On Multivalued Martingales Whose Values May Be Unbounded: Martingale Selectors and Mosco Convergence

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Using classical results on the projective limit of a sequence of subsets, we show the existence of martingale selectors for a multivalued martingale (and supermartingale) with closed values in a separable Banach space \( X \). The existence of \( L^1(X) \)-bounded or uniformly integrable martingale selectors is also discussed. At last, applications to the Mosco convergence of multivalued supermartingales and supermartingale integrands are provided. © 1991 Academic Press, Inc.

1. INTRODUCTION

Multivalued martingales were introduced at the end of the sixties by Van Cutsem and, since, were studied by several authors. The notion of multivalued martingale extends those of real and vector martingale; indeed, the values of the random variables involved are closed convex subsets of a normed space, instead of real numbers or vectors. Of course, this extension is based on the definition and the study of the measurability and integrability of multifunctions (or correspondences), of the multivalued conditional expectation and of set-convergence, which were developed before or at the same time.

The main questions raised by multivalued martingales are: their convergence, their regularity, and the existence of martingales selections (with possibly additional properties); clearly, the third question is proper to the multivalued case.

Multivalued martingales whose values are bounded were studied by a lot...
of authors: Van Cutsem [26a, 26b], Neveu [21b], Daures [12a, 12b, 12c], Hiaï and Umegaki [17], Costé [11b], Luu [19a-19d], Bagchi [3] and Castaing, Touzani, and Valadier [8]. The more recent works among this list also take into account several extensions of the notion of martingale such as quasi-martingales, asymptotic martingales, etc. There are much fewer works on multivalued martingales with unbounded values: Van Cutsem [26c] and recently Choukairi-Dini [10a-10d]. Let us also mention Bismut [5] who studied martingale integrands.

The purpose of the present paper is to go on with the study of multivalued martingales (and, more generally, of supermartingales) with closed convex values, possibly unbounded, in a separable Banach space \( X \). Some preliminaries are given in Section 2. In Section 3, multivalued martingales, submartingales, and supermartingales are defined. Further, using the classical properties of the projective limit of a sequence of subsets, two results on the existence of martingale selections for a multivalued martingale are proved. In Section 4, we ask stronger properties for these martingale selections; the main result is an existence theorem of an \( L^1(X) \)-bounded (resp. uniformly integrable) martingale selection, for a multivalued supermartingale whose distance functions satisfy similar properties. In Section 5, after some preparatory lemmas, a convergence result, for multivalued supermartingales whose values may be unbounded, is given. It extends those obtained before by Van Cutsem [26c] and Choukairi-Dini [10a-10d] (even in the finite dimensional case); indeed, the existence of an \( L^1(X) \)-bounded (resp. uniformly integrable) martingale selection need no longer be taken as an hypothesis, like in [26c] and [10a, 10b], because it has been proved in Section 4. At last, Section 6 provides applications of the results of Section 5 to the convergence of supermartingale integrands.

The results of this paper can have applications in the field of stochastic optimization or control.

2. Notations and Preliminaries

Throughout this paper, \((\Omega, \mathcal{F}, P)\) denotes an abstract probability space, \( X \) a separable Banach space with the dual space \( X^* \). We denote by \( J \) the set of strictly positive integers and by \( \mathbb{R} \) (resp. \( \mathbb{R}^+ \)) the set of real numbers (resp. positive real numbers). For any \( \lambda \in \mathbb{R} \) we put

\[
\lambda^+ := \max(\lambda, 0) \quad \text{and} \quad \lambda^- := \max(-\lambda, 0)
\]

which are, respectively, the positive part and the negative part of \( \lambda \). Let \( 2^X \) be the set of all subsets of \( X \), \( \mathcal{B}(X) \) the Borel \( \sigma \)-field of \( X \), \( \mathcal{E} \) the family of non-empty closed convex subsets of \( X \), and \( \mathcal{X} \) the family of non-empty
weakly compact convex subsets of $X$. Given $C \subseteq 2^X$, the \textit{distance function} $d(\cdot, C)$ and the \textit{support function} $s(\cdot, C)$ of $C$ are defined by

$$
\begin{align*}
  d(x, C) &:= \inf_{y \in C} \|x - y\|, \\
  s(x^*, C) &:= \sup_{x \in C} \langle x^*, x \rangle.
\end{align*}
$$

We also define

$$
  h(C) := \sup_{y \in C} \|y\|
$$

which is the Hausdorff distance between $\{0\}$ and $C$. We also denote by $\overline{C}$ (resp. by $\overline{\overline{C}}$) the closure (resp. the closed convex hull) of $C$.

A \textit{multifunction} $F$, i.e., a map from $\Omega$ into $2^X$, is said to be \textit{measurable} if, for every open set $U$ of $X$, the subset of $2^X$ of $F \cap U$ is measurable.

$$
  F^{-1} U := \{ \omega \in \Omega / F(\omega) \cap U \text{ is non-empty} \}
$$

is a member of the $\sigma$-algebra $\mathcal{A}$. A function $f$ from $\Omega$ into $X$ is called a \textit{selector} if, for any $\omega \in \Omega$, one has $f(\omega) \in F(\omega)$.

A \textit{Castaing representation} of $F$ is a sequence $(f_n)_{n \in J}$ of measurable selection of $F$ such that

$$
  F(\omega) = \overline{\{ f_n(\omega) / n \in J \}} \quad \forall \omega \in \Omega.
$$

It is known (Theorem III.9 of [9]) that a multifunction $F$, with non-empty closed values in $X$, is measurable if and only if it has a Castaing representation or, if and only if, for any $x \in X$, the real function $d(x, F(\cdot))$ is measurable. In the sequel, two multifunctions $F$ and $G$ verifying $F(\omega) = G(\omega)$ for almost all $\omega \in \Omega$, will be identified. Let $L^1(\Omega, \mathcal{A}, P; X) = L^1(X)$ denote the Banach space of (equivalent classes of) measurable functions $f : \Omega \to X$ such that

$$
  \|f\| := \int_{\Omega} \|f(\omega)\| \; dP \quad \text{(also denoted by } E \|f\|)
$$

is finite. $L^1(\mathbb{R})$ is simply denoted by $L^1$. For any measurable multifunction $F$ we put

$$
  S^1(F, \mathcal{A}) := \{ f \in L^1(\Omega, \mathcal{A}, P; X) / f(\omega) \in F(\omega) \ \text{a.s.} \}.
$$

In this definition, $\mathcal{A}$ may be replaced by any sub-$\sigma$-field $\mathcal{B}$ of $\mathcal{A}$. $S^1(F, \mathcal{A})$ is closed if $F$ is closed valued and it is non-empty if and only if the function $d(0, F(\cdot)) \in L^1$. In such a case, we shall say that the multifunc-
tion $F$ is integrable. On the other hand, $F$ is said to be integrably bounded if the function $h(F(\cdot)) \in L^1$. The multivalued integral of $F$ is defined by

$$I(F) := \{ E(f) | f \in S^1(F, \mathcal{A}) \} ,$$

where $E(f) := \int f \, dP$ is the usual Bochner-integral of $f$. Because $I(F)$ is not always closed, we also use the notation $E(F) := \text{cl} \, I(F)$.

