i+1} - x_i) = \max_i h_i \to 0 \) as \( n \to \infty \) and the weights \( \{w^{(k)}_i\}_{i=0}^{n} \) are chosen so that (3) is exact whenever \( f \) is a spline of a given degree \( m \) on the mesh (4) satisfying prefixed end conditions, provided \( m \) is odd and \( m \geq k + 1 \).

We give uniform convergence results for special choices of nodes \( \{x_i\} \) and certain rather general end conditions. We also test our rules on several examples and compare their performance with that of corresponding product rules of interpolatory type based on zeros of classical orthogonal polynomials.

2. Convergence

Let us consider a quadrature formula of the type

\[
I_n(f^{(k)}; y) = \int_a^b w(x) S^{(k)}(x) \frac{f(x)}{x-y} \, dx, (5)
\]

where \( S \) is the unique spline of a given odd degree \( m \geq k + 1 \) interpolating the function \( f(x) \) at the data points \( \{x_i\}_{i=0}^{n} \) and satisfying certain end conditions corresponding to one of the following choices:

(i) Quasi uniform mesh, i.e. mesh (4) is such that the quantity \( \max_i (h/h_i) \) is bounded as \( n \to \infty \), and the end conditions are of one of the following 4 types:

\begin{align*}
(i_1) \quad f^{(k)}(x_j) &= S^{(k)}(x_j), & j = 0, n, \quad k = 1, \ldots, \frac{1}{2}(m-1), \\
(i_2) \quad f^{(k)}(x_j) &= S^{(k)}(x_j), & j = 0, n, \quad k = \frac{1}{2}(m+1), \ldots, m-1, \\
(i_3) \quad S^{(k)}(x^n_+) &= S^{(k)}(x^n_-), & k = 1, 2, \ldots, m-1, \\
(i_4) \quad f^{(2k)}(x_j) &= S^{(2k)}(x_j), & j = 0, n, \quad k = 1, \ldots, \frac{1}{2}(m-1).
\end{align*}
A. Palamara Orsi / Cauchy principal value

(ii) Uniform mesh, i.e. mesh (4) with \( h_i = h \), and

\[(ii_1) \quad S^{(k)}(x_0^+) = S^{(k)}(x_n^-), \quad k = 1, 2, \ldots, m - 1,\]

or

\[(ii_2) \quad \text{"not a knot" end conditions} \quad [7, \text{p.55 and p.211}].\]

**Lemma 1.** Let \( S(x) \) be the unique interpolating spline function of odd degree \( m \) satisfying the “not a knot” end conditions and with equally spaced nodes \( \{ x_i \}_{i=0}^n \) (case \( (ii_2) \)). Then

\[
\| f^{(k)}(x) - S^{(k)}(x) \|_{\infty} \leq K_1 h^{m+1-k}, \quad k = 0, 1, \ldots, m,
\]

where \( K_1 \) is a constant independent of \( h \), for all \( f \in C^m[a, b] \) such that \( f^{(m+1)} \) is uniformly bounded on \([a, b]\).

**Proof.** The thesis (6) follows immediately from [10, Th. 3].

In fact, we have:

\[
\| f^{(k)} - S^{(k)} \|_{\infty} \leq C_1 \omega_{m+1-k} \left( f^{(k)}; \frac{b-a}{n} \right) - C_1 \sup_{0 < \delta \leq (b-a)/n, x, x+(m+1-k)\delta \in [a, b]} | \Delta_{\delta}^{m+1-k} f^{(k)}(x) |
\]

\[
= C_1 \sup_{\delta} \left( \delta \right)^{m+1-k} f^{(m+1)}(\xi) \leq \left( \frac{b-a}{n} \right)^{m+1-k} \| f^{(m+1)} \|_{\infty}
\]

\[
\leq K_1 h^{m+1-k}, \quad k = 0, 1, \ldots, m,
\]

where \( \omega_{m+1-k}(f^{(k)}; (b-a)/n) \) is the modulus of smoothness of \( f^{(k)} \) of order \( m+1-k \), \( \Delta_{\delta}^{m+1-k} f^{(k)}(x) \) is the progressive difference of order \( m+1-k \) with increment \( \delta > 0 \) of the function \( f^{(k)} \) and \( \xi \in (x, x + (m+1-k)\delta) \).

