Linkage of modules

Alex Martsinkovsky a,∗ and Jan R. Strooker b

a Mathematics Department, Northeastern University, Boston, MA 02115, USA
b Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands

Received 18 June 2001
Communicated by Kent R. Fuller

Abstract

It is shown that the notion of linkage of algebraic varieties, introduced by Peskine and Szpiro, can be generalized to finitely generated modules over non-commutative noetherian semiperfect rings. Besides greater generality, the module-theoretic approach brings about new invariants of linkage and yields improved results and simpler proofs even in the traditional settings of commutative algebraic geometry and local algebra. Connections with Auslander–Reiten sequences, singularity theory, derived categories, local cohomology, Buchsbaum modules, and maximal Cohen–Macaulay approximations are discussed.

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Keywords: Linkage; Derived category; Local cohomology; Almost split sequence; Maximal Cohen–Macaulay approximation; Rational double point; Buchsbaum module

1. Introduction

The starting point of this paper is the classical notion of linkage (or liaison) of algebraic varieties. It goes back to the late 19th and early 20th century, when M. Noether, Halphen, and Severi used it to study algebraic curves in $\mathbb{P}^3$. Linkage allows to pass from a given curve to another curve, related in a geometric way to the original one. Iterating the procedure one obtains a whole series of curves in the same “linkage class”. The usefulness of this technique is explained by two observations: (a) certain properties of the curve are preserved under linkage, and (b) the resulting curves may be simpler, and thus easier to handle, than the original one.

∗ Corresponding author.
E-mail addresses: alexmart@neu.edu (A. Martsinkovsky), strooker@math.uu.nl (J.R. Strooker).

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Gradually linkage was applied in more general situations, and during the decades straddling the second world war, significant contributions were made by Dubreil, Apéry, and Gaeta. The breakthrough however came with the 1974 paper [18]. Using sheaves, duality, and homological tools, Peskine and Szpiro reduced general linkage to algebraic questions about certain ideals of regular local rings and thus put linkage theory on a sound algebraic footing. Since then, a great deal of further work has been done by algebraic geometers and commutative algebraists alike. As a short and lucid introduction to a few basic ideas of the subject, we recommend Ulrich’s lectures [22] at the ICTP.\footnote{As is stated on the front page of the ICTP printout, “these are preliminary lecture notes, intended only for distribution to participants”. The reader may want to contact the author for further inquiries.} For a very quick glimpse of linkage, see Section 21.10 of [8]. The monograph [16] covers a lot of material, much of it quite recent.

The goal of the present paper is to show that the notion of linkage of algebraic varieties, as expounded by Peskine and Szpiro [18], is part of a much more general and fundamental formalism which extends to arbitrary finitely generated modules over arbitrary (in particular, non-commutative) noetherian semiperfect rings. The new module-theoretic approach brings about more general and more precise results, sometimes providing simpler proofs of what is already known. Perhaps even more important is the new conceptual perspective. It clearly demonstrates that linkage is a combination of two ring-theoretic operations, forward and backward change of the ring, and a module-theoretic construct, which we call horizontal linkage. The latter offers possibilities for sweeping generalizations, whereas the former almost equally well limit such possibilities. As a consequence, the proposed approach immediately identifies the source of difficulties in any attempt to generalize classical linkage. Figuratively speaking, the horizontal component of linkage, which operates on the abelian argument—the module—is linear, whereas the two vertical components of linkage, which, in addition, operate on the non-abelian argument—the ring—are non-linear. To further justify our point of view, let us mention two standard facts which become almost evident in the new language: even linkage preserves more properties than odd, and the Gorenstein condition is hand in glove with linkage.

What would we like this paper to accomplish? It seems probable that development of linkage theory over non-commutative noetherian semiperfect rings will offer new insights. For instance, for a group ring of a finite group over $p$-adic integers, one would expect ties with representation theory rather than geometry. On the other hand, for algebraic geometers it may be advantageous to start using module-theoretic techniques and fit some of their fundamental results into our framework.\footnote{For these readers we emphasize that our linkage generalizes and extends the algebraic linkage of Peskine and Szpiro. Their more restrictive geometric linkage awaits further work.} This should yield transparent proofs of more general statements, and distinguish these from more specifically geometric considerations.

The paper is broken down into sections as follows. In Section 2 we reformulate the classical definition of Gorenstein linkage to make it amenable to module-theoretic techniques. The operation $\lambda := \Omega Tr$ comes to the forefront.

After recalling some facts about semiperfect rings in Section 3, we give a definition of linkage for arbitrary finitely generated modules over noetherian semiperfect rings and establish basic properties of such linkage.
The module-theoretic approach is immediately tested in Section 4, where we give compact proofs of several basic results on linkage and of their generalizations to modules.

In Section 5 we establish a criterion for horizontal linkage: a module is horizontally linked if and only if it is 1-torsion-free and is stable; the latter means that the module has no projective summands. Thus linkage theory is subsumed into the theory of “cohomological” torsion. The key ingredient here is a very simple calculation of the image of the bidual of a projective cover or, which is the same, the image of the canonical evaluation map. For a finitely generated stable module, it coincides with the result of applying \( \lambda^2 \) to the module.

In Section 6 we show that, over each non-Gorenstein complete Cohen–Macaulay isolated singularity of dimension two, there exists a reflexive (equivalently, maximal Cohen–Macaulay) module whose direct horizontal link is not reflexive (equivalently, not mCM). In fact the same is true for any reflexive module which admits a nontrivial extension by the ring.

In Section 7 we show that, over a Kleinian singularity, every stable maximal Cohen–Macaulay (mCM for short) module is horizontally self-linked. As a consequence, horizontal linkage fixes the non-projective vertices of the Auslander–Reiten quiver of the category of mCM modules. The presented approach is conceptual and completely avoids calculations. The main tool used in the argument is the existence of almost split sequences together with the McKay correspondence.

In Section 8 we determine the action of \( \lambda \) on the mCM approximations of Cohen–Macaulay modules over a Gorenstein commutative local ring, which leads to intriguing duality patterns for mCM approximations. As a consequence, when \( \dim R \) is odd, the stable part of a minimal mCM approximation of the degree \( (\dim R - 1)/2 \) syzygy module of a self-dual CM module (e.g., the residue field) is horizontally self-linked.

In Section 9 we give a derived-categoric criterion of linkage of modules. This generalizes a theorem of Schenzel. In fact, it gives a more precise result even in the original setting. Extending Schenzel’s definition, for a module \( A \) over a noetherian semiperfect ring \( \Lambda \) of finite injective dimension a complex \( J_A \) is defined. It is obtained by the soft truncation of \( \mathbb{R}\text{Hom}_\Lambda(A, \Lambda) \). It turns out that there is always a morphism \( \mathbb{R}\text{Hom}_{\Lambda^{op}}(J_A, \Lambda) \to J_{\lambda A} \) (defined already at the level of complexes) and this morphism is an isomorphism in the bounded derived category \( D^b(\Lambda) \) if and only if \( A \) is horizontally linked. Unlike Schenzel’s approach, where an injective resolution of \( A \) was used in the definition of \( J_A \), we use a projective resolution of \( A \), which results in a very transparent construction of the morphism above. To incorporate non-horizontal linkage one needs to make further assumptions on the ring (see below).

In Section 10 we relate the local cohomology of a module \( A \) over a Gorenstein local ring to the local cohomology of \( \lambda A \), assuming that the homology of \( J_A \) is of finite length. In particular we show that a linked module is Buchsbaum if an only if its link is. Again, these are generalizations of results of Schenzel for ideals.

In Section 11 we generalize a result of J. Herzog and M. Kühl on Bourbaki sequences. The correct conceptual framework for that result is the theory of mCM approximations and we show that even linkage over Gorenstein rings by Gorenstein ideals of finite projective dimension preserves the stable equivalence class of the maximal Cohen–Macaulay approximation. As a consequence, even linkage preserves projective dimension.
In Section 12 we generalize to modules a result of Golod that provides examples of linkage over not necessarily Gorenstein local rings.

2. Horizontal linkage

All rings in this paper are assumed to be noetherian on both sides and all modules are finitely generated left modules. Right modules will be viewed as left modules over the opposite ring. The symbol $\cong$ will be used to denote a natural isomorphism. A commutative ring is said to be Gorenstein if it is of finite injective dimension when viewed as a module over itself. An ideal is said to be Gorenstein if the corresponding factor ring is Gorenstein. For ideals $p$ and $q$ of a ring $R$ the symbol $p : q$ denotes the ideal of all elements $x$ of $R$ such that $x q \subseteq p$.

We begin by recalling the definition of (Gorenstein) linkage. Let $(R, m, k)$ be a commutative Gorenstein local ring with maximal ideal $m$ and residue field $k$.

**Definition 1.** Ideals $a$ and $b$ of $R$ are (algebraically) linked by a Gorenstein ideal $c$ if

(a) $c \subseteq a \cap b$, and 
(b) $a = c : b$ and $b = c : a$.

The following result is immediate.

**Lemma 1.** Under the above assumptions, $a$ and $b$ are linked by $c$ if and only if the ideals $a/c$ and $b/c$ of $R/c$ are linked by the zero ideal of $R/c$.

As a consequence, linkage of ideals by an ideal can always be replaced by linkage by the zero ideal. More pedantically, to check whether or not $a$ and $b$ are linked by $c$, one can extend these ideals along the ring homomorphism $R \to R/c$, check the linkage of the extended ideals by the zero ideal, and contract the extended ideals back to $R$. To simplify the language, we introduce

**Definition 2.** Two ideals in a Gorenstein ring are said to be horizontally linked if they are linked by the zero ideal.

The preceding discussion shows that general linkage can informally be thought of as a three-step procedure consisting of an extension of an ideal, horizontal linkage, and the contraction of an ideal. It is our next goal to show that all three components of this sequence can be defined for arbitrary finitely-generated modules. To this end, we first recall two module-theoretic operations: the syzygy and the transpose.

Let $\Lambda$ be an associative ring, $M$ a (left) $\Lambda$-module, and $P \to M$ an epimorphism such that $P$ is a projective. The kernel of this epimorphism is denoted $\Omega M$ and is called the syzygy module of $M$. By Schanuel’s lemma, the projective equivalence class of this module is uniquely defined. However, if $\Lambda$ admits projective covers, $\Omega M$ can be defined uniquely
up to isomorphism by assuming that $P \to M$ is a projective cover. Unless explicitly stated to the contrary, this assumption will always be made.

Assume now that $M$ is a finitely presented $\Lambda$-module and let $P_1 \to P_0 \to M$ be a finite projective presentation of $M$. The transpose $\text{Tr} M$ of $M$ is defined by the exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to \text{Tr} M \to 0$$

obtained from the presentation above by applying the functor $(-)^*: \text{Hom}_\Lambda(-, \Lambda)$. Again, the transpose is defined only up to projective equivalence (over the opposite ring $\Lambda^{op}$). However, if we assume the existence of projective covers and that the presentation of $M$ above is minimal, then $\text{Tr} M$ is defined uniquely up to isomorphism. Unless explicitly stated to the contrary, this minimality assumption will always be made. In particular, assuming that the ring is commutative local, that the presentation of $M$ above is minimal, and that $M$ has no projective summands, it can easily be shown then $P_0^* \to P_1^* \to \text{Tr} M \to 0$ will be a minimal projective presentation of $\text{Tr} M$. In addition, the transpose of a projective module will be isomorphic to the zero module.

As a first step toward generalizing horizontal linkage, we pass from ideals to modules. The naive way of viewing an ideal as a module is not suitable for our purposes (it also has a drawback that different ideals may be isomorphic as modules). A much better way is to pass from an ideal to the corresponding cyclic module. The precise meaning of this claim is captured in the following obvious statement:

**Lemma 2.** Two ideals in a commutative ring are equal if and only if the corresponding cyclic modules are isomorphic.

**Proof.** Indeed, an ideal in a commutative ring can be recovered as the annihilator of the corresponding cyclic module. \qed

It is immediate, that, in terms of the corresponding cyclic modules, the vertical components of linkage are nothing but the extension of scalars and, respectively, the restriction of scalars. Now, to describe the horizontal component of linkage, we want give a module-theoretic description of the passage from an ideal to its annihilator. It will be formulated in terms of the syzygy and the transpose operators.

**Lemma 3.** Let $a$ and $b \neq R$ be ideals in a commutative local ring $R$. Then $a = 0 : b$ if and only if $R/a$ is isomorphic to $\Omega \text{Tr} R/b$.

**Proof.** If $b = 0$, then the statement is clear. Assume now that $b \neq 0$ and let $R^n \to b$ be a projective cover. Obviously, $R/b$ has no projective summands. Therefore, dualizing a minimal (since $b \neq R$) projective presentation $R^n \to R \to R/b \to 0$ we have an exact sequence $0 \to (R/b, R) \to (R, R) \to (R^n, R) \to \text{Tr} R/b \to 0$ of $R$-modules whose middle terms are a minimal projective presentation of $\text{Tr} R/b$. (Here $(-, R)$ denotes $	ext{Hom}_R(-, R)$.) As a consequence, we have the exact sequence $0 \to (R/b, R) \to (R, R) \to \Omega \text{Tr} R/b \to 0$. Under the canonical isomorphism $(R, R) \to R: f \mapsto f(1)$ the module
(R/b, R) is mapped onto the ideal 0 : b. By Lemma 2, the modules Ω Tr R/b and R/a are
isomorphic if and only if a = 0 : b. □

Returning to linkage over a Gorenstein commutative local ring R we have, as a consequence,

**Proposition 1.** Non-zero ideals a and b of R are horizontally linked if and only if
R/a ≃ Ω Tr R/b and R/b ≃ Ω Tr R/a.