Given a sub-$\sigma$-field $\mathcal{B}$ of $\mathcal{A}$ and an integrable $\mathcal{A}$-measurable multifunction $F$, Hiaï and Umegaki [17] showed the existence of a $\mathcal{B}$-measurable integrable multifunction $G$ such that

$$S^1(F, \mathcal{B}) := \text{cl} \{ E(f | \mathcal{B}) | f \in S^1(F, \mathcal{A}) \} ,$$

the closure being taken in $L^1(X)$. $G$ is the (multivalued) conditional expectation of $F$ relative to $\mathcal{B}$ and is denoted by $E(F | \mathcal{B})$. For the basic properties of the multivalued conditional expectation we refer the reader to [9, 16a, 17]. In the present paper we shall need a notion of convergence for sequences of subsets, which was introduced by Mosco in [20] and which is related to the one of Painlevé-Kuratowski. Let $t$ be a topology on $X$ and $(C_n)_{n \in J}$ be a sequence in $2^X$. We put

$$t-\text{li} \ C_n := \{ x \in X/ x = t-\lim x_n, \ x_n \in C_n \ \forall n \in J \} ,$$

$$t-\text{ls} \ C_n := \{ x \in X/ x = t-\lim x_k, \ x_k \in C_{n(k)} \ \forall k \in J \} ,$$

where $(C_{n(k)})_{k \in J}$ is a subsequence of $(C_n)$. The subsets $t-\text{li} \ C_n$ and $t-\text{ls} \ C_n$ are the lower limit and the upper limit of $(C_n)$, relative to $t$. We obviously have

$$t-\text{li} \ C_n \subset t-\text{ls} \ C_n .$$

A sequence $(C_n)$ is said to converge to $C$, in the sense of Painlevé-Kuratowski, relatively to the topology $t$, if the following equalities are satisfied:

$$C = t-\text{li} \ C_n = t-\text{ls} \ C_n .$$

In this case, we write $C = t-\lim \ C_n$; this relation holds if and only if the two following inclusions are satisfied:

$$t-\text{ls} \ C_n \subset C \subset t-\text{li} \ C_n .$$

Now, denote by $w$ (resp. by $s$) the weak (resp. the strong) topology of $X$. A subset $C$ is said to be the Mosco limit of $(C_n)$ (denoted by $\lim \ C_n$) if we have simultaneously

$$C = w-\lim \ C_n = s-\lim \ C_n .$$
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which is satisfied if and only if

\[ w\text{-}ls\ C_n \subseteq C \subseteq s\text{-}li\ C_n. \]

Concerning the Mosco convergence, we refer to Mosco [20], Wets [27], and Attouch [2]. Now we shall recall the definition and some properties of the projective limit of a sequence of sets. Let \((T_n)_{n \in J}\) be a sequence of sets and, for any \((m, n) \in J^2\) such that \(m \leq n\), a map \(u_{mn} : T_n \to T_m\). Also assume the two following hypotheses:

(a) \(\forall m \in J, u_{nn} = \text{id}_{T_m} = \text{the identity map of } T_m.\)

(b) \(\forall (m, n, p) \in J^3\) such that \(m \leq n \leq p\), \(u_{mp} = u_{mn} \circ u_{np}.\)

The sequence \((T_n)_{n \in J}\), together with the maps \(u_{nn}\), is called a projective system. If the \(T_n\) are topological spaces (resp. uniform spaces) and if the \(u_{nn}\) are continuous (resp. uniformly continuous) we speak of a projective system of topological spaces (resp. of uniform spaces). Let \(T\) be the cartesian product of the \(T_n\), for \(n \in J\), and \(pr_n\), the projection from \(T\) onto \(T_n\). The subset \(S\) of \(T\) defined by

\[ S := \{ x = (x_n)_{n \in J} / pr_m(x) = u_{mn} \circ pr_n(x) \quad \forall (m, n) \in J^2, \ m \leq n \} \]

is called the projective limit of the projective system defined above. The projective limit may be empty; however, the two following results provide instances where it is non-empty.

**Proposition 2.1.** If the \(T_n\) are nonempty compact topological spaces and if the \(u_{nn}\) are continuous, then \(S\) is non-empty and compact.

**Proposition 2.2** (Mittag–Leffler's theorem). If the \(T_n\) are non-empty complete metric spaces, if the \(u_{nn}\) are uniformly continuous and if, for any \(n \in J\), \(u_{n,n+1}(T_{n+1})\) is dense in \(T_n\), then for each \(n \in J\), \(pr_n(S)\) is dense in \(T_n\). This obviously implies the non-vacuity of \(S\).

The proofs of Propositions 2.1 and 2.2 can be found in [6, Proposition 8, p. 1.64; Theorem 1, p. II.17].

**3. Existence of Martingale Selections for a Multivalued Martingale**

In this section we shall present our first results of existence of martingale selections for a multivalued martingale. For this purpose, it will be shown that the set of martingale selections can be viewed as the projective limit
of a suitable projective system of subsets of $L^1(X)$. We also show the existence of martingale selections which are Castaing representations.

Let $X$ be a separable Banach space, $(\Omega, \mathcal{A}, P)$ a probability space, and $(\mathcal{B}_n)_{n \in J}$ an increasing sequence of sub-$\sigma$-fields of $\mathcal{A}$, such that

$$\mathcal{A} = \sigma\text{-field generated by } \bigcup_{n \in J} \mathcal{B}_n.$$  

A sequence $(F_n)_{n \in J}$ of measurable multifunctions with values in $\mathcal{C}$ is said to be adapted to $(\mathcal{B}_n)$ if, for any $n \in J$, $F_n$ is $\mathcal{B}_n$-measurable; i.e., $F_n^{-1}(U) \in \mathcal{B}_n$ for every open set $U$ of $X$. Further, such an adapted sequence is said to be a multivalued martingale if the two following conditions hold:

(a) $\forall n \in J$, $S^1(F_n, \mathcal{B}_n)$ is non-empty

(b) $\forall n \in J$, $F_n = E(F_{n+1} | \mathcal{B}_n)$.

Moreover, if instead of (b), we have $F_n \subseteq E(F_{n+1} | \mathcal{B}_n)$ (resp. $F_n \supseteq E(F_{n+1} | \mathcal{B}_n)$), we shall say that $(F_n)$ is a submartingale (resp. supermartingale).

Remark 3.1. It is worthwhile to observe that, in our definitions, if a multivalued submartingale or supermartingale is single-valued, it is a martingale.

Let $(f_n)_{n \in J}$ be an adapted sequence of measurable functions from $\Omega$ into $X$. We shall say that $(f_n)$ is a martingale selection of $(F_n)$ if it satisfies the two following conditions:

(a) $(f_n)$ is an integrable single-valued martingale

(b) $\forall n \in J$, $f_n \in S^1(F_n, \mathcal{B}_n)$.

The set of martingale selections of the sequence $(F_n)$ will be denoted by $\text{MS}(F_n)$. We begin by proving the existence of a martingale selection for a multivalued martingale with unbounded values.

**Theorem 3.2.** (i) Any multivalued martingale $(F_n)$ with values in $\mathcal{C}$ admits at least a martingale selection;

(ii) For any $k \in J$, $\text{pr}_k(\text{MS}(F_n))$ is dense in $S^1(F_k, \mathcal{B}_k)$.

**Proof.** For each $(m, n) \in J^2$ such that $m \leq n$, define the map $u_{mn}: S^1(F_n, \mathcal{B}_n) \to S^1(F_m, \mathcal{B}_m)$ by

$$u_{mn}(f) = E(f | \mathcal{B}_m) \quad \forall f \in S^1(F_n, \mathcal{B}_n).$$

The sequence $(S^1(F_n, \mathcal{B}_n))_{n \in J}$ together with the $u_{mn}$ is a projective system
of non-empty complete subsets of $L^1(X)$. Moreover, thanks to the definition of the multivalued conditional expectation, it is clear that the subset

$$u_{n,n+1}(S^1(F_{n+1}, \mathcal{B}_{n+1})) = \{ E(f | \mathcal{B}_n) | f \in S^1(F_{n+1}, \mathcal{B}_{n+1}) \}$$

is dense in $S^1(F_n, \mathcal{B}_n)$. Thus, Proposition 2.2 implies that the projective limit of the above projective system is non-empty. Further, any member $(f_k)_{k \in J}$ of the projective limit satisfies, for any $(m, n) \in J^2$ such that $m \leq n$,

$$pr_m((f_k)) = u_{mn} \circ pr_n((f_k))$$

or, equivalently, $f_m = E(f_n | \mathcal{B}_m)$. So we have $(f_k) \in \text{MS}(F_n)$ which proves (i). Statement (ii) is also a consequence of Proposition 2.2. Q.E.D.