Lemma 1 also holds for the spline \( S(x) \) of type \((i_4)\) (see [16]).

Analogous uniform convergence properties hold for the splines \( S(x) \) defined above (see [1]).

**Lemma 2.** Let \( S(x) \) be the unique interpolating spline function of odd degree \( m \) of the type \((ii_2)\) or \((i_4)\). Let \( S^{(k)}(x) = f^{(k)}(x) - S^{(k)}(x) \), where \( f(x) \in C^m[a, b] \) and \( f^{(m+1)} \) is bounded on \([a, b]\). Then, for any \( 0 < \nu < 1 \) and \( y_1 \neq y_2 \), we have

\[
\sup_{y_1, y_2 \in [a, b]} \frac{| r^{(k)}_n(y_2) - r^{(k)}_n(y_1) |}{| y_2 - y_1 |^\nu} \leq K_2 h^{m+1-k-\gamma},
\]

\( \gamma = \nu(m+1-k) \) \( k = 0, 1, \ldots, m-1 \) where \( K_2 \) is a constant independent of \( h \).

**Proof.** First we remark that, for any \( y_1, y_2 \in [a, b] \):

\[
| r^{(k)}_n(y_2) - r^{(k)}_n(y_1) | \leq | f^{(k)}(y_2) - f^{(k)}(y_1) | + | S^{(k)}(y_2) - S^{(k)}(y_1) |
\]

\[
\leq | y_2 - y_1 | \left( \left| f^{(k+1)}(\xi^*) \right| + \left| S^{(k+1)}(\xi^{**}) \right| \right)
\]

\[
\leq C_2 | y_2 - y_1 |, \quad k = 0, 1, \ldots, m - 1,
\]

where \( C_2 \) is a constant independent of \( h \); in fact \( S^{(k+1)} \) is bounded because it converges uniformly to \( f^{(k+1)} \) which is bounded by assumption.
Therefore we may obtain (7) by following the proof in [21, Lemma 5] and taking into account Lemma 1.

Now we are in position to establish the following theorem.

**Theorem 3.** Let us consider the quadrature rules (5) based on the splines $S(x)$ defined as in Lemma 2. Then

$$\lim_{n \to \infty} I_n(f^{(k)}; y) = \int_a^b \frac{w(x)f^{(k)}(x)}{x-y} \, dx, \quad k = 0, 1, \ldots, m-1,$$

for all $f \in C^m[a, b]$ with $f^{(m+1)}$ bounded on $[a, b]$. If $w(x) \in L^1[a, b] \cap H^1[a, b]$ ($0 < \mu \leq 1$) (e.g., $[a, b] = [-1, 1], w(x) = (1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1$), then the quantities

$$\text{lim}_{n \to \infty} \left| \int_a^b \frac{w(x)}{x-y} \, dx \right| \quad \text{and} \quad \int_a^b \frac{|w(x)|}{|x-y|^{1-r}} \, dx, \quad 0 < r < 1, \quad (8)$$

exist and are bounded for all $y \in (a, b)$ and the convergence is uniform in any closed subinterval of $(a, b)$. If, in particular, the quantities (8) exist and are bounded for all $y \in [a, b]$ (e.g., $[a, b] = [-1, 1], w(x) = (1-x)^\alpha(1+x)^\beta, \alpha, \beta > 0$), then the convergence is uniform in the whole closed interval $[a, b]$.

**Proof.** Let us consider the remainder term

$$I(f^{(k)}; y) - I_n(f^{(k)}; y) = \int_a^b \frac{w(x)r_n^{(k)}(x)}{x-y} \, dx$$

and denote by $I$ the closed interval where the quantities (8) exist and are bounded. To prove the theorem, recalling (6) and (7) we write:

$$\| I(f^{(k)}; y) - I_n(f^{(k)}; y) \|_\infty \leq \max_{y \in I} \left| r_n^{(k)}(y) \right| \left| \int_a^b \frac{w(x)}{x-y} \, dx \right| + \int_a^b \frac{|w(x)|}{|x-y|^{1-r}} \left| \frac{r_n^{(k)}(x) - r_n^{(k)}(y)}{|x-y|^{r}} \right| \, dx$$

$$\leq Ch^{m+1-k-r}, \quad k = 0, 1, \ldots, m-1,$$

where $C$ is a constant independent of $h$. □

Analogous theorems hold for the other types of splines mentioned above.