The just proved proposition indicates the importance of the operation Ω Tr. It first
appeared in the Auslander and Bridger treatise [3] on stable module theory, where it was
denoted by D₁. Nowadays, this symbol has acquired a variety of other meanings, and we
shall denote this operation by the symbol λ.

If a and b are horizontally linked, then each of the two ideals is uniquely determined by
the other as its annihilator. It therefore makes sense to say that the ideal a is horizontally
linked. In this terminology we have

**Corollary 1.** A proper ideal a of R is horizontally linked if and only if
R/a is isomorphic to λ²(R/a).

### 3. Linkage of modules: definitions and basic properties

In view of the last two results in the previous section, horizontal linkage can be defined
for arbitrary finitely generated modules over commutative local rings. Moreover, to have
the operation λ well-defined the only condition on the ring that we needed was the
existence of projective covers. All that points to the possibility of working with rings
Λ for which every finitely generated left module has a projective cover. Such rings are
called semiperfect rings. (See [9, Chapters 18 and 22] or [14, Chapter 8] for details.)

They possess the same property for right modules. They are not necessarily noetherian
but, in keeping with our convention, we shall always mean “semiperfect” to include the
noetherian property on both sides. noetherian local rings are semiperfect and, in particular,
the commutative ones, over which the (non-graded version of) linkage of ideals has been
exclusively studied. Before we give a new definition of linkage, we shall recall a basic
property of finitely presented modules over semiperfect rings (see [1, Theorem 32.13]).

**Proposition 2.** Let Λ be a semiperfect ring, M a stable finitely presented Λ-module, and
P₁ → P₀ → M a minimal projective presentation of M. Then P₁ → P₀ → Ω Tr M is a
minimal projective presentation of Tr M. In particular, M* is isomorphic to Ω² Tr M.

At this point we are ready to depart from the classical setting and give a new

**Definition 3.** Let Λ be a (noetherian) semiperfect ring. A finitely generated Λ-module
M and a Λ₀-module N are said to be horizontally linked if M ≃ λ N and N ≃ λ M.
Equivalently, M is horizontally linked (to λ M) if and only if M ≃ λ² M.
Obviously, if $M$ and $N$ are horizontally linked, then $N$ is finitely generated. It follows from this definition that a projective module is linked if and only if it is isomorphic to the zero module. Indeed, the transpose of a projective is zero and therefore $\lambda$ of a projective is also zero. In fact, we can say more.

**Proposition 3.** A horizontally linked $\Lambda$-module is stable.

**Proof.** Suppose $M$ is horizontally linked and $M \cong M' \amalg P$, where $M'$ is stable and $P$ is a projective. If $P_0 \rightarrow M'$ is a projective cover then, by the definition of $\lambda$, there is a projective precove $P_0^* \rightarrow \lambda M$ (in fact, since $M'$ is stable, this is a projective cover). Using the definition of $\lambda$ again, we have a projective precove $P_0 \cong P_0^{**} \rightarrow \lambda^2 M \cong M \cong M' \amalg P$. Thus the projective cover $Q$ of $\lambda^2 M$ is a summand of $P_0$ which is a proper summand of $P_0 \amalg P$. Projecting $P_0 \amalg P$ onto $P_0$ and then onto $Q$ we have a surjective endomorphism (since $\lambda^2 M \cong M$) of a noetherian module. Then it must be an isomorphism. On the other hand, the kernel of this endomorphism contains $P$, whence $P$ is null. $\blacksquare$

As another immediate consequence of the definition, we have

**Proposition 4.** Suppose $M$ is horizontally linked. Then $\lambda M$ is also horizontally linked and, in particular, $\lambda M$ is stable.

**Proof.** Since $M$ is linked, $M \cong \lambda^2 M$. Therefore, $\lambda^2(\lambda M) \cong \lambda(\lambda^2 M) \cong \lambda M$. $\blacksquare$

The proof also shows that horizontal linkage is symmetric in the sense that $M$ is linked to $\lambda M$ if and only if $\lambda M$ is linked to $M$.

Having defined horizontal linkage for modules, we can now define general linkage for modules. Let $\Lambda$ be a semiperfect ring.

**Definition 4.** A finitely generated $\Lambda$-module $M$ is said to be linked to a finitely generated $\Lambda^{\text{op}}$-module $N$ by a two-sided ideal $c$ contained in the annihilators of $M$ and $N$ if $M$ and $N$ are horizontally linked as modules over $\Lambda/c$ and, respectively, over $(\Lambda/c)^{\text{op}} \cong \Lambda^{\text{op}}/c$.

The just defined linkage is also called direct linkage. By varying the ideals $c$, one can build chains of linked modules thus providing the transitive closure for the direct linkage. Abusing the language, we say that the modules in the same chain are also linked. Over a non-commutative ring, modules related by a chain of odd length are in the opposite categories, whereas modules related by a chain of even length are in the same module category. This points, once again, to the special significance of even linkage classes, originally observed in the commutative setting.

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3 Notice that a variant of the traditional definition of linkage of ideals (e.g., [19]) requires that the ideals be of pure height. Over a Gorenstein ring, ideals (and, as we shall see later, modules) linked by Gorenstein ideals are necessarily unmixed. In our definition we have dispensed with that requirement. The purists, who find the omission of purity intolerable, are welcome to complement the given definition with any additional condition of their own.
As it stands now, linkage for modules can be viewed as a sequence of a forward change of ring, horizontal linkage, and a backward change of ring. Each change of ring is along the ring homomorphism $\Lambda \to \Lambda / c$, and it will be colloquially referred to as a \textit{vertical} component of linkage. As we shall see in Section 9, when the ring and the ideal are both Gorenstein, it becomes much easier to transfer information vertically.

\textbf{Remark.} Yoshino and Isogawa \cite{23} have recently proposed the following definition of linkage for Cohen–Macaulay modules over Gorenstein commutative rings: if $M$ is a maximal Cohen–Macaulay module over a Gorenstein ring $R$, then the link of $M$ is $(\Omega M)^*$. This is the same definition that had been given earlier \cite{20} by the second author. (For a non-maximal Cohen–Macaulay modules, one first divides by a maximal regular sequence in the annihilator.) To compare it with our definition, let $0 \to \Omega M \to P_0 \to M \to 0$ be exact with the last map being a projective cover. Dualizing into $R$ we have an exact sequence

$$0 \to M^* \to P_0^* \to (\Omega M)^* \to \text{Ext}^1_R(M, R) \to 0$$

and therefore an exact sequence

$$0 \to \lambda M \to (\Omega M)^* \to \text{Ext}^1_R(M, R) \to 0.$$  

The latter shows that for Cohen–Macaulay modules the definition of Yoshino and Isogawa coincides with our definition but, otherwise, it disagrees even with the traditional definition of linkage for ideals.

4. First examples

The classical analogs (i.e., assuming that the modules in question are cyclic) of the results in the previous section are obviously trivial. We shall now provide a non-trivial, although simple, result admitting a generalization to modules. First recall that a module is said to be \textit{unmixed} if all of its associated primes are of the same height. The following is well-known.

\textbf{Proposition 5.} Let $R$ be a commutative ring, $b$ an ideal of $R$ and $a := \text{Ann} b$. Then $\text{Ass}(R/a) \subseteq \text{Ass}(R)$. If $R$ is unmixed and $b = \text{Ann} a$, then

$$\text{Ass}(R/a) \cup \text{Ass}(R/b) = \text{Ass}(R).$$

\textbf{Proof.} If $b = 0$, the assertion is trivial. If not, let $b = (j_1, \ldots, j_n)$. The kernel of the map $[j_1, \ldots, j_n]^t : R \to R^n$ equals $\text{Ann} b$ and, therefore, $R/\text{Ann} b$ embeds in $R^n$. The first assertion now follows. Now let $p$ be an associated prime of $R$. Since $ab = 0$, $p$ must contain either $a$ or $b$. By the unmixedness assumption, $p$ is minimal and hence is minimal among the primes containing $a$ or $b$. Thus $p$ is an associated prime of $R/a$ or $R/b$. \hfill \Box

A generalization of this result to modules can be stated as follows.
Proposition 6. Let $R$ be a commutative local ring and let $M$ be a finitely generated $R$-module. Then $\text{Ass}(\lambda M) \subseteq \text{Ass}(R)$. If $R$ is unmixed and $M$ is nonzero and horizontally linked, then

$$\text{Ass}(M) \cup \text{Ass}(\lambda M) = \text{Ass}(R).$$

Proof. The first assertion is immediate because $\lambda M$ is a syzygy module. Let $p$ be an associated prime of $R$ and $P_1 \to P_0 \to M \to 0$ a minimal projective (and hence free) presentation of $M$. The latter gives rise to an exact sequence $0 \to M^* \to (P_0)^* \to \lambda M \to 0$. Under the assumption that $p \notin \text{Supp} M \cup \text{Supp} \lambda M$, the end terms and, therefore, the middle term of that sequence would localize to 0 at $p$. This is impossible because $(P_0)^*$ is a nonzero free module (since $M$ is nonzero). Hence $p$ is in the support of either $M$ or $\lambda M$. If $R$ is unmixed, then $p$ is a minimal prime and, therefore, an associated prime of one of the two modules. Finally, if $M$ is linked, then $M \simeq \lambda (\lambda M)$ and $\text{Ass}(M) \subseteq \text{Ass}(R)$. $\square$

The next result gives a large class of horizontally linked modules. To state it, we need to recall the definition of Gorenstein (or G-) dimension (see [3, Proposition 3.8]).

Definition 5. Let $\Lambda$ be a two-sided noetherian ring. A nonzero finitely generated $\Lambda$-module $M$ is said to be of G-dimension zero if

(1) $\text{Ext}_\Lambda^i(M, \Lambda) \simeq 0$ for all $i \geq 1$,
(2) $\text{Ext}_{\Lambda^\text{op}}^i(M^*, \Lambda) \simeq 0$ for all $i \geq 1$, and
(3) $M$ is reflexive.

Classical examples of modules of G-dimension zero are finitely generated projectives over arbitrary rings and arbitrary finitely generated modules over self-injective rings. Over a Gorenstein commutative local ring, modules of G-dimension zero are exactly maximal Cohen–Macaulay modules.

Theorem 1. Let $\Lambda$ be a semiperfect noetherian ring and $M$ a stable $\Lambda$-module of G-dimension zero. Then $M$ is horizontally linked and its link is a stable $\Lambda^\text{op}$-module of G-dimension zero.

Proof. It follows from the definition that the $\Lambda$-dual of a minimal projective resolution

$$\cdots \to Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M^* \to 0$$

of $M^*$ is a minimal projective coresolution of $M$. Splicing it with a minimal projective resolution

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

of $M$, we have a doubly infinite exact complex of projectives,

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} Q_0^* \xrightarrow{\partial^*_1} Q_1^* \to \cdots,$$
called a complete resolution of $M$, which stays exact under the functor $(\cdot, \Lambda)$. By construction and since $M$ has no projective summands, the entire complete resolution is minimal. All this makes the operation $\Omega$ invertible on $M$ up to non-canonical isomorphisms: one defines $\Omega^{-i} M$ as the image of $\partial^i$. It is not difficult to show, using the definition of G-dimension zero given above, that $M^*$ is also of G-dimension zero. It can easily be checked now that $(\Omega M)^* \simeq \Omega^{-1} M^*$ and that $\text{Tr} M \simeq \Omega^{-2} M^*$. Moreover, it is also not difficult to see that a nontrivial syzygy module of a module of G-dimension zero is again of G-dimension zero. Therefore $\lambda^2 M = \Omega \text{Tr} \Omega \text{Tr} M \simeq \Omega \text{Tr} \Omega^{-1} M^* \simeq \Omega \text{Tr}(\Omega M)^* \simeq \Omega^{-1} (\Omega M)^* \simeq \Omega^{-1} \Omega M \simeq M$. The assertion that $\lambda M$ is stable follows from Proposition 4 on p. 593.

**Corollary 2.** Let $R$ be a Gorenstein commutative local ring and $M$ a stable $mCM$ $R$-module. Then $M$ is horizontally linked and its link is again a stable $mCM$ module.

We shall use this corollary to generalize the following basic result of Peskine and Szpiro [18, Proposition 1.3].

**Proposition 7.** Let $R$ be a Gorenstein local ring and $a_1$ a nonzero ideal such that $\dim R/a_1 = \dim R$. Let $a_2$ be the annihilator of $a_1$. The following conditions are equivalent:

(i) $R/a_1$ is Cohen–Macaulay.

(ii) $R/a_2$ is Cohen–Macaulay and $R/a_1$ is unmixed.

Under these conditions, $a_1$ is the annihilator of $a_2$ and $\dim R/a_2 = \dim R$.

**Remark.** In Proposition 1.3 of [18], we find in (ii) the condition “... $R/a_1$ has no embedded components.” That this condition is too weak and has to be replaced can be seen from the following example. Let $S$ be the ring $k[|X, Y, Z|]$ of formal power series in three variables over a field $k$ and $R := S/(XZ)$. The latter is a two-dimensional Gorenstein ring. In that ring, let $a_1 := (yz)$. This is a nonzero ideal consisting entirely of zerodivisors. Let $a_2 := \text{Ann} a_1$. It is immediate that $a_2 = (x)$. The ring $R/a_2$ is two-dimensional regular. In particular it is Cohen–Macaulay. The associated primes of $R/a_1$ are $(z)$ and $(x, y)$ and, therefore, $R/a_1$ has no embedded components. Since these primes are of different heights, $R/a_1$ is not Cohen–Macaulay. Thus (ii) would not imply (i) under the weaker condition. Furthermore, the annihilator of $a_2$ is $(z) \neq a_1$.

In the proposed generalization, we shall replace $R/a_1$ with a finitely generated stable $R$-module $M$ and $R/a_2$ with $\lambda M$.