In the sequel MS($F_n$) will sometimes be denoted by $M$.

**Corollary 3.3.** Under the hypotheses of Theorem 3.2, there exists a countable subset $D$ of MS($F_n$) such that, for any $n \in J$, $pr_n(D)$ is a Castaing representation of $F_n$.

**Proof.** Using Lemma 1.1 of [17] (or Proposition 3 of [15a]) we see that, for each $n \in J$, $F_n$ has a Castaing representation $(f^k_n)_{k \in J}$ whose members are in $S^1(F_n, \mathcal{B}_n)$. Further, Theorem 3.2(ii) asserts that $pr_n(M)$ is dense in $S^1(F_n, \mathcal{B}_n)$. Thus, for any $k \in J$, there exists a sequence $((g^k_n)_{n \in J})_{k \in J}$ in $M$ such that

$$f^k_n = \lim_{i \to \infty} g^k_i \quad \forall (n, k) \in J^2,$$

the limit being taken in $L^1(X)$. Consequently, we can find a negligible subset $N^k_n$ of $(\Omega, \mathcal{B}_n, \mathcal{P})$ and a strictly increasing map $\phi^k_n$ from $J$ into itself verifying that

$$f^k_n(\omega) = \lim_{i \to \infty} g^k_{\phi^k_n(i)}(\omega) \quad \forall \omega \in \Omega \setminus N^k_n,$$

where we have put $j(i) = \phi^k_n(i)$. Now if we set

$$N = \bigcup_{(n,k) \in J^2} N^k_n,$$

it is readily seen that, for each $\omega \in \Omega \setminus N$,

$$\{ g^k_{\phi^k_n(i)}(\omega) / (k, i) \in J^2 \}$$

is dense in $F_n(\omega)$. So, we have proved that for each $n \in J$,

$$\{ g^k_n / (k, i) \in J^2 \}$$
is a Castaing representation of $F_n$ on $\Omega \setminus N$. We end the proof by putting

$$D = \{(g^{ki}_n)_{n \in J} / (k, i) \in J^2\}. \quad \text{Q.E.D.}$$

Remark 3.4. If, for any $n \in J$, $\mathcal{B}_n$ is countably generated (for instance, if $\mathcal{B}_n$ is the sub-$\sigma$-field generated by $F_n$) then $S^1(F_n, \mathcal{B}_n)$ is a strongly separable subset of $L^1(X)$. Therefore $M$ is a separable subset of $L^1(X)'$. Now if $D$ is a countable dense subset of $M$, it is clear that, for each $n \in J$, $\text{pr}_n(D)$ is dense in $S^1(F_n, \mathcal{B}_n)$. Moreover, like in the proof of Corollary 3.3, it is possible to show that $\text{pr}_n(D)$ is a Castaing representation of $F_n$.

Remark 3.5. Similar results to Theorem 3.2 and Corollary 3.3 were first obtained by Van Cutsem [26a, 26b] for weakly compact valued martingales and, later, by Luu [19c, Proposition 2.3] for multivalued martingales whose values are bounded in an infinite dimensional Banach space. Recently, Choukairi also proved results similar to ours (Propositions 2.6 and 2.7 in [10a, 10b]). However, his method, like the one of Luu, is different from the method we use in the present paper and is mainly based on a result of Rao [22, Theorem 1.11 concerning quasi-martingales. We also refer the reader to the pioneer work of Van Cutsem on multivalued martingales, where different methods were used.

The next result concerns the existence of a martingale selection for a multivalued supermartingale, with values in $\mathcal{K}$, the set of all convex weakly compact subsets of $X$. It will be used in Section 4, in order to prove the existence of an $L^1(X)$-bounded martingale selection, for a multivalued supermartingale with values in $\mathcal{K}$.

**Proposition 3.6.** If $(F_n)_{n \in J}$ is a supermartingale with values in $\mathcal{K}$, such that, for any $n \in J$, $\text{Eh}(F_n)$ is finite, then $\text{MS}(F_n)$ is non-empty.

**Proof.** Like in the proof of Theorem 3.2, define the maps $u_{mn}$, for $(m, n) \in J^2$ and $m \leq n$, by

$$u_{mn}(f) = E(f | \mathcal{B}_m) \quad \forall f \in S^1(F_n, \mathcal{B}_n).$$

The supermartingale property implies

$$u_{mn}(S^1(F_n, \mathcal{B}_n)) \subset S^1(F_m, \mathcal{B}_m);$$

hence $u_{mn}$ is a map from $S^1(F_n, \mathcal{B}_n)$ into $S^1(F_m, \mathcal{B}_m)$. Further, for any $n \in J$, the multifunction $F_n$ being integrably bounded, we deduce from Proposition 13 of [1] (see also [18a, 18b]) that $S^1(F_n, \mathcal{B}_n)$ is weakly compact in $L^1(X)$. Therefore, the sequence $(S^1(F_n, \mathcal{B}_n))_{n \in J}$ together with the maps $u_{mn}$ is a projective system of non-empty weakly compact subsets in
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Finally, Proposition 2.1 shows that this projective system admits a non-empty projective limit which is nothing else but MS($F_n$).

**Remark 3.7.** Propositions 2.1 and 2.2, about the projective limit, have already been used in the study of multivalued measures. See for instance A. Costé [11a], N. Serpollier [24, Appendice n° 2] and D. S. Thiam [25]. In [11b], A. Costé also used the projective limit technique in order to prove regularity results for multivalued martingales with closed bounded convex values in a separable Banach space having the Radon–Nikodym property.

4. **Existence of Convergent Martingale Selections for a Multivalued Supermartingale with Unbounded Values**

From Theorem 3.2, we know that for any multivalued martingale $(F_n)_{n \in J}$ such that each $F_n$ is an integrable multifunction with values in $\mathcal{C}$, the set MS($F_n$) of martingale selections is non-empty. Now, it is natural to ask the two following questions:

(a) If $(F_n)_{n \in J}$ is $L^1(X)$-bounded in the sense of distance functions, that is,

$$\sup_{n \in J} Ed(0, F_n) < + \infty,$$

then there exist $(f_n) \in$ MS($F_n$) verifying

$$\sup_{n \in J} E \|f_n\| < + \infty? \quad (4.2)$$

(b) If $(F_n)_{n \in J}$ is uniformly integrable in the sense of distance functions, that is

$$(d(0, F_n))_{n \in J} \text{ is uniformly integrable,} \quad (4.3)$$

then there exist $(f_n) \in$ MS($F_n$) which is uniformly integrable in $L^1(X)$?

**Remark 4.1.** If instead of condition (4.1) (resp. (4.3)) it were assumed the stronger condition

$$\sup_{n \in J} Eh(F_n) < + \infty$$

(resp. $(h(F_n))_{n \in J}$ is uniformly integrable), then each $(f_n) \in$ MS($F_n$) would satisfy (4.2) (resp. would be uniformly integrable). But here, since the values of the $F_n (n \in J)$ are not supposed to be bounded, it is not possible
to keep so strong hypotheses. Only conditions on distance functions are admissible, which makes questions (a) and (b) more difficult to answer.

Theorem 4.4 below will answer affirmatively questions (a) and (b). We begin by two simple lemmas.