3. Numerical applications

In this section we present some numerical results we have obtained from our quadrature formulas (5) based on cubic spline approximation.
In particular, we use the following quadrature formulas:

\[ I_n(f^{(k)}; y) = \oint_a^b w(x) \frac{q^{(k)}(x)}{x-y} \, dx = \sum_{i=0}^{n} w_i^{(k)}(y)f(x_i), \quad k = 0, 1, \tag{9} \]

where \( \phi(x) \) is the unique cubic spline interpolating \( f \) at the equally spaced nodes \( \{ x_i = a + i(b - a)/n \}_{i=0}^{n} \) and satisfying the "not a knot" end conditions (case (ii_2) of Section 2).

The basis functions to represent \( \phi(x) \) are chosen to be the cubic B-splines \( \{ B_i^{(4)} \}_{j=2}^{n+2} \) on the same mesh and with the same end conditions. The use of bases consisting of B-splines appears computationally advantageous, because of their well-known properties [2,3,7,8].

In this case, the weights satisfy the well-conditioned linear system of equations

\[ \sum_{i=0}^{n} w_i^{(k)}(y) B_j^{(4)}(x_i) = I(B_j^{(4)(k)}; y), \quad j = 2, \ldots, n + 2. \tag{10} \]

Since the coefficient matrix is invertible, banded and totally positive, the system (10) may be stably solved by Gauss elimination without pivoting, taking into account the banded nature of the matrix [9].

Note that the evaluation of the integrals \( I(B_j^{(4)(k)}; y) \) reduces, setting \( t_1 = x_0, \{ t_i = x_i \}_{i=2}^{n-1} \), \( t_{n-1} = x_n \), to the evaluation of the following integrals:

\[ q_0(y) = \oint_a^b \frac{w(x)}{x-y} \, dx, \tag{11} \]

\[ \begin{align*}
I_i(r) &= \int_{t_{i-1}}^{t_i} x w(x) \, dx, \quad r = 0, 1, 2, \\
L_i(y) &= \int_{t_{i-1}}^{t_i} \frac{w(x)}{x-y} \, dx, \quad y \in [t_{i-1}, t_i],
\end{align*} \tag{12} \]

which have a closed form expression only for particular choices of \( w(x) \); e.g., \( w(x) = 1, w(x) = (1 - x)^{r+1/2} \) [12,13,15,17]. However, when \( w(x) = (1 - x)^\alpha(1 + x)^\beta (\alpha, \beta > -1) \), Gerasoulis [14] presents a numerically stable and efficient method for the computation of the integrals (12), with \( r = 0 \).

3.1. Example 1

The quadrature formula (9) with \( k = 0 \), \( w(k) = 1 \), is used to approximate the following integrals:

\[ I_1(y) = \int_{-1}^{1} \frac{e^x}{x-y} \, dx, \quad -1 < y < 1, \]

\[ I_2(y) = \int_{-1}^{1} \frac{1-x^2}{x-y} \, dx, \quad -1 < y < 1, \]

\[ I_3(y) = \int_{0}^{1} \frac{|x-0.65|^{1.3}}{x-y} \, dx, \quad 0 < y < 1. \]

A comparison is made with a product quadrature rule based on Legendre polynomials (see [20, algorithm 4.3]).
Table 1
Relative errors in the N-point formula S based on splines and in the N-point product formula P for the integral
\[ I_1(y) = \int_1^\infty \frac{e^y}{x-y} \, dx \]

<table>
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<tr>
<th>( N )</th>
<th>( y = 0.1 )</th>
<th>( y = 0.5 )</th>
<th>( y = 0.9 )</th>
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<td>2.3 \times 10^{-2}</td>
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<tr>
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</tr>
<tr>
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<td>6.0 \times 10^{-14}</td>
<td>5.7 \times 10^{-8}</td>
</tr>
<tr>
<td>64</td>
<td>9.5 \times 10^{-10}</td>
<td>4.3 \times 10^{-14}</td>
<td>6.0 \times 10^{-9}</td>
</tr>
</tbody>
</table>