**Proposition 8.** Let $R$ be a Gorenstein local ring and $M$ a stable finitely generated $R$-module with $\dim M = \dim R$. Then the following conditions are equivalent:

(i) $M$ is maximal Cohen–Macaulay.

(ii) $\lambda M$ is maximal Cohen–Macaulay and $M$ is unmixed.

Under these conditions, $M$ is horizontally linked.
Proof. The implication (i) ⇒ (ii) follows immediately from Corollary 2 on p. 596.

(ii) ⇒ (i). Since $M$ is stable, $M^*$ is isomorphic to $\Omega \lambda M$ and is mCM unless $M^* \simeq 0$. Thus $M^{**}$ is also either mCM or the zero module. The cokernel of the canonical evaluation map $e_M: M \to M^{**}$ is isomorphic to $\text{Ext}^2(\text{Tr} M, R) = \text{Ext}^1(\lambda M, R)$ (see (5.1) in Section 5). Thus, since $\lambda M$ is mCM, $e_M$ is an epimorphism. Since $\dim M = \dim R$ and $M$ is unmixed, the associated primes of $M$ are minimal primes of $R$. Then the same is true for $\ker e_M$. If $p \in \text{Ass}(\ker e_M)$, then $R_p$ is a zero-dimensional Gorenstein ring and, therefore, $M_p$ is reflexive. Therefore, $(\ker e_M)_p \cong \ker e_M_p = 0$. Hence $e_M$ is also a monomorphism and $M$ is isomorphic to $M^{**}$. Since $\dim R = \dim M$, this implies that $M^{**} \neq 0$ and $M$ is mCM. ✷

The last result we want to generalize in this section is due to Ferrand [18, Proposition 2.6]. We recall it now.

Proposition 9. Let $R$ be a regular local ring, $\alpha$ an ideal of $R$ such that $R/\alpha$ is a Cohen–Macaulay ring of codimension $d$, and $f = (f_1, \ldots, f_d)$ an ideal generated by a regular sequence contained in $\alpha$. Let $F_1$ (respectively, $F$) be a projective resolution of $R/\alpha$ (respectively, $R/f$) and $\alpha: F \to F_1$ a morphism of complexes lifting the canonical map $R/f \to R/\alpha$. Then the mapping cone of $\alpha^*: F_1^* \to F^*$ is a projective resolution of the link of $\alpha$ with respect to $f$.

Our generalization will be stated under slightly more general assumptions. The proof however is identical to the original one. It will be given here since it will be needed later in Section 12. We begin by recalling the notion of perfect module. Let $R$ be a commutative ring and $M$ an $R$-module. The grade of $M$, denoted $\text{grade} M$, is defined as $\min \{ i \mid \text{Ext}^i_R(M, R) \neq 0 \}$.

Definition 6. The module $M$ is said to be perfect if $M$ is of finite projective dimension and $\text{grade} M = \text{proj dim} M$.

Notice that: (a) the grade of a nonzero module never exceeds the projective dimension of the module, (b) any perfect module, being of finite projective dimension, must be nonzero (as the projective dimension of the zero module is $-\infty$), and (c) if $M$ is a perfect module, then $\text{Ext}^i_R(M, R)$ is the only nonzero module among all $\text{Ext}^i_R(M, R)$. We can now generalize Ferrand’s result.

Proposition 10. Let $R$ be a Gorenstein local ring, $M$ a perfect $R$-module of projective dimension $g$ linked by a Gorenstein ideal $I$, also of finite projective dimension $g$, and let $\overline{R} := R/I$. Let $P \to M$ be a minimal projective resolution of $M$ and $Q \to P_0$ a projective resolution of $P_0 := P_0/I P_0$ of length $g$. Let $\psi: Q \to P$ be a lifting of the reduction modulo $I$ of the chosen projective cover $P_0 \to M$. Then the mapping cone $\text{Con}(\psi^*)$ is an $R$-projective resolution of $\lambda \overline{M}$.
Proof. The chain map $\varphi^* : P^* \rightarrow Q^*$ gives rise to the short exact sequence of complexes
$$0 \rightarrow Q^* \rightarrow \text{Con}(\varphi^*) \rightarrow P^*[−1] \rightarrow 0.$$ 

The corresponding long cohomology exact sequence degenerates into
$$0 \rightarrow \text{Ext}^g_R(M,R) \rightarrow \text{Ext}^g_R(P_0,R) \rightarrow H^g(\text{Con}(\varphi^*)) \rightarrow 0.$$ (4.1)

Using the functorial isomorphism $\text{Ext}^g_R(−,R) \rightarrow \text{Hom}_R(−,R)$ defined on the category of $R$-modules, we have an exact sequence
$$0 \rightarrow \text{Hom}_R(M,R) \rightarrow \text{Hom}_R(P_0,R) \rightarrow H^g(\text{Con}(\varphi^*)) \rightarrow 0,$$
where the first map is the $R$-dual of the $R$-projective cover $P_0 \rightarrow M$. This shows that $H^g(\text{Con}(\varphi^*))$ is isomorphic to $\lambda^R M$.  

5. Criteria for horizontal linkage

Throughout this section $\Lambda$ is a semiperfect ring and all modules are finitely generated. Our next task is to characterize horizontally linked modules. To state the result, we recall (see [3]) the exact sequence
$$0 \rightarrow \text{Ext}^1_{\Lambda^\text{op}}(\text{Tr}M,\Lambda) \rightarrow M \rightarrow M^* \rightarrow \text{Ext}^2_{\Lambda^\text{op}}(\text{Tr}M,\Lambda) \rightarrow 0,$$ (5.1)

where $M$ is a finitely generated (left) $\Lambda$-module and $e_M$ is the canonical evaluation map. The (left) $\Lambda$-module $\text{Ext}^1_{\Lambda^\text{op}}(\text{Tr}M,\Lambda)$ is called the 1-torsion submodule of $M$. When $\Lambda$ is a commutative domain, the 1-torsion is the usual torsion submodule of $M$. If $\text{Ext}^1_{\Lambda^\text{op}}(\text{Tr}M,\Lambda) = 0$, i.e., the canonical map $e_M$ is a monomorphism, we shall say, following the terminology of [3], that $M$ is 1-torsion-free.4 Every submodule of such a module is also 1-torsion-free, as one easily deduces from the naturality of the evaluation map.

The following is well-known.

Proposition 11. A $\Lambda$-module is 1-torsion-free if and only if it can be embedded in a direct product of copies of $\Lambda$.

Corollary 3. A submodule of a projective module is 1-torsion-free.

The next result can be found in [3, pp. 141, 142].

Lemma 4. Let $M$ be a finitely generated stable $\Lambda$-module. Then the image of the canonical evaluation map $e_M$ is isomorphic to $\lambda^2 M$.

---

4 Some authors will vehemently insist on calling such modules torsionless.
Proof. Let \( P_1 \rightarrow P_0 \xrightarrow{\psi} M \rightarrow 0 \) be a minimal projective presentation of \( M \). The commutative diagram

\[
\begin{array}{c}
P_0 & \xrightarrow{\psi} & M \\
\downarrow{\varepsilon_0} & & \downarrow{\varepsilon_M} \\
P^{**}_0 & \xrightarrow{\psi^{**}} & M^{**}
\end{array}
\]

shows that \( \text{im}(\varepsilon_M) = \text{im}(\psi^{**}) \). Applying the functor \((- , \Lambda)\) to the minimal presentation of \( M \) we have the exact sequences

\[
0 \rightarrow M^* \xrightarrow{\psi^*} P^*_0 \rightarrow P^*_1 \rightarrow \text{Tr}(M) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M^* \xrightarrow{\psi^*} P^*_0 \rightarrow \lambda M \rightarrow 0.
\]

Let \( Q_0 \xrightarrow{\psi} M^* \rightarrow 0 \) be a projective cover and \( \alpha := \psi^* \psi \). Since \( M \) has no projective summands, the sequence \( Q_0 \xrightarrow{\alpha} P^*_0 \rightarrow \lambda M \rightarrow 0 \) is a minimal projective presentation. Dualizing it into \( \Lambda \) we have a commutative diagram with exact row

\[
0 \rightarrow (\lambda M)^* \xrightarrow{\alpha^*} P^{**}_0 \xrightarrow{\alpha^{**}} Q^*_0 \xrightarrow{\psi^*} \text{Tr}(\lambda M) \rightarrow 0.
\]

By construction, \( \text{im}(\alpha^*) \simeq \Omega \text{Tr}(\lambda M) = \lambda^2 M \). On the other hand, since \( \psi^* \) is a monomorphism, \( \text{im}(\alpha^*) \simeq \text{im}(\psi^{**}) \), the latter being isomorphic to \( \text{im}(\varepsilon_M) \). \( \square \)

As a consequence of the lemma, for a stable \( M \) there is a short exact sequence

\[
0 \rightarrow \text{Ext}_A^1(\text{Tr}(M), \Lambda) \rightarrow M \rightarrow \lambda^2 M \rightarrow 0.
\]

We can now characterize horizontally linked modules.

Theorem 2. A finitely generated \( A \)-module \( M \) is horizontally linked if and only if it is stable and is 1-torsion-free, i.e., \( M \) has no projective summands and \( \text{Ext}_A^1(\text{Tr}(M), \Lambda) = 0 \).

Proof. If \( M \) is horizontally linked, then \( M \simeq \lambda^2 M \) and \( M \) is stable by Proposition 3 on p. 593. By the preceding lemma, the canonical evaluation map induces a surjection of \( M \) onto \( \lambda^2 M \). Thus we have a surjective endomorphism of a noetherian module, which then must be an isomorphism (this is true without any assumption on the ring). Therefore \( M \) is 1-torsion-free. The other direction is an immediate consequence of Lemma 4 on p. 598. \( \square \)

Over a Gorenstein commutative local ring, the transpose of an mCM module is again mCM and the higher Ext-groups of such a module with coefficients in the ring vanish.
Combined with the just proved theorem, this gives another proof that stable mCM’s over such rings are horizontally linked.

**Corollary 4.** A module over a (semiperfect) commutative domain is horizontally linked if and only if it is stable and torsion-free.

**Remark.** A commutative ring $R$ is semiperfect if and only if it is a finite direct product of commutative local rings. (See, for example, [14, Theorem 23.11].)

**Corollary 5.** Any stable reflexive module is horizontally linked.

We shall now give an alternative description of horizontally linked modules.

**Corollary 6.** A finitely generated $\Lambda$-module $M$ is horizontally linked if and only if $M$ is a stable syzygy module (i.e., $M$ is a stable submodule of a projective).

**Proof.** If $M$ is linked, then it is stable and $M \cong \lambda^2 M = \Omega \Omega\Omega Tr M$, showing that $M$ is a syzygy module. The converse follows from Corollary 3 on p. 598 and Theorem 2. $\Box$

**Corollary 7.** Suppose that $R_p$ is Gorenstein for every minimal prime $p \in \text{Ass}(R)$. Then any stable unmixed $R$-module $M$ with $\dim M = \dim R$ is horizontally linked.

**Proof.** We need to show that the kernel of the canonical evaluation map $e_M: M \to M^{**}$ is zero. The assumptions on $M$ guarantee that its associated primes are minimal primes of $R$. The same is true for $\ker e_M$. Thus $R_p$ is a zero-dimensional Gorenstein ring for any associated prime $p$ of $\ker e_M$. Over such a ring any module is reflexive and, therefore, $(\ker e_M)_p \cong \ker e_{M_p} = 0$. Thus $\ker e_M = 0$. $\Box$

If $M$ is a syzygy module over a commutative local ring $R$, then $\text{Ass}(M) \subset \text{Ass}(R)$. The following reformulation of a well-known result gives a partial converse.

**Corollary 8.** Suppose $R$ is generically Gorenstein, i.e., $R_p$ is Gorenstein for every prime $p \in \text{Ass}(R)$. An $R$-module $M$ is a syzygy module if and only if $\text{Ass}(M) \subset \text{Ass}(R)$. Thus such a module is horizontally linked if and only if it is stable.

**Proof.** The last assertion follows at once from Corollary 6. The “only if” part is obvious. To prove the “if” part, we need to show that the kernel of $e_M: M \to M^{**}$ is zero. It is enough to show that $(\ker e_M)_p = 0$ for any associated prime $p$ of $\ker e_M$. Any such prime is an associated prime of $M$, and hence of $R$ and, therefore, $R_p$ is Gorenstein. Since $\text{Ass}_{R_p}(M_p) \subset \text{Ass}_{R_p}(R_p)$, the associated primes of the $R_p$-module $M_p$ are all of height zero. The proof of the previous corollary shows now that $\ker e_{M_p} = 0$. $\Box$

As an easy consequence of the just established criteria for horizontal linkage we can now describe a class of modules that can be linked by nonzero ideals.
Corollary 9. Let $M$ be a module over a commutative local ring $R$ which is stable as an $R/\text{Ann } M$-module. Suppose that $\dim M = \dim R/p$ for every $p \in \text{Ass } M$. Let $c \subseteq \text{Ann } M$ be an ideal such that $\dim R/c = \dim M$. If $R/c$ is generically Gorenstein, then $M$ is linked by $c$. If, in addition, $R/c$ is Gorenstein and $M$ is a CM $R$-module, then the link of $M$ is CM of the same dimension.

Proof. Let $p$ be an associated prime of $M$. Then $p$ is minimal among primes containing $c$ because $\dim R/c = \dim M = \dim R/p$. Thus $\text{Ass}_{R/c} M \subseteq \text{Ass } R/c$. The first claim now follows from the previous corollary. The second claim follows from Corollary 2 on p. 596. ☐

6. Examples of (non-)invariance under horizontal linkage

Let $R$ be a Gorenstein commutative local ring and $M$ a horizontally linked $R$-module. Corollary 2 on p. 596 shows that $M$ is mCM if and only if $\lambda M$ is mCM. Thus over a Gorenstein ring horizontal linkage preserves and reflects the property of a module to be mCM.

