



Thinning on cell complexes from polygonal tilings[☆]

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ABSTRACT

This paper provides a theoretical foundation of a thinning method due to Kovalevsky for 2D digital binary images modelled by cell complexes or, equivalently, by Alexandrov spaces, whenever these are constructed from polygonal tilings. We analyze the relation between local and global simplicity of cells, and prove their equivalence under certain conditions. For the proof we apply a digital Jordan theorem due to Neumann–Lara/Wilson which is valid in any connected planar locally Hamiltonian graph. Therefore we first prove that the incidence graph of the cell complex constructed from any polygonal tiling has these properties, showing that it is a triangulation of the plane. Moreover, we prove that the parallel performance of Kovalevsky's thinning method preserves topology in the sense that the numbers of connected components, for both the object and of the background, remain the same.

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1. Introduction

This paper provides a theoretical foundation of thinning in 2D digital binary images modelled by cell complexes, or equivalently, by topological Alexandrov spaces. Thinning is an important preprocessing method widely used in digital image processing to facilitate the classification or recognition of objects of interest. In the case of binary images where the set of objects has already been determined, thinning is an iterative procedure which produces a particular subset named skeleton, from the set of all object elements. The skeleton should represent topological properties like connectedness as well as geometrical properties related to the size and form of the object, and it should have as few as possible elements. During the thinning, so-called simple and non-end object elements are deleted from the “frontier” of the remaining object in each iteration. Due to classical work by Rosenfeld [24], end elements are situated at the ends of arcs, that should be preserved as part of the skeleton, whereas simple elements are those whose deletion preserves the connectedness of the object and of the background. In the spirit of this idea we define the following:

Definition 1. A thinning method, to be applied to an object within a 2D binary image, is said to **preserve topology** whenever it preserves both the number of connected components of the object and the number of connected components of the background.

Important theoretical questions about the thinning are,

- (1) the characterization of simplicity by local properties, considered only within a certain neighborhood of the element.
- (2) the question is if a proposed method can be parallelized, that means, if the parallel implementation of the method preserves topology.

[☆] Extended version of “Thinning on quadratic, triangular, and hexagonal cell complexes”, presented at IWCIA'2008.

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What “connected component”, “frontier”, “simple”, and “end” mean, depends on the mathematical model used for the domain of (definition of) the digital image. The domain of a nD (n -dimensional) digital image usually is supposed to be a discrete set in \mathbb{R}^n , that is, a discrete subset of the Euclidean topological space \mathbb{R}^n , and within the context of Digital Image Processing and Analysis, it is most frequently modelled by an adjacency graph, see [8,22]. For 2D images, this graph is usually related to the quadratic tiling of the plane, but hexagonal and triangular tilings (see [4,29]), and even much more general families of subsets of \mathbb{R}^2 (see [3,11,25]) have also been considered to be theoretically and practically valuable and useful.

The domain of a nD digital image can be alternatively modelled by an n -dimensional cell complex, or, equivalently, by an Alexandroff T_0 space, where usually the discrete set is identified with the set of n -cells. This model has been proposed and applied by Kovalevsky, see [12–17], but also by other authors, see for example [10,28] and [30]. In contrast to the cell model used in [8] where the lower-dimensional cells are used to define the adjacency relation between n -cells, Kovalevsky uses all cells, even in algorithms. In this regard, in a short note within the paper [13] a thinning method was proposed which seems to be the first (and unique, so far) proposal of a thinning algorithm on cell complexes. The same method was shortly described within [14,15,17], with slightly distinct definitions but without profound details, without proofs, and presenting the same unique example. These publications of Kovalevsky opened interesting theoretical and algorithmic questions which provided the motivation of our investigations.