Table 2
Relative errors in the N-point formula S based on splines and in the N-point product formula P for the integral
\[ I_2(y) = \int_1^\infty \frac{\sqrt{1-x^2}}{x-y} \, dx \]

<table>
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<tr>
<th>( N )</th>
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</tbody>
</table>

Table 3
Relative errors in the N-point formula S based on splines and in the N-point product formula P for the integral
\[ I_3(y) = \int_1^\infty \frac{1.1.\sqrt{x-0.65}}{x-y} \, dx \]

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</table>

Tables 1–3 show the relative errors evaluated by applying the rule (9) (denoted by S) and the product rule based on Legendre polynomials (denoted by P), respectively, with the same total number \( N \) of nodes.

In the first case (see Table 1), because of the high degree of smoothness of the function \( f \) (entire function), the product rule based on polynomial approximations converges faster than the rule based on splines.
In the second case (see Table 2), the function $f$ is regular in $(-1, 1)$ but has singularities at $x = \pm 1$ in the first derivative; also in this case the polynomial rule performs better than the spline rule; this is not surprising since in the case of the polynomial rule the mesh is nonuniform and, as $n \to \infty$, the nodes concentrate faster towards the endpoints than in the middle of the integration interval (see [28]). We have also applied to integral $I_2(y)$ a quadrature formula of type (5) where $S$ is the interpolating cubic spline satisfying the "not a knot" end conditions associated with the following nonuniform mesh partitioning the interval $[-1, 1]$

\[ \left\{ \tau_i = -1 + \left( \frac{2i}{n} \right)^q \right\}_{i=0}^{n/2} \cup \left\{ \tau_i = -\tau_{n-i} \right\}_{i=n/2} \text{, } n \text{ even}, \quad (13) \]

with $q = 8$ (see, for instance, [7,26]).

However, in this case, the quadrature formula does not seem to be convergent and, moreover, the behaviour of the error is very similar to that of the corresponding interpolation process (for this latter, see [7, pp.180–196]).

In the third case (see Table 3), the function $f$ has a jump discontinuity in the first derivative inside the integration interval and the rate of convergence of the two formulas appears comparable.

### 3.2. Example 2

We now apply the quadrature formula (9), with $k = 0$, $\omega(x) = \sqrt{1 - x^2}$, to the solution of the following integral equation

\[ \sin(\frac{1}{2}\pi y) \omega(y) g(y) + \frac{1}{\pi} \int_{-1}^{1} \frac{\cos(\frac{1}{2}\pi x) \omega(x) g(x)}{x - y} \, dx = \sqrt{1 - y^2}, \quad (14) \]

\[ \omega(x) = e(1 - x)^{(x - 1)/2}(1 + x)^{-(x + 1)/2}, \quad (15) \]

written in the form

\[ g(y) = \sqrt{1 - y^2} \sin(\frac{1}{2}\pi y) - \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - x^2} F(x)}{x - y} \, dx + \frac{1}{\pi}, \quad (16) \]

with $F(x) = (1 - x)^{(1-x)/2}(1 + x)^{(1+x)/2} \cos(\frac{1}{2}\pi x)$ (see [14]).

In [14] Gerasoulis applies a piecewise polynomial quadrature to (16) and compares his results with those of Welstead [29, p.103] obtained by solving the equation (14) with the use of orthogonal polynomials associated with the weight function $\omega(x)$ in (15). He remarks that the use of orthogonal polynomials for nonclassical weight functions requires, as $n$ increases, a higher computational effort than the method he applies.

Nevertheless we have solved equation (16) using the quadrature rule (9) based on splines and the one based on the Chebyshev polynomials of second kind [20, algorithm 4.3].