As to be expected and as will be now shown, over non-Gorenstein rings mCM’s need not be preserved under horizontal linkage. The example we are about to construct will also show that reflexivity of a module need not be preserved. Both facts are known in the traditional setting of linkage of ideals. However the existing examples are based on calculations with explicitly given ideals. Our example, in the spirit of this paper, is based on a conceptual approach and completely avoids calculations. Moreover we actually produce infinitely many examples.

We begin with the following simple observation.

Lemma 5. Let $M$ be a finitely presented $\Lambda$-module such that the canonical evaluation map $e_M : M \to M^{**}$ is onto. Then $\text{Ext}^1(\lambda M, \Lambda) = 0$.

Proof. $\text{Ext}^1(\lambda M, \Lambda) \cong \text{Ext}^2(\text{Tr } M, \Lambda) = 0$, by the exact sequence (5.1) on p. 598. ☐

We are now ready to describe the promised example.

Example. Let $R$ be a commutative two-dimensional complete normal domain which is a quotient of a formal power series ring over a field. In particular $R$ is local. By Serre’s criterion, $R$ is a Cohen–Macaulay ring with an isolated singularity. Moreover, the category of mCM modules coincides with the category $\text{Ref}(R)$ of reflexive modules. Assume furthermore that $R$ is not Gorenstein. Let $M$ be any mCM module such that $\text{Ext}^i(M, R) \neq 0$. Such modules do exist because if $\text{Ext}^i(N, R) = 0$ for each positive integer $i$ and for each mCM $R$-module $N$, then $R$ would be of finite injective dimension (because the second syzygy module of any module is mCM or zero), contrary to the assumption that $R$ is not Gorenstein. Without loss of generality we may assume that $M$ is stable. Being also reflexive, $M$ is horizontally linked. If $\lambda M$ were reflexive, then, $\text{Ext}^1(M, R) \cong \text{Ext}^1(\lambda^2 M, R)$ would vanish by the previous lemma, a contradiction. Thus the link of any
stable reflexive (respectively, an mCM) \( R \)-module \( M \) such that \( \text{Ext}^1(M, R) \neq 0 \) is not reflexive (respectively, mCM).

7. The operator \( \lambda \) and Kleinian singularities

On a horizontally linked module the operation \( \lambda \) is, by definition, an involution. When the ring is commutative it is of interest, therefore, to ask whether there are modules which are fixed by \( \lambda \) itself. Let \( R \) be a commutative local ring.

**Definition 7.** A finitely generated \( R \)-module \( M \) is said to be horizontally self-linked if \( M \cong \lambda M \).

**Lemma 6.** If \( M \) is horizontally self-linked, then \( \Omega M \cong M^* \). If \( R \) is Gorenstein and \( M \) is a stable mCM, then the converse holds.

**Proof.** Suppose \( M \) is horizontally self-linked. Then it has no projective summands and, therefore, \( M^* \cong \Omega^2 \text{Tr} M \cong \Omega \lambda M \cong \Omega M \). Suppose now that \( R \) is Gorenstein and \( M \) is a stable mCM. Then, as we already saw, \( \Omega \) is invertible on \( M \) and the isomorphisms \( \Omega M \cong M^* \cong \Omega^2 \text{Tr} M \) imply that \( M \cong \lambda M \).

For the rest of this section we shall assume that \( R \) is a complete Kleinian singularity. Equivalently, \( R \) is the complete two-dimensional hypersurface ring \( \mathbb{C}[x,y,z]/(f) \), where \( f \) is one of Arnold’s simple singularities. Recall that such singularities are indexed by the Dynkin diagrams of types \( A, D, \) and \( E \):

\[
\begin{align*}
A_n & \quad f = x^2 + y^2 + z^{n+1}, \quad n \geq 1, \\
D_n & \quad f = x^2 + y^2 z + z^n, \quad n \geq 4, \\
E_6 & \quad f = x^2 + y^3 + z^4, \\
E_7 & \quad f = x^2 + y^3 + yz^3, \\
E_8 & \quad f = x^2 + y^3 + z^5.
\end{align*}
\]

The category of mCM modules over \( R \) coincides with the category of reflexive modules. An important fact is that this category admits almost split sequences (also known as Auslander–Reiten sequences), a notion we shall now recall. This will be done for the category \( \text{mCM}(R) \) of maximal Cohen–Macaulay modules over a complete Cohen–Macaulay singularity \( R \). A short exact sequence

\[
0 \to A \to B \to C \to 0
\]

of mCM \( R \)-modules is said to be almost split if the following three conditions are satisfied:
(1) The sequence is not split.
(2) Any map $X \to C$ which is not a splittable epimorphism with $X$ an indecomposable $mCM$ can be lifted to $B$.
(3) Any map $A \to Y$ which is not a splittable monomorphism with $Y$ an indecomposable $mCM$ can be extended to $B$.

If such a sequence exists, then the end terms are both indecomposable and each one determines the other up to isomorphism. One says that the category of $mCM$ $R$-modules has almost split sequences if for each indecomposable $mCM$ module $A$, which is not isomorphic to the dualizing module $\omega_R$ of $R$, there is an almost split sequence in $mCM(R)$ that starts with $A$. Equivalently, $mCM(R)$ has almost split sequences if for each indecomposable $mCM$ module $C$, which is not isomorphic to $R$, there is an almost split sequence in $mCM(R)$ that ends with $C$.

Returning to the Kleinian singularities, in any almost split sequence, the first and the last terms are isomorphic. Recall also that the (isoclasses of) indecomposable non-projective $mCM$ modules over such a singularity are in one-to-one correspondence with the vertices of the underlying Dynkin diagram. It can be deduced from [15] and [2] that the rank of such a module is equal to the coefficient of the corresponding vertex in the maximal root, as is shown below (see [6, Tables I, IV, and V–VII]):

Dynkin diagrams of types $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$. 
For further details, see [2].

**Theorem 3.** Let $R$ be a complete Kleinian singularity. Then any indecomposable non-projective $mCM$ module is horizontally self-linked.

**Proof.** Let $c(M)$ denote the divisor of a stable reflexive $R$-module $M$ and $\beta(M)$ the minimal number of generators of $M$. Since the multiplicity of a simple Arnold singularity is two, a result of Herzog and Kühl [13, Corollary 1.3], shows that $\beta(M) = 2 \text{rank} M$. Therefore the exact sequence $0 \to \Omega M \to R(\beta(M)) \to M \to 0$ shows that $\text{rank} \Omega M = \text{rank} M$ and therefore $\text{rank} \Omega M = \text{rank} M^* = \text{rank} M$. Since $R$ is Gorenstein, $M$ is indecomposable if and only if $\Omega M$ is. As a consequence, if $M$ is the only indecomposable reflexive of a given rank, then $M$ is horizontally self-linked by the previous lemma. We also have (see [5, Chapter VII, §4, Ex. 8.d]), that $c(\Omega M) = -c(M) = c(M^*)$ (the first equality holds by the additivity of the divisor). In particular, if $\text{rank} M = 1$ then, since $\Omega M$ and $M^*$ are both reflexive of rank one and since such modules are uniquely determined up to isomorphism by their divisors, we conclude that $\Omega M \simeq M^*$. Thus, by the previous lemma, any rank one reflexive is horizontally self-linked.

Now suppose that $M$ is an indecomposable non-projective reflexive module which is horizontally self-linked. Let

$$0 \to M \to N \to M \to 0 \quad (7.1)$$

be an almost split sequence ending with $M$. The module $N$ need not be indecomposable, it may even have a projective summand. Decomposing $N$ into a direct sum of a stable module and a projective we claim that the stable part is horizontally self-linked. This will be proved in three steps.

**Step 1.** We want to show that the dual sequence

$$0 \to M^* \to N^* \to M^* \to 0 \quad (7.2)$$

is almost split. Because all the modules in question are reflexive, it is clear that the dualized sequence is not split. To show that the lifting problem for non-isomorphisms from indecomposable reflexives to the contravariant copy of $M^*$ has a solution, we dualize the new sequence into $R$. It then turns back into the original almost split sequence and the lifting problem becomes the extension problem for non-isomorphisms from the covariant copy of $M$ into indecomposable reflexives. Solving this problem and dualizing the solution, we have a solution for the lifting problem. A similar argument shows that the extension problem is also solvable. Thus the dual of the almost split sequence is almost split.

**Step 2.** Using a minimal projective resolution of $M$ and the Horse–Shoe Lemma, we apply $\Omega$ to (7.1). The result can be written as

$$0 \to \Omega M \to \Omega' N \xrightarrow{\gamma'} \Omega M \to 0, \quad (7.3)$$
where $\Omega'N$ may differ from $\Omega N$ by a projective summand. We claim that this sequence is also almost split. First of all we have to show that it is not split. Suppose it is. Then $\Omega'N$ is isomorphic to $\Omega M \oplus \Omega M$. The latter module is stable and therefore $\Omega'N \simeq \Omega N$.

Since $\Omega$ is invertible on stable mCM's and since $M$ is stable, the stable part of $N$ is isomorphic to $M \oplus M$. Since rank is additive, the original almost split sequence shows that $N$ itself is isomorphic to $M \oplus M$. A result of Miyata [17, Theorem 1] then shows that sequence (7.1) is split, contrary to the assumption. Now we have to show that the lifting problem for non-isomorphisms from an indecomposable reflexive $X$ to the contravariant copy of $\Omega M$ has a solution. If $X$ is projective, we have nothing to prove. Suppose $X$ is not projective. Then any non-isomorphism $\alpha': X \to \Omega M$ can be realized as a lifting, along the corresponding projective resolutions, of a non-isomorphism $\Omega^{-1}X \to M$, where $\Omega^{-1}X$ is again indecomposable. This non-isomorphism can be lifted to $N$ and the result can then be lifted along the projective resolutions to a morphism $\beta': X \to \Omega'N$. A tedious but straightforward diagram chase shows that this morphism differs from a solution to the original lifting problem by a map factoring through a projective cover $Q$ of $\Omega^{-1}X$. In other words, there are maps $\psi'$ and $\tau$, shown in the diagram below.

```
X \psi' \rightarrow \Omega M
\Omega'N \tau \rightarrow \Omega M
```

such that $\alpha' = \gamma'\beta' + \tau\psi'$. Since $Q$ is projective, there exists a map $\delta: Q \to \Omega'N$ such that $\tau = \gamma'\delta$. Then $\alpha' = \gamma'(\beta' + \delta\psi')$. This solves the lifting problem. A similar argument applies to the extension problem. Thus sequence (7.3) is almost split.

**Step 3.** Having started with the almost split sequence (7.1), we have two new almost split sequences $0 \to M^* \to N^* \to M^* \to 0$ and $0 \to \Omega M \to \Omega'N \to \Omega M \to 0$. We have also assumed that $M$ is self-linked. Hence $\Omega M$ and $M^*$ are isomorphic. But the isomorphism class of an almost split sequence is uniquely determined by any of its end terms and therefore $\Omega'N$ is isomorphic to $N^*$. Using the Krull–Remak–Schmidt Theorem, we deduce that the stable part of $N$ is horizontally self-linked.

Now we can finish the proof of the theorem by examining the ranks of the indecomposable mCM's. Since all indecomposable reflexives over $A_n$ are of rank one, we are done in this case. To deal with the remaining cases, we need one more fact about almost split sequences for mCM's over a rational double point. Namely, the middle term of an almost split sequence ending with $M$ is the direct sum of the modules corresponding to the vertices adjacent to the vertex corresponding to $M$ in the extended Dynkin diagram of $R$ with the extra vertex corresponding to $R$ (see [2]). In the case of $D_n$, we start with the almost split sequences ending with the rank one modules and move to the adjacent modules until we cover all modules. In the case $E_6$, the same method shows that all indecomposables, except, possibly, the rank two module $A$ adjacent to the extra vertex,
are horizontally self-linked. But then $A$ must also be self-linked because both $\Omega$ and dualization into $R$ are involutions on the isomorphism classes of stable reflexives. In the case $E_7$ we start with the unique rank one module and show that the four modules on the long arm of the diagram are horizontally self-linked. Examining the almost split sequence ending with the rank four module, we deduce that the direct sum of the adjacent modules is self-linked. Their ranks are 3, 2, and 3. The first module is on the long arm, and is therefore horizontally self-linked. Since the remaining modules have different ranks, the Krull–Remak–Schmidt Theorem shows that each one of them is also horizontally self-linked. Finally, to treat the case of $E_8$, we notice that the rank six module, being the only module of this rank, must be horizontally self-linked. The adjacent vertices are all of different ranks and thus are also self-linked. The remaining modules can be reached, one by one, by almost split sequences.

Corollary 10. The operation $\lambda$ fixes the non-projective vertices of the Auslander–Reiten quiver of the category of mCM $R$-modules.

8. The operator $\lambda$ and mCM approximations of CM modules

Throughout this section $R$ will be a Gorenstein commutative local ring. We begin by recalling the definition of mCM approximations introduced by Auslander and Buchweitz [4]. An mCM approximation of a finitely generated $R$-module $C$ is a short exact sequence $0 \rightarrow Y_C \rightarrow X_C \xrightarrow{f} C \rightarrow 0$, where $X_C$ is mCM and $Y_C$ is of finite injective (i.e., finite projective) dimension. It is said to be minimal if any endomorphism $h : X_C \rightarrow X_C$ such that $f = fh$ is necessarily an automorphism.