This paper develops a theoretical foundation of Kovalevsky’s thinning method, based on a study of cell complexes constructed from polygonal plane tilings. We answer both theoretical questions cited above: First, we prove that local simplicity is sufficient to satisfy global simplicity, and that both are equivalent under certain conditions (Theorem 12 and Corollary 13 in Section 6). Second, we show that Kovalevsky’s thinning method can be parallelized, by showing that the parallel performance of the method preserves topology (Theorem 15 and Corollary 16 in Section 8). For the proof of the equivalence between local and global simplicity, we apply a digital Jordan theorem due to Neumann-Lara/Wilson [21] which is valid in any connected planar locally Hamiltonian graph. Therefore, we first prove that the incidence graph of the cell complex constructed from any polygonal tiling has these properties, showing that it is a triangulation of the plane (Theorem 5 in Section 5). This latter fact is of interest in the study of cell complexes and Alexandroff spaces, independently from thinning. This paper pretends to contribute to digital topology; it does not pretend neither a computational analysis of Kovalevsky’s thinning method nor an analysis of properties and practical relevance of Kovalevsky skeletons.

The novelties of this paper with respect to the previous version published in the Proceedings of IWICIA 2008 [29] are the following ones:

1. Our cell complex is generated from any polygonal tiling, applying a general construction due to [30]. In contrast, the cell complex in [29] was constructed only from the quadratic, the triangular, and the hexagonal tilings in an intuitive way.
2. All proofs of the present paper are valid for the general supposition of cell complexes constructed from polygonal tilings. In contrast, many proofs of [29] used intuitive arguments valid only for the three specific cell complexes considered, and were less detailed.
3. Our Theorem 4 was not proved in [29]. Our Theorem 5, which is one of the main results of this paper and has importance independently from thinning theory, was not even mentioned in [29].

The paper is organized as follows: In Sections 2 and 3, preliminaries about cell complexes, Alexandroff topological spaces and important suppositions are presented. Section 4 reports the construction of a cell complex from any polygonal tiling from [30]. In Section 5, we prove that the incidence graph of this cell complex is a special triangulation of the plane (Theorem 5). This fact is applied in Section 6 where the relation between simplicity and local simplicity as well as a characterization of simplicity by a connectivity number are studied (Theorem 12). Section 7 presents and analyzes Kovalevsky’s thinning algorithm. Section 8 presents a proof of the fact that the parallel implementation of Kovalevsky’s algorithm preserves topology (Theorem 15). Section 9 contains concluding remarks.

Throughout the paper, \mathbb{N} , \mathbb{Z} , \mathbb{R} denote the sets of natural, integer, and real numbers, respectively. In a topological space X , for $M \subseteq X$, $cl_X(M)$ denotes the closure of M , $int_X(M)$ its interior, $fr_X(M)$ its frontier, but we omit the index X if possible. \mathbb{R}^2 denotes the Euclidean plane equipped with the standard topology. For a finite set M , $|M|$ denotes the number of its elements.

2. Cell complexes

Recall the definition of a cell complex from [23], as it has been used in many papers of Kovalevsky:

Definition 2. An (abstract) cell complex is a structure (X, \leq, \dim) where (X, \leq) is a poset (partially ordered set, that is, \leq is a binary reflexive transitive and antisymmetric relation on the set X), and $\dim : X \rightarrow \mathbb{N} \cup \{0\}$ is a function such that $x \leq y$ implies $\dim(x) \leq \dim(y)$, for any $x, y \in X$. The elements of X are called cells, and, for $x \in X$, if $\dim(x) = k$, x is named k -cell. The dimension of (X, \leq, \dim) is defined by $\sup\{\dim(x) : x \in X\}$.

If (X, \leq, \dim) is a cell complex, then a subcomplex $M = (M, \leq_M, \dim_M)$ of X is entirely determined by the subset $M \subseteq X$, by defining \leq_M as the restriction of \leq onto $M \times M$, and \dim_M as the restriction of \dim onto M .