**Lemma 4.2.** If $r$ is a positive integrable function and $\mathcal{B}$ a sub-$\sigma$-field of $\mathcal{A}$, then for any $x \in X$ one has

$$E(B(x, r) \mid \mathcal{B}) = B(x, E(r \mid \mathcal{B})) \quad \text{a.s.},$$

where $B(x, r)$ denotes the closed ball of radius $r$, centered at $x$.

**Proof.** Apply Theorem 5.3(1°) of [17] to the multifunction $F$ defined by

$$F(\omega) := B(x, r(\omega)) \quad \forall \omega \in \Omega$$

(a direct proof is also possible). Q.E.D.

**Lemma 4.3.** If $F$ is an integrable measurable multifunction with closed values in $X$ and $\mathcal{B}$ a sub-$\sigma$-field of $\mathcal{A}$, then, for any $x \in X$, we have

$$d(x, E(F \mid \mathcal{B})) \leq E(d(x, F) \mid \mathcal{B}) \quad \text{a.s.}$$

**Proof.** Using the measurable choice theorem (see, for example, Theorem III.6 in [9]), it is not hard to show that, for each $\delta > 0$, there exists $g \in S^1(F, \mathcal{A})$ satisfying

$$\|x - g(\omega)\| \leq d(x, F(\omega)) + \delta \quad \text{a.s.} \quad (4.4)$$

Then, passing to conditional expectations, we obtain

$$E(\|x - g\| \mid \mathcal{B}) \leq E(d(x, F) \mid \mathcal{B}) + \delta.$$ 

Further, applying Jensen's inequality,

$$\|x - E(g \mid \mathcal{B})\| = E((x - g) \mid \mathcal{B}) \leq E(d(x, F) \mid \mathcal{B}) + \delta. \quad (4.5)$$

Then noting that

$$E(g \mid \mathcal{B})(\omega) \in E(F \mid \mathcal{B})(\omega) \quad \text{a.s.}$$

and using (4.5) we obtain

$$d(x, E(g \mid \mathcal{B})) \leq E(d(x, F) \mid \mathcal{B}) + \delta \quad \text{a.s.}$$

which implies the desired conclusion, because $\delta$ is arbitrary. Q.E.D.
Now, it is convenient to introduce a new class of subsets of $X$, by putting

$$
\mathcal{R} := \{ C \in \mathcal{G} / C \cap \bar{B}(0, r) \in \mathcal{X}, \forall r > 0 \}.
$$

In this definition $r$ can be restricted to integer values and it is clear that $\mathcal{R}$ contains the members of $\mathcal{G}$ which are weakly locally compact. If $X$ is reflexive we have $\mathcal{R} = \mathcal{G}$. 

**Theorem 4.4.** (1°) Let $(F_n)_{n \in J}$ be a multivalued supermartingale, with values in $\mathcal{R}$, which satisfies the following condition:

$$
\sup_{n \in J} E d(0, F_n) < + \infty. \quad (4.6)
$$

Then, there exists $(f_n)_{n \in J} \in \text{MS}(F_n)$ such that

$$
\sup_{n \in J} E \|f_n\| < + \infty. \quad (4.7)
$$

(2°) Moreover, if $(d(0, F_n))_{n \in J}$ is uniformly integrable in $L^1$, there exists $(f_n)$ in $\text{MS}(F_n)$ which is uniformly integrable in $L^1(X)$.

**Proof.** (1°) For any $n \in J$ define the function $v_n$ by

$$
v_n(\omega) := d(0, F_n(\omega)) + 1, \quad \omega \in \Omega.
$$

Using Lemma 4.3 and the supermartingale property we obtain for every $n \in J$,

$$
E(d(0, F_{n+1}) | \mathcal{R}_n) \geq d(0, E(F_{n+1} | \mathcal{R}_n)) \geq d(0, F_n) \quad \text{a.s.}
$$

which implies that $(v_n)_{n \in J}$ is a positive integrable submartingale verifying

$$
\sup_{n \in J} E(v_n) < + \infty. \quad (4.8)
$$

Now, using Krickeberg's decomposition (see Theorem IV.1.2 in [21a]), it is possible to write for each $n \in J$

$$
v_n = r_n - s_n \quad \text{a.s.,}
$$

where $(r_n)_{n \in J}$ is a positive integrable martingale and $(s_n)_{n \in J}$ is a positive integrable supermartingale. Further, define for every $n \in J$ the multifunction $G_n$ by

$$
G_n(\omega) := F_n(\omega) \cap \bar{B}(0, r_n(\omega)), \quad \omega \in \Omega.
$$
Observe, first, that, for any $n \in J$, multifunction $G_n$ is $\mathcal{B}_n$-measurable. This can be seen by appealing to Proposition 3.3.3 of [15d]. Moreover, the inequality
\[ h(G_n(\omega)) \leq r_n(\omega) \quad \forall n \in J, \forall \omega \in \Omega, \tag{4.9} \]
shows that each $G_n$ is integrably bounded. At last, using the monotonicity of the conditional expectation, Lemma 4.2, and the fact that $(r_n)_{n \in J}$ is a positive integrable martingale, we obtain, for any $n \in J$,
\[
E(G_{n+1} | \mathcal{B}_n) = E(F_{n+1} \cap \overline{B}(0, r_{n+1}) | \mathcal{B}_n) \\
\subset E(F_{n+1} | \mathcal{B}_n) \cap E(\overline{B}(0, r_{n+1}) | \mathcal{B}_n) \\
\subset F_n \cap \overline{B}(0, E(r_{n+1} | \mathcal{B}_n)) = F_n \cap \overline{B}(0, r_n) = G_n.
\]
This calculation shows that $(G_n)_{n \in J}$ is a multivalued supermartingale with values in $\mathcal{X}$ because each $F_n$ is $\mathcal{A}$-valued. Now, Proposition 3.6 implies that $\text{MS}(G_n)$ is non-empty. Moreover, an inspection of the proof of Krickeberg’s decomposition for an integrable real submartingale shows that
\[
\sup_{n \in J} E r_n \leq \sup_{n \in J} E v_n < + \infty.
\]
Consequently, every $(f_n)$ in $\text{MS}(G_n)$ satisfies
\[
\sup_{n \in J} E \|f_n\| \leq \sup_{n \in J} E v_n < + \infty
\]
which gives the desired conclusion.

$(2^o)$ In the same way, if $(d(0, F_n))_{n \in J}$ is assumed to be uniformly integrable in $L^1(\mathbb{R})$, it is not hard to see that $(f_n)$ is uniformly integrable in $L^1(\mathcal{X})$. Q.E.D.

Remark 4.5. It is not difficult to see that, if $(F_n)$ is a multivalued supermartingale, the sequence of multifunctions $(H_n)$ defined by
\[
H_n(\omega) = \overline{B}(0, d(0, F_n(\omega))), \quad \omega \in \Omega, \quad n \in J,
\]
is a multivalued submartingale. Indeed, it suffices to apply Lemma 4.3. However, in general, even if $(F_n)$ is a multivalued martingale, the sequence $(G_n)$ defined by
\[
G_n(\omega) := F_n(\omega) \cap H_n(\omega)
\]
is neither a multivalued supermartingale, nor a multivalued submartingale. This is why, in the proof of Theorem 4.4, Krickeberg’s decomposition is needed.
5. CONVERGENCE OF MULTIVALUED SUPERMARTINGALES WITH UNBOUNDED VALUES

In this section we shall present a convergence result for multivalued supermartingales whose values are unbounded. This will be achieved in two steps. At the first one, we shall show convergence results for multivalued supermartingales with values in $\mathcal{H}$ (Proposition 5.7). At the second one, we shall use a method of truncation in connection with simple properties of Mosco convergence (Lemma 5.11).