In the beginning of this paper we saw that a stable mCM $R$-module is horizontally linked and its link is again mCM. Trivially, since an mCM module is its own mCM approximation, this can be interpreted as a result about the link of an mCM approximation. Thus we can pose a more general question: what happens with mCM approximations of arbitrary modules under horizontal linkage? In this section we shall answer this question for non-maximal Cohen–Macaulay modules and for their arbitrary syzygy modules.

Let $A$ be a CM $R$-module of codepth $n := \text{depth } R - \text{depth } A \geq 1$ and

$$(P, d) : \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow A \rightarrow 0$$

a minimal projective resolution of $A$. By the duality for CM modules, the groups $\text{Ext}^i_R(A, R)$ vanish for all $i \neq n$. In particular, $A^* = 0$ and the sequence $0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr} A \rightarrow 0$ is exact. Hence $\lambda A \simeq P_0^*$ is projective and, by Proposition 4, p. 593, $A$ itself is not horizontally linked. However we can ask the more meaningful question of what happens with mCM approximations of $A$ and of its syzygy modules $\Omega^iA$, $i = 1, \ldots, n - 1$, under $\lambda$. For $i = n$ the corresponding syzygy module is mCM and we have already answered the question above.

Our main technical tool for answering that question will be the gluing construction for CM modules of Herzog and Martsinkovsky [12]. We briefly recall the relevant basic facts.
Let $A^\vee$ denote the $R$-module $\text{Ext}^n(A, R)$ and let

$$(Q, \partial) : \cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{} A^\vee \to 0$$

be a minimal projective resolution of it. By the duality for CM modules, $A^\vee$ is a CM module of codepth $n$. Hence the homology of $(Q^\ast, \partial^\ast)$ is concentrated in degree $-n$.

Consequently,

$$\begin{align*}
\lambda(\Omega^i(A^\vee)) &\simeq \text{im}(\partial^\ast_{i+1}) \simeq \ker(\partial^\ast_{i+2}) \quad \text{for } i \neq n-1, \\
\text{proj dim}(\partial^\ast_{i+1}) &\equiv i \quad \text{for } i = 0, \ldots, n-1, \\
\ker(\partial^\ast_{i+1}) &\text{ is } m\text{CM for } i \geq n.
\end{align*} \quad (8.1)$$

Recall also that, by the duality for CM modules, $A^{\vee\vee}$ is isomorphic to $A$. As a consequence, dualizing $Q$ into $R$ and shifting it $n$ steps to the left, we can find a quism $\nu : P \to Q^\ast[-n]$ (called a gluing for $A$):

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{} P_1 \xrightarrow{d_1} P_0 \xrightarrow{} 0$$

Its mapping cone, which we denote $\text{Con}(\nu)$, is an exact complex of projectives. Thus the homology of any truncation of $\text{Con}(\nu)$, being an infinite syzygy module, is mCM. Since mCM's have trivial higher Ext-modules with coefficients in $R$, we conclude that $\text{Con}(\nu)$ is in fact a complete resolution, i.e., besides being exact, it also remains exact when dualized into $R$. Associated with $\nu$ is the corresponding cone-cylinder diagram

$$\begin{array}{ccccccc}
0 & \to & Q^\ast[-n] & \xrightarrow{} & \text{Con}(\nu) & \xrightarrow{} & P[-1] & \to & 0 \\
\alpha & & \downarrow & & \downarrow & & \downarrow & & = \\
0 & \to & P & \xrightarrow{} & \text{Cyl}(\nu) & \xrightarrow{} & \text{Con}(\nu) & \to & 0 \\
\beta & & \downarrow & & \downarrow & & \downarrow & & \beta \\
P & \to & Q^\ast[-n]
\end{array}$$

where the top and the middle rows are exact, and $\alpha$ and $\beta$ are homotopy inverses of each other. Applying to the top row the degree $i+1$ truncation functor $\tau_{i+1}$ we have an exact sequence

$$0 \to \tau_{i+1}Q^\ast[-n] \to \tau_{i+1}\text{Con}(\nu) \to \tau_{i+1}P[-1] \to 0$$

---

5 Henceforth, the term quism will be used in place of the term quasiisomorphism.
for each integer $i$. For $i = 0, \ldots, n - 1$ the corresponding long cohomology exact sequence degenerates into a short exact sequence which is a minimal mCM approximation of $\Omega^i A$. In view of the first isomorphism in the first line of (8.1), which holds for all $i$, this approximation can be written as
\[ 0 \rightarrow \lambda(\Omega^{n-i-1}(A^\vee)) \rightarrow X_{\Omega^i A} \rightarrow \Omega^i A \rightarrow 0, \]
where $X_{\Omega^i A}$ is isomorphic to $\text{im}[\begin{array}{c} -d_i \\
\nu_i \\
\delta_{n-i}^{i} \end{array}]$, $i = 0, \ldots, n - 1$. Replacing $A$ by $A^\vee$ and $i$ by $n - i - 1$ we have, for $i = 0, \ldots, n - 1$, the exact sequences
\[ 0 \rightarrow \lambda(\Omega^i A) \rightarrow X_{\Omega^{n-i-1}(A^\vee)} \rightarrow \Omega^{n-i-1}(A^\vee) \rightarrow 0. \]
These approximations are obtained by the gluing construction for $A^\vee$. Since the complete resolution $\text{Con}(v)$ remains exact when dualized into $R$, the chain map $v^n[n]: Q \rightarrow P^n[-n]$ is a gluing for $A^\vee$. Taking the appropriate truncation of $\text{Con}(v^n[n])$ we obtain a projective resolution of $X_{\Omega^{n-i-1}(A^\vee)}$. Notice that this resolution is not, in general, minimal. (Its deviation from minimality is measured by Auslander’s deltas.) If we compute $\lambda$ (i.e., both $\Omega$ and $\text{Tr}$) with this non-minimal resolution, then we find that $\lambda X_{\Omega^{n-i-1} A^\vee}$ is isomorphic to $\text{im}[\begin{array}{c} -d_i \\
\nu_i \\
\delta_{n-i}^{i} \end{array}]$, i.e., to $X_{\Omega^i A}$. By symmetry, using the truncations of the (in general, non-minimal) complete resolution $\text{Con}(v)$ we have that $\lambda X_{\Omega^i A}$ is isomorphic to $X_{\Omega^{n-i-1}(A^\vee)}$. Returning to minimal resolution we have that the obtained isomorphisms should be replaced by stable isomorphisms. As a result, we have proved the following

**Theorem 4.** Let $R$ be a Gorenstein commutative local ring and $A$ a Cohen–Macaulay module of codepth $n \geq 1$. Let
\[ 0 \rightarrow Y_{\Omega^i A} \rightarrow X_{\Omega^i A} \rightarrow \Omega^i A \rightarrow 0 \]
and
\[ 0 \rightarrow Y_{\Omega^{n-i-1} A^\vee} \rightarrow X_{\Omega^{n-i-1} A^\vee} \rightarrow \Omega^{n-i-1}(A^\vee) \rightarrow 0, \]
where $i = 0, \ldots, n - 1$, be minimal mCM approximations. Then $\lambda(\Omega^i A) \cong Y_{\Omega^{n-i-1}(A^\vee)}$, and $\lambda X_{\Omega^i A}$ is isomorphic to the stable part of $X_{\Omega^{n-i-1}(A^\vee)}$ for $i = 0, \ldots, n - 1$. If $i = 0, \ldots, n - 2$, then $\lambda(Y_{\Omega^i A}) \cong \Omega^{n-i-1}(A^\vee)$. If $\lambda$ is computed using the truncations of the not necessarily minimal complete resolution $\text{Con}(v)$ for $A$, then $\lambda X_{\Omega^i A}$ and $X_{\Omega^{n-i-1} A^\vee}$ are isomorphic.

**Corollary 11.** Under the assumptions of the theorem, suppose $A$ is a self-dual CM module (e.g., the residue field) of codepth at least one. Then $\lambda X_{\Omega^i A}$ is isomorphic to the stable part of $X_{\Omega^{n-i-1} A}$ for $i = 0, \ldots, n - 1$. If $\lambda$ is computed using the truncations of the (non-minimal) complete resolution $\text{Con}(v)$ for $A$, then $\lambda X_{\Omega^i A}$ and $X_{\Omega^{n-i-1} A}$ are isomorphic. In particular, if $n$ is odd, then the stable part of $X_{\Omega^{(n-1)/2} A}$ is a horizontally self-linked mCM module.
Remarks. (1) The last corollary can also be proved by direct examination of the gluing construction for $A$.

(2) Using a very compact, if not ambiguous, notation it is possible to prove the statement of Theorem 4 regarding the approximating mCM module in just a couple of lines. Namely, let $D$ denote the duality for CM modules and $X$ the operation of taking an mCM approximation. Then the statement of the theorem can be encoded as $\Omega^{-1}DX\Omega^iA \simeq X\Omega^{n-i-1}DA$. The gluing construction shows that $\Omega$ and $D$ commute with $X$ and that $\Omega^{n-i}D$ equals $D\Omega^i$. The desired result now follows after straightforward manipulations. The details are left to the reader.

9. The operator $\lambda$ and the bounded derived category

Let $R$ be a Gorenstein commutative local ring. Our goal is to generalize a theorem of Schenzel which gives an invariant of linkage with values in the derived category of $R$. We begin by recalling that result.

Let $a$ be an ideal of $R$ of pure height $g$ and $E_R$ a minimal injective resolution of $R$ as an $R$-module. Of primary importance to us is the dualizing complex for $R/a$, defined as (see [11, Chapter V])

$$I_a := \text{Hom}_R(R/a, E_R),$$

whose homology is, by definition, $\text{Ext}_R^g(R/a, R)$. In the derived category of $R$, this complex is isomorphic to $\mathbb{R}\text{Hom}_R(R/a, R)$. It is well known that $I_a$ is concentrated between (upper) degrees $g$ and $\dim R$. The first non-vanishing cohomology module $\text{Ext}_R^g(R/a, R)$ is the canonical module $K_a$ for $R/a$. Viewing $K_a[-g]$ as a subcomplex of $I_a$ we have a short exact sequence of complexes

$$0 \to K_a[-g] \to I_a \to J_a \to 0.$$

By construction,

$$H^i(J_a) \simeq \begin{cases} 
H^i(I_a) & \text{if } i \neq g, \\
0 & \text{if } i = g.
\end{cases}$$

We can now state the theorem of Schenzel [19, Theorem 3.1].

Theorem 5. Let $a$ and $b$ be ideals of $R$ linked by a Gorenstein ideal $c$ of height $g$. Then there exists a canonical isomorphism

$$J_b[g] \cong \mathbb{R}\text{Hom}_R(J_a, R)$$

in the derived category of $R$.

Thus the truncated dualizing complexes of linked ideals are isomorphic up to a shift and duality. The generalization of this theorem will be accomplished in two steps. First
we prove it for horizontal linkage of modules over noetherian semiperfect rings of finite injective dimension and then for general linkage over Gorenstein rings. Moreover, not only shall we generalize the above theorem, but we shall also give a more precise statement, turning, in particular, our result into a derived-categoric criterion for linkage. We shall also see that the horizontal component of linkage is responsible for the duality, whereas the vertical components are responsible for the shift.

Let $\Lambda$ be a noetherian semiperfect ring of finite injective dimension, $A$ a finitely generated $\Lambda$-module (this is the analog of $R/\alpha$), and

$$
\cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varphi} A \rightarrow 0
$$

a minimal projective resolution of $A$. Similar to the case of ideals, we define a complex $I_A$ as the $\Lambda$-dual of the projective resolution of $A$:

$$
I_A := 0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \cdots
$$

As above, this complex is isomorphic to $\mathbb{R} \text{Hom}_\Lambda(A, \Lambda)$ in the bounded derived category $D^b(\Lambda^{\text{op}})$. Notice however that we are using a projective resolution of the contravariant argument rather than an injective resolution of the covariant argument. The degree zero homology of this complex is isomorphic to $A^*$. Viewing it as a subcomplex of $I_A$ concentrated in degree zero, we have a short exact sequence of complexes

$$
0 \rightarrow A^* \rightarrow I_A \rightarrow J_A \rightarrow 0,
$$

where $J_A$ is isomorphic to the complex

$$
0 \rightarrow \lambda A \rightarrow P_1^* \rightarrow P_2^* \rightarrow \cdots
$$

with $\lambda A$ in degree zero. By construction, $J_A$ is exact in degree zero and, since $\Lambda$ is of finite injective dimension, this complex has bounded homology.

Before we state the first theorem we need the following auxiliary results:

**Lemma 7** [7, Proposition 5.2.4]. Let $u : C' \rightarrow C$ be a quism, $P$ a complex of projectives, and $E$ a complex of injectives.

(a) If $P$ is bounded on the right or if both $C$ and $C'$ are bounded on the right, then $\text{Hom}_\Lambda(1, u) : \text{Hom}_\Lambda(P, C') \rightarrow \text{Hom}_\Lambda(P, C)$ is also a quism.

(b) If $E$ is bounded on the left or if both $C$ and $C'$ are bounded on the left, then $\text{Hom}_\Lambda(u, 1) : \text{Hom}_\Lambda(C, E) \rightarrow \text{Hom}_\Lambda(C', E)$ is also a quism.

**Lemma 8.** Let $\Lambda$ be a ring of finite injective dimension, $\iota : A \rightarrow I$ a finite injective resolution of $\Lambda$, and $\pi : P \rightarrow C$ a quism, where $P$ is a complex of projective $\Lambda$-modules. Then $\text{Hom}_\Lambda(C, I)$ is quasiisomorphic to $\text{Hom}_\Lambda(P, \Lambda)$.

\[\text{See the footnote on p. 607.}\]
Proof. By Lemma 7(b), Hom\(_{\Lambda}(C, I)\) is quasiisomorphic to Hom\(_{\Lambda}(P, I)\). Since both \(\Lambda\) and \(I\) are bounded, Lemma 7(a), shows that Hom\(_{\Lambda}(P, I)\) quasiisomorphic to Hom\(_{\Lambda}(P, \Lambda)\). 