We suppose the domain of a 2D digital image to be modelled by a two-dimensional cell complex C where an object of interest is modelled by a finite subcomplex T , such that the image function assigns the value 1 to any cell of T , whereas each cell of $T^c = C \setminus T$ has the value 0; T^c is named the background. To model the digital image by a cell complex C , supposing that the 2D image is defined on a discrete set $D \subset \mathbb{R}^2$, we follow the usual idea that D is identified with the set of 2-cells

of C . In [12–17], $D = \mathbb{Z}^2$ (standard case) was identified with the set of 2-cells of a quadratic cell complex, but note that \mathbb{Z}^2 could be identified with the set of 2-cells of a triangular and of a hexagonal cell complex, too. For defining the subcomplex which models the object, the lower-dimensional cells have to have been generated additionally. We suppose C to form a decomposition of \mathbb{R}^2 . Then the natural quotient map $\pi : \mathbb{R}^2 \rightarrow C$ which assigns to each $x \in \mathbb{R}^2$ the (unique) cell of C which contains x , is an example of a digitization map. For $M \subseteq \mathbb{R}^2$, $\pi(M)$ is the set of all cells which, considered as subsets of \mathbb{R}^2 , intersect M ; this is an analog to the Gauss digitization defined for a set of pixels, see page 56 of [8]. We will apply the map π in the construction of a cell complex from any polygonal tiling in Section 4. Under this philosophy, each object is the digital version of some subset of the Euclidean space. Kovalevsky generated the lower-dimensional cells of the object in a different way, independently of \mathbb{R}^2 , applying some (heuristically founded) rule; for example, a maximum rule is applied in [12–15] which implies that any object is a closed subcomplex.

Since in practice, a digital image is modelled by a finite portion M of the cell complex C , for many implementations of algorithms to be correctly working, the object T has to be supposed to not touch the boundary of the image domain M . Our proofs do not need such a supposition, because they work in the cell complex C corresponding to a whole plane tiling, and any object T is supposed to be finite (and hence has a bounded pre-image, under the digitization map). So, T is “surrounded” by cells of $M \setminus T$.

3. Cell complexes and Alexandroff spaces

An Alexandroff space is a topological space in which any element is contained in a minimal open neighborhood given as the intersection of all open sets which contain this element [1]. Alexandroff T_0 spaces and posets are equivalent structures (actually, there are isomorphic categories): For a given poset (X, \leq) , the set $st(x) = \{y \in X : x \leq y\}$, named the *open star* of x , is the minimal open neighborhood of x , and the family $\{st(x) : x \in X\} \cup \{\emptyset\}$ is a base of an Alexandroff T_0 topology τ on X . Conversely, for a given Alexandroff T_0 space (X, τ) , denoting by $U(x)$ the minimal open neighborhood of x , the corresponding partial order, called the *specialization order* of (X, τ) , is defined by $x \leq y \Leftrightarrow x \in cl(\{y\}) \Leftrightarrow y \in U(x)$. For this reason, a cell complex is a topological model for digital images. The specialization order \leq of (X, τ) determines completely the topology τ , for example, $cl(M) = \{y \in X : y \leq m \text{ for some } m \in M\}$, $int(M) = \{m \in M : m \leq y \text{ implies } y \in M\}$. For details, see [1,10].

Now denote by (X, τ) an Alexandroff T_0 space and by (X, \leq) its corresponding poset. We will also use the following concept from [13]: For $M \subseteq X$, its *open frontier* is defined by $of(M) = \{y \in X : cl(\{y\}) \cap M \neq \emptyset \text{ and } cl(\{y\}) \cap M^c \neq \emptyset\}$; clearly $of(M) = \{y \in X : m \leq y \text{ for some } m \in M \text{ and } m \leq y \text{ for some } m \in M^c\}$. The open frontier of M is the frontier of M in the dual topological space which is determined by the reversed order \geq .

In the following we recall some other known concepts and facts, for details we refer the reader to [1,10,31,30]: Two elements $x, y \in X$ are *incident* or *comparable* if $x \leq y$ or $y \leq x$. The set $in(x) = \{y \in X : x \text{ is incident with } y\}$, for $x \in X$, is named *incidence set* of x . The (reflexive symmetric) incidence relation gives rise to an undirected graph on X called *incidence graph* (or, comparability graph) which provides a well-known graph theoretical connectedness concept based on paths: A subset M of X is *incidence-connected* if for any $p, q \in X$, there exists $\{p_0, p_1, \dots, p_{n-1}, p_n\} \subseteq M$ such that $p_0 = p$ and $p_n = q$, and p_i is incident with p_{i+1} , $0 \leq i \leq n-1$ (a pq -path). A *component* (in the incidence graph) is defined to be a maximal incidence-connected subset of X . It is well-known, see [10], that **for any $M \subseteq X$, M is topologically connected** (there are no two disjoint non-empty proper subsets A and B of M , both open in the subspace M , such that $A \cup B = M$) **if and only if M is incidence-connected**. Hence, a subset of X is a component in the incidence graph if and only if it is a topologically connected component.