If $X^*$ stands for the dual space of $X$ and $B^*$ for the closed unit ball of $X^*$, we denote by $D^*_\mathbb{Q}$ a countable subset of $B^*$ which is dense for the Mackey topology. $D^*$ will denote the set of all rational linear combinations of members of $D^*_\mathbb{Q}$. It is clear that $D^*$ is a countable dense subset of $X^*$ for the Mackey topology. The two following lemmas were already used in [7b].

**Lemma 5.1.** Let $(C_n)_{n \in J}$ be a sequence in $\mathcal{H}$ verifying conditions (i) and (ii):

(i) there exists $K \in \mathcal{H}$ such that $C_n \subseteq K$, $\forall n \in J$,

(ii) $\lim_{n \to \infty} s(x^*, C_n)$ exists for each $x^* \in D^*$.

Then, there exists $C \in \mathcal{H}$ such that, for any $x^* \in X^*$,

$s(x^*, C) = \lim_{n \to \infty} s(x^*, C_n)$.

**Proof.** Condition (i) implies that $(s(\cdot, C_n))_{n \in J}$ is an equicontinuous sequence, for the Mackey topology. Thus, using (ii) we deduce that it admits a limit at each point $x^*$ of $X^*$. Now define the function $r$ by

$$r(x^*) := \lim_{n \to \infty} s(x^*, C_n), \quad x^* \in X^*;$$

$r$ is sublinear and continuous for the Mackey topology. Consequently, there exists $C \in \mathcal{H}$ such that

$$r(x^*) = s(x^*, C) \quad \forall x^* \in X^*.$$

Q.E.D.

**Lemma 5.2.** Let $(F_n)_{n \in J}$ be a sequence of measurable multifunctions with values in $\mathcal{H}$. Assume hypotheses (i) and (ii):

(i) there exists a multifunction $H$ with values in $\mathcal{H}$, such that

$$F_n(\omega) \subseteq H(\omega) \quad \forall \omega \in \Omega, \forall n \in J;$$

(ii) for any $x^* \in D^*$, the sequence $(s(x^*, F_n(\omega)))_{n \in J}$ is almost surely convergent. Then
(a) There exist a measurable multifunction $F$ with values in $\mathcal{H}$ and a
negligible subset $N$ verifying

$$s(x^*, F(\omega)) = \lim_{n \to \infty} s(x^*, F_n(\omega))$$

for any $x^* \in X^*$ and $\omega \in \Omega \setminus N$.

(b) Moreover, if

$$\sup_{n \in J} Eh(F_n) < +\infty$$

is assumed, then $F$ is integrably bounded.

Proof. (a) Since $D^*$ is countable it is possible to find a negligible sub-
set $N$ such that $\lim_{n \to \infty} s(x^*, F_n(\omega))$ exists for any $x^* \in D^*$ and $\omega \in \Omega \setminus N$. Then apply Lemma 5.1 for every $\omega \in \Omega \setminus N$ with $K := H(\omega)$ and $C_n := F_n(\omega)(n \in J)$. This entails the existence of a multifunction $F$, with values in $\mathcal{H}$, which satisfies

$$s(x^*, F(\omega)) = \lim_{n \to \infty} s(x^*, F_n(\omega)) \quad \forall \omega \in \Omega \setminus N, \forall x^* \in X^*.$$ 

The measurability of $F$ is a consequence of this equality and of Lemma 7
in [15b].

(b) Since for $C \in \mathcal{H}$,

$$h(C) := \sup \left\{ s(x^*, C) / x^* \in D^*_1 \right\},$$

we have

$$\lim_{n \to \infty} \inf h(F_n(\omega)) \geq h(F(\omega)), \quad \omega \in \Omega \setminus N.$$ 

Hence Fatou's lemma gives the conclusion. Q.E.D.

Remark 5.3. If condition (5.1) in the above lemma is replaced by

$$\sup_{n \in J} Ed(0, F_n) < +\infty,$$  

we obtain, as in the proof of (b), that $Ed(0, F)$ is finite, which implies that $F$ admits at least one integrable selector. Indeed, it suffices to use the equality

$$d(0, C) = \sup \left\{ -s(x^*, C) / x^* \in D^*_1 \right\}$$

which is valid for any $C \in \mathcal{H}$. 

The next lemma, which is a special case of Proposition 6.4.4 in \[15c\], provides a sufficient condition of Mosco convergence for a sequence in \( \mathcal{K} \).

**Lemma 5.4.** Let \((C_n)_{n \in J}\) be a sequence in \( \mathcal{K} \) and \( C \) in \( \mathcal{K} \). If the two following conditions hold:

(i) \( d(x, C) = \lim_{n \to \infty} d(x, C_n) \ \forall x \in X \)

(ii) \( s(x^*, C) = \lim_{n \to \infty} s(x^*, C_n) \ \forall x^* \in X^* \),

then \( C = \lim_{n \to \infty} C_n \).

**Remark 5.5.** Lemma 5.4 remains true if \( \mathcal{K} \) is replaced by the family of all closed convex subsets of \( X \), which are weakly locally compact and contain no line. On the other hand, if \( X^* \) is strongly separable, \( \mathcal{K} \) may be replaced by the family of all closed bounded convex subsets of \( X \). See Proposition 6.4.4 in \[15c\] for a more general result.

**Remark 5.6.** In \[4\], Beer studied the topology on \( \mathcal{K} \), which is the upper bound of the two following topologies:

(a) the topology of simple convergence of distance functions on \( X \)

(b) the topology of simple convergence of support functions on \( X^* \).

Beer showed that, when \( \mathcal{K} \) is endowed with this topology, then addition, closed convex hull of the union, and scalar multiplication of subsets are continuous operations. This led him to call this topology: the “linear topology.” He also showed that it is stronger than the Mosco topology. In the present paper, Lemma 5.4 above is a special case of this property. Also note that Proposition 5.7(b) below provides convergence results for the linear topology.

The next result concerns multivalued supermartingales with values in \( \mathcal{K} \). It will be useful when we establish the convergence of multivalued supermartingales whose values are unbounded.

**Proposition 5.7.** Let \((F_n)_{n \in J}\) be a multivalued supermartingale with values in \( K \), which satisfies the two following hypotheses:

\[
\sup_{n \in J} Es(x^*, F_n) < +\infty \quad \forall x^* \in D^* \tag{5.3}
\]

\[
K(\omega) := \bar{\text{co}} \left( \bigcup_{x^* \in D^*} F_n(\omega) \right) \in \mathcal{K} \quad \text{a.s.} \tag{5.4}
\]

Then, we can assert:
(a) There exist a measurable multifunction $F$ with values in $\mathcal{X}$ and a negligible subset $N$ of $\Omega$ such that

$$
\lim_{n \to \infty} s(x^*, F_n(\omega)) = s(x^*, F(\omega)) \quad \forall \omega \in \Omega \setminus N, \forall x^* \in X^*. \tag{5.5}
$$

(b) If condition (5.3) is replaced by the stronger one

$$
\sup_{n \in J} Ed(0, F_n) < +\infty \tag{5.6}
$$

and if for any $n \in J$, $x^* \in X^*$, $s(x^*, F_n(\cdot))$ is integrable, then $F$ is an integrable multifunction and we have

$$
d(x, F(\omega)) = \lim_{n \to \infty} d(x, F_n(\omega)) \quad \forall \omega \in \Omega \setminus N, \forall x \in X. \tag{5.7}
$$

In particular, this yields

$$
F(\omega) = \lim_{n \to \infty} F_n(\omega) \quad \text{a.s.} \tag{5.8}
$$

(c) If (5.6) is replaced by the stronger hypothesis

$$
\sup_{n \in J} Eh(F_n) < +\infty \tag{5.9}
$$

then $F$ is integrably bounded.