The just proved lemma shows that for rings \(\Lambda\) of finite injective dimension, one can use unbounded projective complexes as contravariant arguments of \(\text{RHom}_{\Lambda}(-, \Lambda)\).

Now we can state and prove the promised criterion for horizontal linkage.

**Theorem 6.** Let \(\Lambda\) be a noetherian semiperfect ring of finite injective dimension and \(A\) a finitely generated stable \(\Lambda\)-module. Then there is a morphism

\[ \tau : \text{RHom}_{\Lambda}(J_A, \Lambda) \to J_{\lambda A} \]

in \(D^b(\Lambda)\) which induces an isomorphism on homology in degrees different from zero. Moreover the following are equivalent:

(i) \(A\) is horizontally linked.

(ii) \(\tau\) is an isomorphism in \(D^b(\Lambda)\).

(iii) \(\text{RHom}_{\Lambda}(J_A, \Lambda)\) and \(J_{\lambda A}\) are isomorphic in \(D^b(\Lambda)\).

Proof. We begin by constructing the morphism \(\tau\). Let

\[ \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\psi} A^* \to 0 \]

be a minimal projective resolution of \(A^*\). Splicing it with \(0 \to A^* \xrightarrow{\varphi^*} P_0^* \xrightarrow{d_1^*} P_1^* \to \cdots\) we have a complex

\[ \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\varphi^*} P_0^* \xrightarrow{d_1^*} P_1^* \to \cdots \]

which we identify with the mapping cone \(\text{Con}(\alpha)\) of \(\alpha := \varphi^* \psi\) (this is the map which we have already encountered in the proof of Lemma 4 on p. 598. There is also an obvious quism \(\text{Con}(\alpha) \to J_A\) given by the commutative diagram

\[ \cdots \to Q_0 \xrightarrow{\alpha = \varphi^* \psi} P_0^* \xrightarrow{d_1^*} P_1^* \xrightarrow{\lambda \Lambda} P_2^* \to \cdots \]

where the left-most vertical map is the image of \(d_1^*\). Dualizing the top row into \(\Lambda\) we have a complex

\[ \cdots \to P_1^{**} \xrightarrow{\alpha^*} Q_0^* \xrightarrow{\partial_1^*} Q_1^* \to \cdots \]
We shall identify this complex with \( \text{Con}(\psi^*\varphi^{**})[1] \). By Lemma 8, it is quasiisomorphic to \( R\text{Hom}_\Lambda(\mathcal{J} A, \Lambda) \). Since \( \Lambda \) is of finite injective dimension, this quism is an isomorphism, call it \( \rho \), in \( D^b(\Lambda) \).

Switching to \( \lambda A \), we compute
\[
\mathcal{J}_{\lambda A} \cong 0 \to \lambda^2 A \to Q_0^* \to Q_1^* \to \cdots
\]
(thus \( \lambda^2 A \) is in degree 0 and \( Q_0^* \) is in degree \(-1\)) and define a morphism \( \tau := \varsigma \rho : R\text{Hom}_\Lambda(\mathcal{J} A, \Lambda) \to \mathcal{J}_{\lambda A} \) by composing the just defined isomorphism \( \rho \) with the morphism
\[
\varsigma : \text{Con}(\psi^*\varphi^{**})[1] \to \mathcal{J}_{\lambda A}
\]
(9.1) given by the commutative diagram
\[
\begin{array}{ccccccc}
\cdots & P_2^{**} & P_1^{**} & P_0^{**} & \sigma^* = \psi^*\varphi^{**} & Q_0^* & Q_1^* & \cdots \\
\downarrow \psi^{**} & \downarrow \delta_1^* & \downarrow \delta_1^* & \downarrow \delta_1^* & \downarrow \delta_1^* & \downarrow \delta_1^* & \downarrow \delta_1^* & \cdots \\
0 & \lambda^2 A & \psi^* & Q_0^* & Q_1^* & \cdots
\end{array}
\]
where the degree zero map, abusively denoted \( \varphi^{**} \), is obtained from \( \varphi^{**} \) by restricting the codomain to \( \lambda^2 A \). Because \( A \) is assumed to be stable, Lemma 4 on p. 598 shows that this map, being the image of \( \varphi^{**} \), is well-defined. Similarly, the degree 0 part of the differential of \( \mathcal{J}_{\lambda A} \), abusively denoted \( \psi^{*} \), should be understood as the restriction of \( \psi^{*} : A^{**} \to Q_0^* \) to \( \lambda^2 A \). Again, by Lemma 4 on p. 598, this map is well-defined.

It is immediate that \( \tau \) induces an isomorphism on homology in each degree except, possibly, in degree zero. Since \( \mathcal{J}_{\lambda A} \) is exact in degree zero, \( \tau \) is an isomorphism in the derived category if and only if the top complex is also exact in degree zero. Thus conditions (ii) and (iii) are equivalent. It remains to show that conditions (i) and (ii) are also equivalent. Because \( \psi^{*} \) is a monomorphism, \( \ker \psi^{*} \varphi^{**} = \ker \varphi^{**} \). Since finitely generated projectives are reflexive, \( \varphi^{**} \) can be written as the composition
\[
P_0 \xrightarrow{\varphi} A \xrightarrow{e_A} A^{**}.
\]
But \( P_1 \to P_0 \xrightarrow{\psi} A \) is exact at \( P_0 \) and therefore \( P_1 \to P_0 \xrightarrow{\psi^{**}} A^{**} \) is exact at \( P_0 \) if and only if the canonical evaluation map \( e_A : A \to A^{**} \) is a monomorphism. Since \( A \) is stable by assumption, we are done by Theorem 2 on p. 599. □

As a consequence of the proof of the theorem we have an immediate

**Corollary 12.** Under the assumption of the theorem, the following are equivalent:

(i) \( A \) is horizontally linked.
(ii) $\mathbb{R} \text{Hom}_{\Lambda}^{\text{op}}(J_A, \Lambda)$ and $J_{\lambda A}$ have isomorphic homology.

(iii) $\mathbb{R} \text{Hom}_{\Lambda}^{\text{op}}(J_A, \Lambda)$ and $J_{\lambda A}$ have isomorphic homology in degree zero.

Remarks. (1) If $\Lambda$ is not of finite injective dimension, then both the theorem and the corollary remain true if $D^b(\Lambda)$ is replaced by the category of complexes of $\Lambda$ modules, $\mathbb{R} \text{Hom}_{\Lambda}^{\text{op}}(J_A, \Lambda)$ is replaced by $\text{Con}(\psi^* \phi^{**})[1]$, and isomorphisms are replaced by quisms.

(2) Using the homotopy uniqueness of a projective resolution of a module, it is easy to see that the homotopy types of complexes $\text{Con}(\psi^* \phi^{**})$ and $J_A$, and therefore of the complexes $\text{Con}(\psi^* \phi^{**})[1]$ and $J_{\lambda A}$, are uniquely determined. It is equally straightforward to check that the morphism $\varsigma : \text{Con}(\psi^* \phi^{**})[1] \rightarrow J_{\lambda A}$ is canonical in the homotopy category of complexes.

We already mentioned in the proof of the theorem that $\tau$ is a quism if and only if it induces an isomorphism on the degree zero homology groups. But $J_{\lambda A}$ is exact in degree zero. As a consequence, it suffices to check only the degree zero homology of the complex

$$\cdots \rightarrow P_{1}^{**} \rightarrow P_{0}^{**} \xrightarrow{\alpha^*} Q_{0}^* \rightarrow Q_{1}^* \rightarrow \cdots$$

And, of course, we no longer need the finiteness assumption on the injective dimension of $\Lambda$. This is summarized in

**Theorem 7.** Let $\Lambda$ be a noetherian semiperfect ring and $A$ a finitely generated $\Lambda$-module. Then $A$ is horizontally linked if and only if the complex

$$\cdots \rightarrow P_{1}^{**} \rightarrow P_{0}^{**} \xrightarrow{\alpha^*} Q_{0}^* \rightarrow Q_{1}^* \rightarrow \cdots$$

is exact in degree zero (i.e., at $P_{0}^{**} \cong P_{0}$).

Now we shall treat the case of non-horizontal linkage. First we want to recall some preparatory results. By a soft truncation $\sigma_n K$ of a complex

$$K \cdots \rightarrow K_n \xrightarrow{d_n} K_{n-1} \rightarrow \cdots$$

of modules we shall understand the complex

$$0 \rightarrow \text{im} d_n \rightarrow K_{n-1} \rightarrow K_{n-2} \rightarrow \cdots$$

obtained from $K$ by replacing $d_n$ by the inclusion of its image in $K_{n-1}$, replacing each $K_i$ for $i \geq n+1$ by the zero module, and leaving all other components unchanged. Extending this definition to chain maps in an obvious way, one easily checks that $\sigma_n$ becomes an endofunctor on the category of complexes. Notice that for any complex $K$ the complex $\sigma_n K$ is exact in degree $n$. The commutation relation between truncation and shift functors is given by
Lemma 9. $\sigma_i(K[n]) \cong (\sigma_{i+n}K)[n]$.

In the rest of this section we shall work over a Gorenstein commutative local ring $R$. Let $c$ be a Gorenstein ideal of $R$ of height $g$, $\overline{R} := R/c$, and $A$ an $\overline{R}$-module. The next two results are well-known.

Lemma 10. $\mathbb{R}\text{Hom}_R(\overline{R}, R)$ is isomorphic to $\overline{R}[g]$.

Lemma 11. Let $K$ be a complex of $\overline{R}$-modules. Then $\mathbb{R}\text{Hom}_R(K, R)$ is isomorphic to $\mathbb{R}\text{Hom}_R(K, \overline{R})[g]$.

Proof.

$\mathbb{R}\text{Hom}_R(K, R) \cong \mathbb{R}\text{Hom}_R(K \otimes_R \overline{R}, R) \cong \mathbb{R}\text{Hom}_R(K, \mathbb{R}\text{Hom}_R(\overline{R}, R))$.

By the previous lemma, the last complex is isomorphic to $\mathbb{R}\text{Hom}_R(K, \overline{R})[g]$. $\square$

Recalling the complex $I_A$ constructed earlier, we shall refer to it as $\mathbb{R}\text{Hom}_R(A, R)$. We then define $J_A$ as $\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R)$. Because of the presence of two rings $R$ and $\overline{R}$, we need a more precise notation and we shall denote the above complexes as $I_A^R$ and $J_A^R$, respectively. Similarly, we define $I_A^R$ as $\mathbb{R}\text{Hom}_R^\ast(A \otimes_R \overline{R})$ and $J_A^R$ as $\sigma_{-g}\mathbb{R}\text{Hom}_R^\ast(A \otimes_R \overline{R})$.

Lemma 12. In $D^\bullet(R)$, there is an isomorphism

$\mathbb{R}\text{Hom}_R^\ast(\sigma_{-g}\mathbb{R}\text{Hom}_R^\ast(A, \overline{R}), \overline{R}) \cong \mathbb{R}\text{Hom}_R^\ast(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R), \overline{R})$.

Proof.

$\mathbb{R}\text{Hom}_R^\ast(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R), \overline{R})$

$\cong \mathbb{R}\text{Hom}_R(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, \overline{R}), \overline{R})[-g]$ (by Lemma 11)

$\cong \mathbb{R}\text{Hom}_R(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R)[-g], R)[-g]$ (by Lemma 11)

$\cong \mathbb{R}\text{Hom}_R(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R)[-g], R)[-g]$ (by Lemma 11)

$\cong \mathbb{R}\text{Hom}_R(\sigma_{-g}\mathbb{R}\text{Hom}_R(A, R), \overline{R})$. $\square$

In the next lemma we shall make use of the operation $\lambda$ over the ring $\overline{R}$ applied to $A$. To avoid confusion, we denote the result by $\lambda\overline{R}A$.

Lemma 13. There is an isomorphism of complexes of $R$-modules

$\sigma_{-g}\mathbb{R}\text{Hom}_R^\ast(\lambda\overline{R}A, \overline{R}) \cong (\sigma_{-g}\mathbb{R}\text{Hom}_R(\lambda\overline{R}A, R))[-g]$.

$^7$ Thus $J_A$ depends not only on $A$ but on the non-negative integer $g$. 
Proof. We use the same arguments as above: by Lemma 11, there is an isomorphism 
\[ \sigma_0 R \text{Hom}_R(\lambda \mathfrak{P} A, R) \cong \sigma_0 (R \text{Hom}_R(\lambda \mathfrak{P} A, R)[-g]). \] By Lemma 9, the last module is isomorphic to \((\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g]. \)

Combining Theorem 6 on p. 611 and the last two lemmas we have

**Theorem 8.** Let \( R \) be a Gorenstein commutative local ring, \( \mathfrak{r} \) a Gorenstein ideal of \( R \) of height \( g \), \( \mathfrak{R} := R/\mathfrak{r} \), and \( A \) a finitely generated stable \( \mathfrak{R} \)-module. Then there exists a morphism

\[ \chi : R \text{Hom}_R(\sigma_{-g} R \text{Hom}_R(A, R), R) \rightarrow (\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g] \]

in \( D^b(R) \) which induces an isomorphism on homology in degrees different from zero. Moreover the following are equivalent:

(i) \( A \) is linked by \( \mathfrak{r} \) (i.e., \( A \) is horizontally linked over \( \mathfrak{R} \)).

(ii) \( \chi \) is an isomorphism in \( D^b(R) \).

(iii) \( R \text{Hom}_R(\sigma_{-g} R \text{Hom}_R(A, R), R) \) and \( (\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g] \) are isomorphic in \( D^b(R) \).