In particular, in Table 4 we have reported the absolute errors, with respect to the value $g(1) = 0.518592$, obtained by using the following approximant functions: nonclassical orthogonal polynomials (the corresponding result is taken from [14, p.901] and was derived by Welstead [29]), piecewise-linear polynomials (the corresponding results are taken from [14, Table 2, p.901]), Chebyshev polynomials of second kind and splines.
We can see that, in this case, the method based on Chebyshev polynomial approximation (2229 operations and a timing of 0.77 seconds to compute the Cauchy principal value with an absolute error of order of magnitude $10^{-6}$ ($n = 20$ is needed) performs better than the spline rule on uniform mesh (about 13086 operations and a timing of 2.09 seconds to compute the Cauchy principal value with the same absolute error ($n = 80$ is needed)). This is in accordance with the results of Example 1, since $F(x) \in C^1[-1,1]$ and its second derivative has integrable singularities at $x = \pm 1$.

3.3. Example 3

The last application of the formulas (9) is to the well-known Prandtl’s integral equation

$$
U(y) = \frac{1}{2\pi} \int_{-1}^{1} \frac{U'(x)}{x-y} \, dx = g(y), \quad -1 \leq y \leq 1,
$$

where $D(y)$ and $g(y)$ are given functions assumed to satisfy the Hölder’s condition on $[-1,1]$, with $D(y)$ nowhere zero, and we will solve the equation (17) by the following collocation method.

It is known that the solution is of the form

$$
U(x) = \sqrt{1-x^2} u(x),
$$

where $u(x) \in C^1[-1,1]$, $u'(x) \in H^\beta[-1,1] (0 < \beta \leq 1)$ (see [18,23,24]).

We approximate the function $u(x)$ by the cubic spline $u_n(x)$ on the uniform mesh $\{x_i\}_{i=0}^n$, satisfying the “not a knot” end conditions, interpolating $u(x)$ at the nodes $\{x_i\}_{i=0}^n$ and furthermore we assume

$$
u_n(x) = \sum_{j=2}^{n+2} \gamma_j B_j^{(4)}(x).
$$

(Notice that the value of $u_n(x)$ at a point depends only on four of the coefficients $\gamma_j$).

We obtain the equation

$$
\frac{\sqrt{1-y^2}}{D(y)} u_n(y) - \frac{1}{2\pi} \sum_{i=0}^{n} w_i(y) u_n(x_i) = g(y),
$$

Table 4

<table>
<thead>
<tr>
<th>n</th>
<th>Nonclassical orthogonal polynomials</th>
<th>Chebyshev polynomials of second kind</th>
<th>Piecewise-linear polynomials</th>
<th>Splines</th>
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<td>Uniform mesh</td>
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<td>1.2 $\cdot$ 10$^{-3}$ *</td>
<td>2.6 $\cdot$ 10$^{-3}$ *</td>
</tr>
<tr>
<td>40</td>
<td>9.0 $\cdot$ 10$^{-6}$ *</td>
<td>2.7 $\cdot$ 10$^{-7}$</td>
<td>3.4 $\cdot$ 10$^{-4}$ *</td>
<td>6.6 $\cdot$ 10$^{-4}$ *</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>3.8 $\cdot$ 10$^{-7}$</td>
<td>9.0 $\cdot$ 10$^{-5}$ *</td>
<td>1.7 $\cdot$ 10$^{-4}$ *</td>
</tr>
</tbody>
</table>

* Results taken from [14, p.901].
Table 5

Solution of Prandtl's equation \( U(y)/(2 - y)- (2\pi)^{-1} \int_{x}^{1} \frac{\partial U'(x)}{\partial y} (x - y) \, dx = 1 \) by spline collocation method C and by Multhopp's method M

<table>
<thead>
<tr>
<th>N</th>
<th>x = 0</th>
<th>x = 0.2</th>
<th>x = 0.4</th>
<th>x = 0.6</th>
<th>x = 0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>M</td>
<td>C</td>
<td>M</td>
<td>C</td>
</tr>
<tr>
<td>4</td>
<td>0.7008405</td>
<td>0.6966501</td>
<td>0.6910945</td>
<td>0.6887644</td>
<td>0.6709246</td>
</tr>
<tr>
<td>8</td>
<td>0.6974269</td>
<td>0.6970238</td>
<td>0.6893327</td>
<td>0.6647683</td>
<td>0.6634905</td>
</tr>
<tr>
<td>16</td>
<td>0.6971507</td>
<td>0.6970617</td>
<td>0.6893823</td>
<td>0.6641279</td>
<td>0.6640099</td>
</tr>
<tr>
<td>32</td>
<td>0.6970837</td>
<td>0.6970618</td>
<td>0.6893826</td>
<td>0.6640391</td>
<td>0.6640099</td>
</tr>
<tr>
<td>64</td>
<td>0.6970837</td>
<td>0.6970618</td>
<td>0.6893826</td>
<td>0.6640391</td>
<td>0.6640099</td>
</tr>
</tbody>
</table>