(d) If the sequence $(h(F_n))_{n \in J}$ is uniformly integrable, we have

$$
E(F|\mathcal{B}_n) \subset F_n \quad \forall n \in J \text{ a.s.}
$$

Proof. (a) For any $x^* \in D^*$, $(s(x^*, F_n))_{n \in J}$ is a real supermartingale and condition (5.3) shows that it converges a.s. (see, for example, the book of Neveu [21a or 21b]). Using Lemma 5.2(a), we obtain the desired conclusion.

(b) Let $D$ be a countable dense subset of $X$. For any $x \in D$ and $C \in \mathcal{X}$, the following equality holds true:

$$
d(x, C) = \sup \left[ \langle x^*, x \rangle - s(x^*, C)/x^* \in D^*_1 \right].
$$

Then applying Lemma V.2.9 in [21a] (or Lemma 4 in [21b]) to the countable family of integrable submartingales,

$$
(\langle x^*, x \rangle - s(x^*, F_n))_{n \in J}, \quad x^* \in D^*_1,
$$

we deduce the existence of a negligible subset $N$ such that

$$
d(x, F(\omega)) = \lim_{n \to \infty} d(x, F_n(\omega)) \tag{5.10}
$$
for all $x \in D$ and $\omega \in \Omega \setminus \mathcal{N}$. But the sequence $(d(\cdot, F_n(\omega)))_{n \in J}$ being equicontinuous, for each $\omega \in \Omega \setminus \mathcal{N}$, we deduce that (5.10) remains valid for any $x \in X$. At last, Lemma 5.4 yields relation (5.8).

(c) is an obvious consequence of Lemma 5.2(b).

(d) The result of part (a), the properties of the conditional expectation, and the uniform integrability assumption imply, for any $x \in D$ and $n \in J$,

$$s(x^*, E(F|\mathcal{B}_n)) = E(s(x^*, F)|\mathcal{B}_n)$$

$$= \lim_{k \to \infty} E(s(x^*, F_k)|\mathcal{B}_n) \leq s(x^*, F_n) \quad a.s.,$$

which gives the desired conclusion. Q.E.D.

Remark 5.8. If $(F_n)_{n \in J}$ is a multivalued martingale instead of a supermartingale, then part (d) of Proposition 5.7 becomes

$$E(F|\mathcal{B}_n) = F_n \quad \forall n \in J \quad a.s.$$

Remark 5.9. In [10a, 10c], assuming that $X$ has the Radon–Nikodym property and that $X^*$ is strongly separable, Choukairi proved variants of Proposition 5.7.

Remark 5.10. Concerning part (c) of Proposition 5.7, consider a subset $\mathcal{K}'_1$ of $\mathcal{K}'$, which is separable for the topology generated by the classical Hausdorff distance (denoted by $h$). If the values of supermartingale $(F_n)$ lie in $\mathcal{K}'_1$, then it is not hard to prove that

$$\lim_{n \to \infty} h(F_n(\omega), F(\omega)) = 0 \quad a.s.$$

Indeed, it suffices to use the same method as in [21b].

The following lemma will be useful. It provides a sufficient condition of Mosco convergence for a sequence of possibly unbounded subsets, in terms of Mosco convergence of sequences of bounded subsets. See also [2] for a formulation in terms of functions.

**Lemma 5.11.** Let $(C_n)_{n \in J}$ be a sequence in $2^X$ and $(r_k)_{k \in J}$ an increasing sequence of positive real numbers such that $\lim_{k \to \infty} r_k = + \infty$. Assume that, for every $k \in J$, the sequence $(C_n \cap B(0, r_k))_{n \in J}$ has a Mosco limit denoted by $C^k$. If we set

$$C := \bigcup_{k \in J} C^k$$
then
\[ C = \lim_{n \to \infty} C_n \]
(in particular, \( C \) is closed).

Proof. For any \( x \in C \) there exists \( j \in J \) such that \( x \in C_j \). From the following inclusions
\[ C_j \subseteq s\text{-}li(C_n \cap \bar{B}(0, r_j)) \subseteq s\text{-}li C_n, \]
we deduce
\[ C \subseteq s\text{-}li C_n. \]

Let us show now the inclusion \( w\text{-}ls C_n \subseteq C \). If \( x \in w\text{-}ls C_n \), there exists a sequence \( (x_i)_{i \in J} \) in \( X \) satisfying
\[ x_i \in C_{n(i)} \text{ for each } i \in J \quad \text{and} \quad x = w\text{-}lim x_i, \]
where \( (C_{n(i)})_{i \in J} \) is a subsequence of \( (C_n) \). Because every weakly convergent sequence is bounded, it is possible to find \( j \in J \) verifying
\[ \|x_i\| \leq r_j \quad \forall i \in J \]
which implies
\[ x \in w\text{-}ls(C_n \cap \bar{B}(0, r_j)) \subseteq C_j \subseteq C. \]
Q.E.D.

The next theorem is the main result of this section. It concerns the Mosco convergence of a multivalued supermartingale with unbounded values.

**Theorem 5.12.** Let \( X \) be a separable Banach space and let \((F_n)_{n \in J}\) be a multivalued supermartingale, with values in \( \mathcal{C} \), which satisfies the two conditions:

(i) \[ \sup_{n \in J} Ed(0, F_n) < +\infty, \]
(ii) there exists a multifunction \( H \) with values in \( \mathcal{A} \) such that
\[ F_n(\omega) \subseteq H(\omega) \quad \forall n \in J \ a.s. \]

Then, we can find an integrable measurable multifunction \( F \), with values in \( \mathcal{A} \), satisfying
\[ F(\omega) = \lim_{n \to \infty} F_n(\omega) \quad a.s. \]
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Proof. For each \( k \in J \) define the positive submartingale \( (v_n^k)_{n \in J} \) by
\[
v_n^k(\omega) := d(0, F_n(\omega)) + k, \quad \omega \in \Omega
\]
and apply Krickeberg's decomposition theorem to it, as in the proof of Theorem 4.4. This yields
\[
v_n^k = r_n^k - s_n^k,
\]
where \( (r_n^k)_{n \in J} \) is a positive integrable martingale and \( (s_n^k)_{n \in J} \) is a positive integrable supermartingale. Further, define the multivalued supermartingale \( (F_n^k)_{n \in J} \) by
\[
F_n^k(\omega) := F_n(\omega) \cap \bar{B}(0, r_n^k(\omega)), \quad \omega \in \Omega.
\]
For any \((n, k) \in J^2\) we have
\[
h(F_n^k(\omega)) \leq r_n^k(\omega) \quad \text{a.s.}
\]
Since, for any \( k \in J \), \( \sup_{n \in J} E(r_n^k) \) is finite, the positive martingale \( (r_n^k)_{n \in J} \) is almost surely convergent, hence bounded. More precisely, for any \( k \in J \) and \( \omega \in \Omega \), there exists \( w_k(\omega) > 0 \) such that
\[
r_n^k(\omega) \leq w_k(\omega) \quad \forall n \in J.
\]
From this inequality and hypothesis (ii), we deduce that, for any \((n, k) \in J^2\),
\[
F_n^k(\omega) \subset H(\omega) \cap \bar{B}(0, w_k(\omega)), \quad \omega \in \Omega,
\]
where the right-hand side is weakly compact. Therefore, Proposition 5.7(b) applied for each \( k \in J \) to the sequence \( (F_n^k)_{n \in J} \) shows the existence of a negligible subset \( N \) of \( \Omega \) such that, for any \( \omega \in \Omega \setminus N \), the sequence \( (F_n^k(\omega))_{n \in J} \) converges, in the sense of Mosco, to some subset denoted by \( F^k(\omega) \). Moreover, Proposition 5.7(c) also shows that \( F^k \) is an integrably bounded measurable multifunction with values in \( \mathcal{X} \). At last, define the multifunction \( F \) by
\[
F(\omega) := \begin{cases} 
\bigcup_{k \in J} F^k(\omega) & \text{if } \omega \in \Omega \setminus N \\
\{0\} & \text{if } \omega \in N
\end{cases}
\]
and apply Lemma 5.11 which yields
\[
F(\omega) = \lim_{n \to \infty} F_n(\omega) \quad \forall \omega \in \Omega \setminus N.
\]
Clearly, the measurability of each multifunction \( F^k (k \in J) \) implies that \( F \) is measurable too. \( \Box \)

**Corollary 5.13.** Assume the same hypotheses as in Theorem 5.12 but replace condition (i) by

(i) the sequence \((d(0, F_n))_{n \in J}\) is uniformly integrable.