Proof. We can construct \( \chi \) as the composition of the following maps:

\[ R \text{Hom}_R(\sigma_{-g} R \text{Hom}_R(A, R), R) \]

\[ \xrightarrow{\sim} R \text{Hom}_R(\sigma_0 R \text{Hom}_R(A, R), \mathfrak{R}) \] (by Lemma 12)

\[ \xrightarrow{=} R \text{Hom}_R(\mathfrak{J}_{\mathfrak{P} A}, \mathfrak{R}) \] (by the definition of \( \mathfrak{J}_{\mathfrak{P} A} \))

\[ \xrightarrow{\tau} \mathfrak{J}_{\mathfrak{P} A} \] (by Theorem 6 on p. 611)

\[ \xrightarrow{=} \sigma_0 R \text{Hom}_R(\lambda \mathfrak{P} A, \mathfrak{R}) \] (by the definition of \( \mathfrak{J}_{\mathfrak{P} A} \))

\[ \xrightarrow{=} (\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g] \] (by Lemma 13).

Now the theorem trivially follows from Theorem 6 on p. 611 and from the fact that, by definition, \( A \) is linked by \( \mathfrak{r} \) if and only if \( A \) is horizontally linked over \( \mathfrak{R}. \)

**Corollary 13.** Under the assumptions of the theorem, the following are equivalent:

(i) \( A \) is linked by \( \mathfrak{r} \).

(ii) \( R \text{Hom}_R(\sigma_{-g} R \text{Hom}_R(A, R), R) \) and \( (\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g] \) have isomorphic homology.

(iii) \( R \text{Hom}_R(\sigma_{-g} R \text{Hom}_R(A, R), R) \) and \( (\sigma_{-g} R \text{Hom}_R(\lambda \mathfrak{P} A, R))[-g] \) have isomorphic homology in degree zero.

The following result is based on Theorem 7 on p. 613.
Theorem 9. Under the assumptions of Theorem 8 on p. 615, \( A \) is linked by \( c \) if and only if \( R \text{Hom}_R(\sigma - g R \text{Hom}_R(A, R), R) \) is exact in degree zero.

Remark. The theorems proved in this section illustrate the difference between the horizontal and vertical components of linkage. The former is module-theoretic in nature and is amenable to a variety of techniques, thus making possible a high level of generality (semi-perfectness). The latter, a change of ring, is essentially ring-theoretic which makes it much harder to transfer information, whence much more stringent restrictions on the rings (Gorenstein commutative local). The difference between vertical and horizontal components is similar to the one between linear and non-linear phenomena.

We conclude this section by recording the following simple observations the first two of which will be used later.

Lemma 14. Let \( (R, \mathfrak{m}, k) \) be a Gorenstein local ring, \( c \) a Gorenstein ideal, and \( A \) an \( R \)-module linked by \( c \). Then the height of \( c \) equals the grade of \( A \), i.e., the smallest number \( g \) such that \( \text{Ext}^i_R(A, R) \neq 0 \). In particular, the height of \( c \) is uniquely determined by \( A \).

Proof. Let \( \overline{R} := R/c \). By Lemma 11 on p. 614, \( \text{Ext}^i_R(A, \overline{R}) \) is isomorphic to \( \text{Ext}^i_R(A, \overline{R}) \) for all integers \( i \). Thus it suffices to show that \( \text{Hom}_R(\sigma, \overline{R}) \neq 0 \). Suppose this is not true. Since \( A \) is horizontally linked over \( \overline{R} \), it has no projective summands and therefore \( \text{Hom}_R(\sigma, \overline{R}) \) is the first syzygy module of \( \lambda \overline{R}A \), which, under our assumption would mean that \( \lambda \overline{R}A \) is projective. But \( \lambda \overline{R}A \) is horizontally linked to \( A \), a contradiction. \( \square \)

The just proved lemma has an immediate consequence for G-dimension. Recall [3, (4.13)] that over a Gorenstein ring the G-dimension of a module equals its grade. Whence

Corollary 14. Over Gorenstein rings, G-dimension is preserved under linkage.

Remark. When the projective dimension of a module is finite, it is equal to the G-dimension. However, in contrast with G-dimension, projective dimension need not, as we saw in Theorem 4 on p. 608, be preserved already under horizontal linkage. At the same time we shall see later that an even number of links by Gorenstein ideals of finite projective dimension does preserve the projective dimension.

As another consequence of the just proved results we have

Corollary 15. Under the assumptions of Lemma 14, the module \( A \) is Cohen–Macaulay of codepth \( g \) if and only if its link is Cohen–Macaulay of codepth \( g \).

Proof. The lemma shows that \( A \) is Cohen–Macaulay if and only if \( A \) is maximal Cohen–Macaulay over \( R/c \). Now the result follows from Corollary 2 on p. 596 and the fact that the codepth of a Cohen–Macaulay module equals its grade. \( \square \)
10. The operator λ and local cohomology

Our next goal is to relate local cohomology of the modules $A$ and $\lambda A$ over a Gorenstein commutative local ring $R$. This can be done under the additional assumption that the homology of $J_A$ is of finite length. Then we shall extend the obtained results by incorporating vertical operations (i.e., change of rings). The motivation comes from a result of Schenzel [19, Corollary 3.3] which can be formulated as follows:

**Proposition 12.** Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring, $a$ and $b$ ideals of $R$ linked by a Gorenstein ideal $c$, and $E$ an injective envelope of $k$. Suppose that the homology of $J_a$ is of finite length. Then there are canonical isomorphisms

$$H^i_m(R/b) \cong \text{Hom}_R(H^{n-i}_m(R/a), E),$$

where $i = 1, \ldots, n - 1$ and $n = \dim R/a = \dim R/b$ (assuming $n \geq 2$).

While our proof follows the idea of Schenzel’s proof, the specialization of our result to the case of ideals yields a more precise statement: the above modules are isomorphic when $b$ is the annihilator of $a$, even without linkage.

First we need to recall the local duality (see [11]).

**Proposition 13.** Let $(R, \mathfrak{m}, k)$ be a commutative local ring which is a factor ring of a Gorenstein local ring $S$ of dimension $n$. Let $A$ be a finitely generated $R$-module and $E$ the injective envelope of the $R$-module $k$. Then, for each integer $i$, there is a natural isomorphism

$$H^i_m(A) \cong \text{Hom}_R(\text{Ext}^{n-i}_S(A, S), E),$$

where the $S$-module structure on $A$ is induced by the homomorphism $S \to R$.

The above proposition is a particular case of the local duality for complexes, which we shall now state. First recall [21, Corollary 3.3] that if a ring $R$ has a dualizing complex, then it has a dualizing complex $D$ with

$$D^i \cong \bigsqcup_{p \in \text{Spec } R \atop \dim R/p = -i} E(R/p),$$

where $E(R/p)$ is an injective envelope of $R/p$. Also recall that, for a module $M$ and ideal $a$ of $R$, the symbol $\Gamma_a(M)$ denotes the submodule of $M$ of all elements that can be annihilated by various powers of $a$. It is immediate that $\Gamma_a(-)$ is a subfunctor of the identity functor and is thus left-exact. The right-derived functors of $\Gamma_a(-)$ are denoted $R\Gamma_a(M)$. The resulting values are known the local cohomology functors of $M$ at $a$. This construction can be adapted to a bounded below complex $X$ in place of the module $M$. To this end, choose a quasi $X \to E$, where $E$ is a bounded below complex of injective modules, and define $R\Gamma_2(X)$ as $\Gamma_a(E)$.
**Proposition 14** [21, Theorem 3.4]. Let \((R, m, k)\) be a commutative local ring with dualizing complex \(D\) as above, \(E\) an injective envelope of the \(R\)-module \(k\), and \(X\) a bounded below complex with finitely generated homology modules. Then there exists a quism

\[
\mathcal{R} \Gamma_m(X) \to \text{Hom}_R(\text{Hom}_R(X, D), E).
\]

We also need the following trivial observation.

**Lemma 15.** Let \(A\) be a finitely generated module and \(\pi : I_A \to J_A\) the canonical surjection (i.e., \(\pi\) is the identity map in negative degrees). Then, for any module \(M\), the induced map \((\pi, M) : (I_A, M) \to (I_A, M)\) gives rise to an isomorphism on homology in positive degrees.

**Proof.** Apply the functor \((- , M)\) to the exact sequence \(0 \to A^* \to I_A \to J_A \to 0\) and use the fact that the induced sequence in degree zero is left-exact. \(\square\)

Now we can state the main result of this section.

**Theorem 10.** Let \((R, m, k)\) be a Gorenstein commutative local ring of dimension \(n \geq 2\), \(A\) a finitely generated stable \(R\)-module such that the homology of \(J_A\) is of finite length, and \(E\) an injective envelope of \(k\). Then, for each \(i = 1, \ldots, n - 1\) there is an isomorphism

\[
H^i_m(\lambda A) \xrightarrow{\cong} \text{Hom}_R(H^{n-i}_m(A), E).
\]

**Proof.** Under the above assumptions, Theorem 6 on p. 611 yields a morphism

\[
\tau = \varsigma \rho : \mathbb{R} \text{Hom}_R(J_A, R) \to J_{\lambda A}.
\]

The explicit construction of \(\varsigma\) shows that \(\tau\) is a homology isomorphism in each degree except, possibly, zero. Since \(E\) is injective, the same is true for the map

\[
(\tau, E) : \text{Hom}_R(J_{\lambda A}, E) \to \text{Hom}_R(\mathbb{R} \text{Hom}_R(J_A, R), E).
\]

The assertion of the theorem will be proved by identifying the local cohomology modules with the homology of the two complexes.

Let \(D\) be the normalized dualizing complex of \(R\). Then \(R\) is isomorphic to \(D[n]\) and therefore \(\text{Hom}_R(\mathbb{R} \text{Hom}_A(J_A, R), E) \cong \text{Hom}_R(\mathbb{R} \text{Hom}_A(J_A, D), E)[-n]\). By the local duality for complexes, the latter is isomorphic to \(\mathbb{R} \Gamma_m(J_A)[-n]\). Since the homology of \(J_A\) is, by assumption, of finite length, this complex is isomorphic in \(D^b(R)\) to \(J_A[-n]\). As a result,
$$H_i\left( \text{Hom}_R(\mathbb{R} \text{Hom}_A(J_A, R), E) \right) \cong H_{i-n}(J_A) = \begin{cases} \text{Ext}^{n-i}_R(A, R) & \text{for } i = 1, \ldots, n-1, \\ 0 & \text{for } i = n. \end{cases}$$

By the local duality for modules these groups are isomorphic $\text{Hom}_R(H^i_m(A), E)$ and 0, respectively. On the other hand, since $E$ is injective,

$$H_i\left( \text{Hom}_R(J_A, E) \right) \cong \text{Hom}_R(H^i_m(J_A), E) \cong \text{Hom}_R(\text{Ext}_R^i(\lambda A, R), E) \quad \text{for } i = 1, \ldots, n.$$

(If $i = 0$, the corresponding group is zero.) Using local duality again, we have that the homology in question is isomorphic to $H^{n-i}_m(\lambda A)$. Interchanging $i$ and $n - i$, we have the desired assertion. 

Now we can extend the just proved result to include a change of ring.

**Theorem 11.** Let $(R, m, k)$ be a Gorenstein commutative local ring, $\mathfrak{c}$ a Gorenstein ideal of $R$ of height $g$, $\overline{R} := R/\mathfrak{c}$, $d := \dim \overline{R}$. $A$ a finitely generated stable $\overline{R}$-module such that the homology of $J_A$ is of finite length, and $E$ an injective envelope of $k$. Assume that $d \geq 2$.

Then, for each $i = 1, \ldots, d-1$ there is an isomorphism

$$H^i_m(\lambda \overline{R}A) \xrightarrow{\cong} \text{Hom}_R(H^{d-i}_m(A), E).$$

The proof of this theorem is similar to the proof of its horizontal prototype except that one starts with the morphism $\chi$ of Theorem 8 on p. 615. The details are left to the reader.

**Remark.** Specializing to the case of the ideals we have an improvement of Schenzel’s result mentioned at the beginning of this section. In colloquial terms, under the assumptions of the theorem the local cohomology modules of an ideal and its annihilator are dual to each other.

**Corollary 16.** Under the assumptions of the theorem, suppose that $H^i_m(A)$ is of finite length for $i = 0, 1, \ldots, d - 1$. Then the same is true for $H^i_m(\lambda \overline{R}A)$.

**Proof.** Since $\lambda \overline{R}A$ is a first syzygy module, $H^0_m(\lambda \overline{R}A) = 0$. For the remaining values of $i$ the result follows from the theorem and Matlis duality.

Recall that a noetherian module of positive dimension is said to be Buchsbaum if its local cohomology modules in degrees from 0 to $d - 1$ are $k$-vector spaces. As an immediate consequence of the theorem we have

**Corollary 17.** Under the assumptions of the theorem, suppose that $A$ is a Buchsbaum module. Then $\lambda \overline{R}A$ is also Buchsbaum.
11. Even linkage classes

We keep the notation of the previous section. Thus \((R, m, k)\) is a Gorenstein local ring. Recall that two \(R\)-modules \(M\) and \(M'\) are said to be in the same even linkage class, or evenly linked, if there is a chain of linked modules of even length that starts with \(M\) and ends with \(M'\).

In [13], Herzog and Kühl proved the following result (Theorem 2.1).