Table 6

Solution of Prandtl's equation \( U(y)/(2 - y)- (2\pi)^{-1} \int_{x}^{1} \frac{\partial U'(x)}{\partial y} (x - y) \, dx = 1 \) by spline collocation method C and by Multhopp's method M

<table>
<thead>
<tr>
<th>N</th>
<th>x = 0</th>
<th>x = 0.2</th>
<th>x = 0.4</th>
<th>x = 0.6</th>
<th>x = 0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>M</td>
<td>C</td>
<td>M</td>
<td>C</td>
</tr>
<tr>
<td>4</td>
<td>0.2473866</td>
<td>0.2305316</td>
<td>0.2689822</td>
<td>0.2518747</td>
<td>0.3465638</td>
</tr>
<tr>
<td>8</td>
<td>0.2654942</td>
<td>0.2558533</td>
<td>0.2996778</td>
<td>0.2878116</td>
<td>0.3599161</td>
</tr>
<tr>
<td>16</td>
<td>0.2730465</td>
<td>0.2687130</td>
<td>0.3065784</td>
<td>0.3049650</td>
<td>0.3648823</td>
</tr>
<tr>
<td>32</td>
<td>0.2757049</td>
<td>0.2760909</td>
<td>0.3076171</td>
<td>0.3076967</td>
<td>0.3611342</td>
</tr>
<tr>
<td>64</td>
<td>0.2765696</td>
<td>0.2765383</td>
<td>0.3079097</td>
<td>0.3079097</td>
<td>0.3612020</td>
</tr>
</tbody>
</table>
where, for a fixed \( y \in [-1, 1] \), the weights \( \{ w_i \}_{i=0}^{n} \) satisfy the linear system

\[
\sum_{i=0}^{n} w_i(y) B_j^{(4)}(x_i) = \int_{-1}^{1} \frac{-x}{\sqrt{1-x^2}} B_j^{(4)}(x) \, dx + \int_{-1}^{1} \frac{\sqrt{1-x^2} B_j^{(4)}(x)}{x-y} \, dx.
\]

In order to determine the coefficients \( \{ \gamma_j \}_{j=0}^{n} \), we collocate the equation (20) at the nodes \( \{ x_k \}_{k=0}^{n} \).

The resulting linear system to be solved is:

\[
\sum_{j=2}^{n+2} \left( \frac{\sqrt{1-x_k^2}}{D(x_k)} B_j^{(4)}(x_k) - \frac{1}{2\pi} \sum_{i=0}^{n} B_j^{(4)}(x_i) w_i(x_k) \right) \gamma_j = g(x_k), \quad k = 0, \ldots, n.
\]  

We remark that if \( D(x) = D(-x) \) and \( g(x) = g(-x) \), then \( U(x) = U(-x) \).

We have considered (17) for \( D(y) = g(y) = 1 \) and for \( D(y) = 2 - y, \ g(y) = |y| \).

Tables 5 and 6 show the behaviour of the solution of the integral equation obtained by the collocation method under consideration (denoted by C) and by the Multhopp's method (denoted by M) [18].

The convergence of the Multhopp's method is faster for well-behaved function \( u(x) \) (see Table 5), whereas, when the function \( u(x) \) has a lower degree of smoothness, the rate of convergence of the two methods is basically the same (see Table 6), according to what Tables 1 and 3 show.

All the numerical results presented were carried out on a IBM PS/2 computer working with about 16-digit double precision arithmetic.

Acknowledgement

I'd like to thank G. Monegato for many helpful and valuable suggestions.

References