Then \( \mathbb{E}(F|\mathcal{B}_n) \subset F_n \ \forall n \in J \ a.s. \)

**Proof.** For any \( n \in J \), \((F^k_n)_{n \in J}\) is a multivalued supermartingale with values in \( \mathcal{K} \). Thus, condition (i) and Proposition 5.7(d) imply that

\[
\mathbb{E}(F^k | \mathcal{B}_n) \subset F^k_n \ \forall (n, k) \in J^2 \text{ a.s.}
\]

Further, using the relations

\[
F_n(\omega) = \bigcup_{k \in J} F^k_n(\omega) \quad \forall n \in J
\]

\[
F(\omega) = \bigcup_{k \in J} F^k(\omega)
\]

valid a.s., and Theorem 2.1 of [16b] we obtain for any \( n \in J \) and a.s.

\[
\mathbb{E}(F|\mathcal{B}_n) = \mathbb{E}\left( \bigcup_{k \in J} F^k | \mathcal{B}_n \right) = \text{cl}\left( \bigcup_{k \in J} \mathbb{E}(F^k | \mathcal{B}_n) \right)
\subset \text{cl}\left( \bigcup_{k \in J} F^k_n \right) = F_n.
\]

**Remark 5.14.** It is worthwhile to observe that condition (ii) of Theorem 5.12 is automatically fulfilled if \( X \) is reflexive. Indeed, it suffices to put

\[
H(\omega) := X \quad \forall \omega \in \Omega.
\]

The same remark is valid when \( X \) is a separable dual, i.e., \( X = Y^* \), where \( Y \) is a separable Banach space and if the values of multifunctions \( F_n \) are closed in the weak-* topology of \( Y^* \). In this case, the weak topology on \( X = Y^* \) is replaced by the weak-* topology \( \sigma(Y^*, Y) \).

**Remark 5.15.** In [10a, Theorem 2.16], Choukairi proved results similar to ours, when \( X \) is reflexive. But he also needed the following extra assumption: there exists \((f_n)\) in \( \text{MS}(F_n) \) such that \( \sup_{n \in J} \|f_n\| \in L^1 \).

More recently in [10c, 10d], using Theorem 4.4, Choukairi independently obtained a slightly less general version of Theorem 5.12, for a multivalued martingale with values in a separable reflexive Banach space.
In fact, Theorem 4.4 was included in an oral communication that I gave in Montpellier in April 1988. Anyway, it is important to note the crucial part played by Krickeberg decomposition result in Theorem 4.4, as well as in Theorem 5.12.

**Remark 5.16.** On the measurability of the Mosco-limit of a sequence $(F_n)_{n \in J}$ of measurable multifunctions and, more generally, on the measurability of $w$-$ls$ $F_n$ see [15c].

**Remark 5.17.** The results of Sections 3 to 5 lead us to ask the following question: is it possible to prove a version of Proposition 3.6 (resp. Theorem 5.12) for multivalued submartingales?

6. **APPLICATIONS TO THE MOSCO CONVERGENCE OF SUPERMARTINGALE INTEGRANDS**

The results of Section 5, being valid for multivalued martingales with unbounded values in any separable Banach space $X$, apply to the special case where the measurable multifunctions are epigraphic multifunctions associated with integrands. So, the main goal of this section is to reformulate Theorem 5.12 in terms of supermartingale integrands. First, we recall some definitions and known facts.

If $u$ is a numerical function (i.e., with values in $\mathbb{R} = [-\infty, +\infty]$), defined on $X$, its *epigraph*, denoted by $\text{epi}(u)$, is the subset of $X \times \mathbb{R}$,

$$\text{epi}(u) := \{(x, \lambda) \in X \times \mathbb{R} / u(x) \leq \lambda\};$$

$u$ is said to be *proper* if it is not the constant $+\infty$ and if it does not take the value $-\infty$. The *conjugate* (or polar) function of $u$ is denoted by $u^*$ and defined on $X^*$ by

$$u^*(x^*) := \sup[\langle x^*, x \rangle - u(x)/x \in X].$$

Let $u, u_n$, for $n \in J$, be numerical functions defined on $X$. The sequence $(u_n)_{n \in J}$ is said to be *Mosco-convergent* to $u$, if $\text{epi}(u)$ is the Mosco limit of $(\text{epi}(u_n))_{n \in J}$ in $X \times \mathbb{R}$. In such a case, we also say that the sequence $(u_n)$ *epi-converges* to $u$ (in the Mosco sense). In this section this convergence will be denoted by

$$u - M\text{-lim}_e u_n.$$

The Mosco-convergence of a sequence $(u_n)$ can also be defined by the equality of the two functions

$$w\text{-li}_e u_n \quad \text{and} \quad s\text{-ls}_e u_n.$$
which are, respectively, the weak-epi-lower limit and the strong-epi-upper limit of the sequence \((u_n)_{n \in J}\). Moreover, the two following formulas hold true (see [2]).

\[
\begin{align*}
\text{epi}(w\text{-}li \ u_n) &= w\text{-}ls(\text{epi}(u_n)) \\
\text{epi}(s\text{-}ls \ u_n) &= s\text{-}li(\text{epi}(u_n)).
\end{align*}
\]

Following Rockafellar [23], we shall say that an application \(\phi\), defined on the product space \(\Omega \times X\) with values in \(\mathbb{R}\), is a normal integrand if it satisfies the two following properties:

(a) \(\phi(\omega, \cdot)\) is lower semi-continuous for almost all \(\omega \in \Omega\)

(b) the multifunction \(\omega \to \text{epi} \phi(\omega, \cdot)\), with closed values in \(X \times \mathbb{R}\), is measurable.