**Theorem 12.** Let \(R\) be a local Gorenstein domain with infinite residue field \(k\). Let \(0 \to F' \to M' \to I' \to 0\) and \(0 \to F'' \to M'' \to I'' \to 0\) be any two Bourbaki sequences (i.e., \(F'\) and \(F''\) are free, \(M'\) and \(M''\) are maximal Cohen–Macaulay modules, and \(I'\) and \(I''\) are Cohen–Macaulay ideals of codimension 2). Then the following statements are equivalent:

(i) \(M'\) and \(M''\) are stably isomorphic.
(ii) \(I'\) and \(I''\) are evenly linked by complete intersections.

In the present day terminology, a Bourbaki sequence is just a particular case of an \(m\text{CM}\) approximation. Thus a natural question arises whether the theorem above is a particular case of a more general result. The main goal of this section is to show that the implication (ii) \(\Rightarrow\) (i) is true for arbitrary modules, without any assumption on the modules being Cohen–Macaulay and on their codimensions, and without any assumption on the residue field. Here is the precise statement.

**Theorem 13.** Let \((R, m, k)\) be a Gorenstein local ring, \(c_1\) and \(c_2\) Gorenstein ideals of finite projective dimension, and \(A_1, A_2\) \(R\)-modules such that \(A_1\) is linked to \(A_2\) by \(c_1\) and \(A\) is linked to \(A_2\) by \(c_2\). Then the \(m\text{CM}\) approximations of \(A_1\) and \(A_2\) are stably isomorphic.

**Proof.** Let \(R_1 := R/c_1\) and \(R_2 := R/c_2\). By the assumption, the \(R\)-modules \(R_1\) and \(R_2\) are of finite projective dimension. By Lemma 14 on p. 616, \(ht(c_1) = ht(c_2)\). Call this number \(g\). We begin with the claim that the first syzygy modules \(\Omega^1_{R_1} A_1\) and \(\Omega^1_{R_2} A_2\), computed, respectively, over the rings \(R_1\) and \(R_2\), are isomorphic over \(R\). Indeed, \(\Omega^1_{R_1} A_1 = \Omega^1_{R_1} \lambda_{R_1} A = \text{Hom}_{R_1}(A, R_1)\), the latter, by virtue of Lemma 11 on p. 614, being isomorphic \(\text{Ext}^1_R(A, R)\). The same argument applies to \(\Omega^1_{R_2} A_2\). The claim is proved. To simplify notation, we shall use the symbol \(\Omega^1\) to denote either of the two isomorphic modules.

Now we want to construct an \(m\text{CM}\) approximation of \(A_1\) as an \(R\)-module. Let \(R^n \to A\) be a projective cover of \(A\) as an \(R\)-module. By Nakayama’s lemma, the extension \(R^n_1 \to A\) of this map to \(R_1\) is a projective cover of \(A\) as an \(R_1\)-module. As the definition of \(\lambda\) shows, the projective cover of \(A_1 = \lambda_{R_1} A\), being the \(R_1\)-dual of \(R^n_1\), is isomorphic (non-canonically) to \(R^n_{A_1}\). Thus we have an exact sequence
of $R_1$-modules which we want to view as a sequence of $R$-modules. Let

$$0 \to \Omega^1 \to Y \Omega^1 \to X \Omega^1 \to 0$$

be a hull of finite projective dimension of $\Omega^1$ over $R$. This means that $Y \Omega^1$ is of finite projective dimension and $X \Omega^1$ is mCM. Taking the push-out of the two maps from $\Omega^1$ we have a commutative diagram

\[
\begin{array}{ccccccccc}
\text{0} & \text{0} & \text{0} \\
\downarrow & \downarrow & \downarrow \\
0 & \Omega^1 & R^n_1 & A_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & Y \Omega^1 & T & A_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
X \Omega^1 & X \Omega^1 & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & & & \\
\end{array}
\]

of $R$-modules with exact rows and columns. Since $R^n_1$ is of finite projective dimension over $R$ and $X \Omega^1$ is an mCM, the middle column is split and, therefore, $T \simeq X \Omega^1 \amalg R^n_1$. Thus the middle row can be rewritten as

$$0 \to Y \Omega^1 \to X \Omega^1 \amalg R^n_1 \to A_1 \to 0.$$

The exact sequence

$$0 \to \Omega^1 \amalg R^n_1 \to R^n \xrightarrow{\pi} R^n_1 \to 0,$$

where the last map is the canonical surjection with kernel of finite projective dimension, gives rise to the exact sequence

$$0 \to \Omega^1 \amalg R^n_1 \to X \Omega^1 \amalg R^n \xrightarrow{0, \pi} X \Omega^1 \amalg R^n_1 \to 0.$$
Taking the pull-back of the two maps into \( X \Omega^1_1 \oplus R^n_1 \) we have another commutative diagram

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
\Omega^1_1 R^n_1 & \Omega^1_1 R^n_1 \\
\downarrow & \downarrow \\
0 & L & X \Omega^1_1 & \oplus & R^n_1 & A_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & y \Omega^1_1 & X \Omega^1_1 & \oplus & R^n_1 & A_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

The usual properties of the pull-back and the Snake Lemma show that the rows and columns of this diagram are exact. Since \( L \) is an extension of modules of finite projective dimension, it is itself of finite projective dimension. Therefore the middle row is an mCM approximation of \( A_1 \). But the approximating module \( X \Omega^1_1 \oplus R^n_1 \) depends only on \( A_1 \). Since an mCM approximation is defined up to stable equivalence, we are done. \( \Box \)

**Corollary 18.** Under the assumptions of the theorem, \( A_1 \) is of finite projective dimension if and only if \( A_2 \) is.

**Proof.** A module is of finite projective dimension if and only if its mCM approximation is projective. \( \Box \)

Now we want to strengthen the last corollary.

**Proposition 15.** Under the assumptions of Theorem 13 on p. 620, \( \text{proj dim} \, A_1 = \text{proj dim} \, A_2 \).

**Proof.** By the preceding corollary, we may assume that both dimensions are finite. Then they coincide with the respective Gorenstein dimensions, which are preserved under linkage by Corollary 14 on p. 616. \( \Box \)

**Remarks.** (1) Thus even linkage over Gorenstein rings by Gorenstein ideals of finite projective dimension preserves projective dimension. Also notice that direct linkage over Gorenstein rings by arbitrary Gorenstein ideals preserves the property of being Cohen–Macaulay. Indeed any CM \( R \)-module linked by a Gorenstein ideal \( c \) is necessarily a maximal CM \( R/c \)-module (as its grade over \( R/c \) must be zero) and this property is preserved under horizontal linkage.
(2) In the special case of Cohen–Macaulay modules, the assertion of Theorem 13 on p. 620 was proved in [23, Corollary 1.6]. See the remark at the end of Section 3 for the definition of linkage used in that paper.

We conclude this section with three questions.

**Question 1.** Is there a generalization of the implication (1) ⇒ (2) in the Herzog–Kühl theorem? In other words, under what conditions the stable equivalence of the mCM approximations of two modules would imply that the modules are evenly linked? An obvious necessary condition is that the modules have the same grade. Another invariant of even linkage by Gorenstein ideals is given by

**Proposition 16.** Let \((R, m, k)\) be a Gorenstein local ring, \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\) Gorenstein ideals, and \(A_1, A, A_2\) \(R\)-modules such that \(A_1\) is linked to \(A\) by \(\mathfrak{c}_1\) and \(A\) is linked to \(A_2\) by \(\mathfrak{c}_2\). Then the \(R\)-modules \(\text{Ext}^i_{R_1}(A_1, R_1)\) and \(\text{Ext}^i_{R_2}(A_2, R_2)\) are isomorphic for each positive integer \(i\).

**Proof.** Let \(g\) be the common height of the ideals \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\). For each \(i > 0\) we have

\[
\text{Ext}^i_{R_1}(A_1, R_1) = \text{Ext}^{i+g}_{R}(A_1, R) = H^i(\lambda_{-g} R \text{Hom}_R(\lambda_{R_1} A, R)[-g]).
\]

By Theorem 8, the latter module is isomorphic to \(H^i(\lambda_{-g} R \text{Hom}_R(\lambda_{R_1} A, R)[-g])\) and this expression depends only on \(A\) and \(g\). \(\Box\)

**Question 2.** Under what assumptions can stable isomorphism of the mCM approximations in Theorem 13 on p. 620 be replaced by isomorphism? When the approximating mCM modules are isomorphic, Auslander’s delta-invariant of the evenly linked modules are equal. Of critical importance, since this situation occurs in a great variety of applications, is the case when the delta-invariant vanishes.

**Question 3.** Does Theorem 13 on p. 620 generalize further to linkage by Gorenstein ideals?

12. A generalization of a theorem of Golod

The goal of this section is to provide a module-theoretic generalization of the following result of Golod [10].

**Theorem 14.** Let \(R\) be a commutative noetherian local ring and \(K\) a perfect ideal of grade \(n\) such that \(\text{Ext}^n_R(R/K, R)\) is cyclic. Let \(I \supseteq K\) be another perfect ideal of grade \(n\). Then \(J := K :_R I\) is a perfect ideal of grade \(n\) and \(K :_R J = I\).

Before proving the theorem, Golod proves three lemmas which we shall also need. They are already stated in the language of modules. However the proof of Golod’s theorem itself only works for cyclic modules, and it is not clear if it can be modified to accommodate
non-cyclic modules. To circumvent the problem, we shall give a simple argument based on the Proposition 10 on p. 597, which works for any module. As always, all modules are assumed to be finitely generated.

We begin by recalling the auxiliary results from [10].

**Lemma 16.** Let $M$ be a perfect $R$-module of grade $n$. Then $\text{Ext}^n_R(M, R)$ is also a perfect $R$-module of grade $n$, and $\text{Ext}^n_R(\text{Ext}^n_R(M, R), R) \cong M$. In particular, $\text{Ann}_R \text{Ext}^n_R(M, R) = \text{Ann}_R M$.

**Corollary 19.** If $K$ is a perfect ideal of grade $n$ such that $\text{Ext}^n_R(R/K, R)$ is cyclic, then $\text{Ext}^n_R(R/K, R) = R/K$.

**Lemma 17.** Let $I$ be an ideal of grade $n$ and $M$ an $R$-module with $\text{Ann}_R M \supseteq I$. Then there is a functorial in $M$ isomorphism

$\text{Ext}^n_R(M, R) \cong \text{Hom}_R(M, \text{Ext}^n_R(R/I, R))$.

**Lemma 18.** Let $M$ be a perfect $R$-module of grade $n$ and $I$ an ideal of grade $n$ such that $\text{Ann}_R M \supseteq I$. Then the canonical map

$e : M \to \text{Hom}_R(M, \text{Ext}^n_R(R/I, R))$.

is bijective. In particular, if $I$ is perfect and $\text{Ext}^n_R(R/I, R)$ is cyclic, then $M$ is a reflexive $R/I$-module.

We are now ready to state and prove the promised generalization.

**Theorem 15.** Let $R$ be a commutative noetherian local ring, $I$ a perfect ideal of grade $n$ such that $\text{Ext}^n_R(R/I, R)$ is cyclic, and $\overline{R} = R/I$. Let $M$ be a perfect $R$-module of grade $n$ such that $\text{Ann}_R M \supseteq I$. Assume that $M$, as an $\overline{R}$-module, is stable. Then:

1. $M$ is linked by $I$.
2. The mapping cone $\text{Con}(\phi^*)$ constructed in Proposition 10 on p. 597 is a projective resolution of $\lambda_{\overline{R}} M$.
3. $\lambda_{\overline{R}} M$ is perfect of grade $n$.

**Proof.** Since $M$ is $\overline{R}$-stable, the first assertion follows immediately from Lemma 18 on p. 624. Lemma 17 on p. 624 and the proof of Proposition 10 on p. 597 show that, in the notation of that proposition, $\text{Con}(\phi^*)$ is a projective resolution of $\lambda_{\overline{R}} M$, thus proving the second assertion. It now follows that proj dim $\lambda_{\overline{R}} M \leq n + 1$. Since the map $\phi_0 : Q_0 \to P_0$ is an isomorphism, the last differential in $\text{Con}(\phi^*)$ is a split monomorphism and therefore proj dim $\lambda_{\overline{R}} M \leq n$. Now to prove the last assertion it suffices to show that the grade of $\lambda_{\overline{R}} M$ is $n$. Notice that $\text{Hom}_R(\text{Con}(\phi^*), R) \cong \text{Con}(\phi)[n + 1]$. The long homology exact sequence corresponding to the short exact sequence

$0 \to P \to \text{Con}(\phi) \to Q[-1] \to 0$
shows that the only possibly non-vanishing homology of $\text{Con}(\varphi)$, isomorphic to the first syzygy module of $M$ over $R$, is in lower degree 1. It is indeed nonzero since $M$ is assumed stable over $R$. This means that the first non-vanishing homology $\text{Hom}_R(\text{Con}(\varphi^*), R)$ is in upper degree $n$. In other words, grade $\lambda R M = n$. This finishes the proof of the theorem.

Acknowledgments

The question of whether linkage can be defined for modules was, to the best of our knowledge, first raised by Peter Schenzel in a private conversation with the second author who, shortly thereafter, succeeded in defining linkage for maximal Cohen–Macaulay modules over Gorenstein commutative local rings [20]. Our work on the present paper started as an attempt to deconstruct Schenzel’s paper [19] on liaison and duality. His approach to linkage was of great help to us. Thanks are also due to Ragnar-Olaf Buchweitz, Hans-Bjørn Foxby, and Jürgen Herzog for helpful discussions. Finally, we thank the referee for his/her careful reading of the manuscript and very useful comments on the style of the exposition.

Most of the results of this paper were obtained in the fall semester of 1996 which the first author spent at Utrecht University as a Visiting Fellow of the Dutch Research Council NWO. The work continued during visits of the second author to Northeastern University in March 1998 and of the first author to Utrecht University in March 1999. All these sojourns were partly supported by the hosting Departments. It is our pleasure to thank all three institutions for their help.

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