It is called the epigraphical multifunction of \(\phi\). It is also possible to call \(\phi\) a random lower semicontinuous function. \(\phi\) is said to be convex if \(\phi(\omega, \cdot)\) is convex, for almost all \(\omega \in \Omega\). A normal integrand \(\phi\) is said to be integrable if its epigraphical multifunction is integrable in the sense given in Section 2. Appealing to the definition of an integrable multifunction and considering on \(X \times \mathbb{R}\) the norm

\[
\| (x, \lambda) \| = \| x \| + | \lambda |, \quad x \in X, \lambda \in \mathbb{R},
\]

it is easy to see that \(\phi\) is integrable if and only if there exists \(f \in L^1(X)\) such that

\[
\phi(\cdot, f(\cdot))^+ \in L^1.
\]

Moreover, it is readily seen that \(\phi\) is integrable if and only if the positive function

\[
\omega \to d(0, \text{epi} \phi(\omega, \cdot)) = \inf \| x \| + \phi(\omega, x)^+ / x \in X
\]

is integrable (here 0 denotes the zero vector of \(X \times \mathbb{R}\)). For convenience this function will be denoted by \(d(\phi)(\cdot)\). The conditional expectation, relative to the sub-\(\sigma\)-field \(\mathcal{B}\), of an integrable normal integrand \(\phi\) is the normal integrand \(\psi\), denoted by \(E(\phi | \mathcal{B})\), whose epigraphical multifunction

\[
\omega \to \text{epi} \psi(\omega, \cdot)
\]

is the conditional expectation of the integrable multifunction \(\omega \to \text{epi} \phi(\omega, \cdot)\) (see [9, Chap. VIII, Section 9]). A martingale integrand is a sequence \((\phi_n)_{n \in J}\) of convex integrable normal integrands such that
(epi $\phi_n$)$_{n \in J}$ is a multivalued martingale with closed convex values in $X \times \mathbb{R}$; or equivalently, if for any $n \in J$, we have

$$\phi_n = E(\phi_{n+1} | \mathcal{B}_n).$$

More generally, $(\phi_n)_{n \in J}$ is called a submartingale integrand (resp. a supermartingale integrand) if (epi $\phi_n$)$_{n \in J}$ is a multivalued submartingale (resp. supermartingale) with closed convex values in $X \times \mathbb{R}$. A martingale integrand is sometimes called an "epi-martingale."

Remark 6.1. If $(\phi_n)_{n \in J}$ is a submartingale integrand then, for almost all $\omega \in \Omega$, one has in $X \times \mathbb{R}$,

$$\text{epi } \phi_n(\omega, \cdot) \subseteq E(\text{epi } \phi_{n+1} | \mathcal{B}_n)(\omega, \cdot)$$

(6.1)

which, by the definition of the conditional expectation of an integrand, via the epigraph, is equivalent to

$$\phi_n(\omega, \cdot) \geq E(\phi_{n+1} | \mathcal{B}_n)(\omega, \cdot) \quad \text{a.s.}$$

(6.2)

We observe that inclusion (6.1) and inequality (6.2) are inverted; (6.2) is the inequality of supermartingales instead of submartingales. However, considering the support function of epi $\phi_n$, we get again a submartingale. More precisely, for any $(x^*, r) \in X^* \times \mathbb{R}$, the sequence

$$\omega \to s((x^*, r), \text{epi } \phi_n(\omega, \cdot)), \quad n \in J,$$

is a real-valued submartingale. Further, taking $r = -1$, we obtain, for all $\omega \in \Omega$ and $n \in J$,

$$s((x^*, -1), \text{epi } \phi_n(\omega, \cdot)) = \phi_n^*(\omega, x^*),$$

where

$$\phi_n^*(\omega, x^*) := \sup \{ \langle x^*, x \rangle - \phi_n(\omega, x) | x \in X \}.$$ 

Consequently, for every $x^* \in X^*$, the sequence $(\phi_n^*(\cdot, x^*))_{n \in J}$ is a real-valued submartingale. A similar remark holds for a supermartingale integrand.

We go on with a simple lemma which provides two equivalent formulations of hypothesis (i) of Theorem 5.12, in the case of normal integrands.

Lemma 6.2. Let $(\phi_n)_{n \in J}$ be an adapted sequence of convex normal integrands. The two following statements are equivalent:

(a) $\sup_{n \in J} Ed(\phi_n) < +\infty$, 

...
(b) there exists an adapted bounded sequence \((u_n)_{n \in J}\) in \(L^1(X)\) such that
\[
\sup_{n \in J} \int_{\bar{\Omega}} \phi_n(\omega, u_n(\omega))^+ \, dP < +\infty.
\]

Proof. (a) \(\Rightarrow\) (b) For each \(n \in J\), the function
\[
\omega \to d(\phi_n)(\omega) := d(0, \text{epi } \phi_n(\omega, \cdot))
\]
is \(\mathcal{B}_n\)-measurable. Thus, using the measurable choice theorem (see, for instance, Theorem III.6 in [9]), it is not hard to show the existence of a \(\mathcal{B}_n\)-measurable function \(u_n\) verifying
\[
\|u_n(\omega)\| + \phi_n(\omega, u_n(\omega))^+ \leq d(\phi_n)(\omega) + 1.
\]
Since the sequence \((d(\phi_n))_{n \in J}\) is \(L^1\)-bounded, so are the sequences \((\|u_n(\cdot)\|)_{n \in J}\) and \((\phi_n(\cdot, u_n(\cdot))^+)_{n \in J}\).

(b) \(\Rightarrow\) (a) is an obvious consequence of the following inequality
\[
d(\phi_n)(\omega) \leq \|u_n(\omega)\| + \phi_n(\omega, u_n(\omega))^+ \quad \forall \omega \in \Omega, \forall n \in J. \quad \text{Q.E.D.}
\]

Now we give the analog of Theorem 5.12 for supermartingale integrands.

**Theorem 6.3.** Let \(X\) be a separable Banach space and \((\phi_n)_{n \in J}\) a supermartingale integrand which satisfies the two following conditions:

(i) \(\sup_{n \in J} Ed(\phi_n) < +\infty\);

(ii) there exists a multifunction \(L\) with values in \(\mathcal{R}\) such that
\[
\text{epi } \phi_n(\omega, \cdot) \subset L(\omega) \times \mathbb{R} \quad \forall n \in J \text{ a.s.}
\]

Then, there exists an integrable convex normal integrand \(\phi\) such that
\[
\phi(\omega, \cdot) = \lim_{n \to \infty} \phi_n(\omega, \cdot) \quad \text{a.s.}
\]

Proof. For any \(n \in J\), define the measurable multifunction \(F_n\), with convex values in \(X \times \mathbb{R}\), by setting
\[
F_n(\omega) := \text{epi } \phi_n(\omega, \cdot) \quad \forall \omega \in \Omega.
\]

Since \((\phi_n)_{n \in J}\) is a supermartingale integrand, \((F_n)_{n \in J}\) is a multivalued supermartingale with closed convex values in \(X \times \mathbb{R}\). Further, condition (i) implies that \((F_n)_{n \in J}\) satisfies condition (i) of Theorem 5.12. Similarly, condition (ii) implies that \((F_n)_{n \in J}\) satisfies condition (ii) of Theorem 5.12.
Indeed, for any \((r, s) \in \mathbb{R}^2_+\) and for any \(\omega \in \Omega\),
\[
(L(\omega) \times \mathbb{R}) \cap (\mathbb{B}(0, r) \times [-s, +s]) = (L(\omega) \cap \mathbb{B}(0, r)) \times (\mathbb{R} \cap [-s, +s])
\]
is weakly compact in \(X \times \mathbb{R}\), so that multifunction \(\omega \to L(\omega) \times \mathbb{R}\) enjoys the same property as multifunction \(H\) in Theorem 5.12. Therefore, we can apply Theorem 5.12 which proves the existence of a multivalued multifunction \(F\), with closed convex values in \(X \times \mathbb{R}\), such that
\[
F(\omega) = \lim_{n \to \infty} F_n(\omega) \quad \text{a.s.}
\]
Obviously, for almost all \(\omega \in \Omega\), \(F(\omega)\) is the epigraph of an \(\mathbb{R}\)-valued function \(\phi(\omega, \cdot)\) defined on \(X\) and \(\phi\) is an integrable convex normal integrand.

Q.E.D.

Remark 6.4. Using Theorem 6.3 and the well-known variational properties of the Mosco-convergence (see for example, Section 2.2 in [2; 20, or 28a, 28b] for the case where \(X\) is finite dimensional), it is possible to derive approximation results for some stochastic optimization problems.

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REFERENCES


Jørgensen et application aux amarts multivoques. In Séminaire d'Analyse Convexe de l'Université de Montpellier, Exposé n° 8.