For any element x of X , its *local order dimension* in X (or height [2]) is defined to be the maximum length k of chains of the form $x_0 < x_1 < \dots < x_k = x$ in (X, \leq) , where $x < y$ means $x \leq y, x \neq y$; and it is defined to be infinite if this maximum does not exist. The *order dimension* of X is defined to be the supremum over the local order dimensions of all its elements. An element x of the cell complex X will be named *k-cell* if its local order dimension in X is equal to k . Note that, for a subcomplex M of X and $x \in M$, its local dimension in X can differ from its local dimension in M , because \leq_M is the restriction of \leq_X onto M . We mention that the order dimension is a topological dimension: it was proved in [31], that the order dimension of any poset coincides with the small inductive dimension (known from general topology) of the corresponding Alexandroff T_0 space.

4. Cell complexes from polygonal tilings

Let us report some well-known concepts (see [7] and [26]): A *polyhedron* is a bounded subset of \mathbb{R}^n which is the intersection of finitely many closed halfspaces, hence it is compact and convex. A two-dimensional polyhedron in \mathbb{R}^2 is named a *polygon*. For any non-empty convex set $M \subseteq \mathbb{R}^n$, there is precisely one affine subspace H of \mathbb{R}^n called the *carrier* of M which contains M and satisfies $int_H(M) \neq \emptyset$. A *plane tiling* is a countable family of closed subsets of the Euclidean plane \mathbb{R}^2 , named *tiles*, whose union covers the plane, and whose interiors are pairwise disjoint. A plane tiling is *locally finite* if for any $x \in \mathbb{R}^2$, there is an open disc which intersects only a finite number of tiles. A locally finite plane tiling is *normal* if each tile is homeomorphic to a closed disc in \mathbb{R}^2 , and *monohedral* if each tile is congruent to the same tile (*prototile*). A locally finite plane tiling is a *polygonal tiling* if each tile is a polygon. A polygonal tiling is *edge-to-edge* if the intersection of any two

tiles is a whole side of each of these tiles. It is well known that there exist only three monohedral edge-to-edge polygonal tilings (2.1.1 of [7]): the quadratic tiling (the prototile is a square), the triangular tiling (the prototile is a triangle), and the hexagonal tiling (the prototile is a hexagon).

Let \mathcal{T} be now a polygonal tiling. Note that we do not require neither the edge-to-edge nor the monohedral conditions. The construction of the cell complex, or equivalently, of the Alexandroff T_0 space, resumed in this section, follows a general proposal due to Kronheimer [18], but was developed in [30] (for any polyhedral tiling of \mathbb{R}^n).

Define the family $\mathcal{W} = \{int(T) : T \in \mathcal{T}\}$. Then, \mathcal{W} is also locally finite, and each element W of \mathcal{W} is the interior of a (convex) polygon. Moreover, the union of the elements of \mathcal{W} is dense in \mathbb{R}^2 (because of the local finiteness, the union of the tiles equals the closure of the point set union of \mathcal{W}). In consequence, \mathcal{W} is a “fenestration” of \mathbb{R}^2 due to Kronheimer [18], and it is a “polyhedral fenestration” of \mathbb{R}^2 as defined in [30].

For $x \in \mathbb{R}^2$, define $N_x = \{W \in \mathcal{W} : x \in cl(W)\}$, $P_x = \cap \{cl(W) : W \in N_x\}$. Note that N_x is finite, and hence P_x is a polyhedron whose carrier will be denoted by H_x . Clearly P_x is a polygon, or a line segment, or a single point, whose carrier H_x is \mathbb{R}^2 , or the whole line containing the segment, or the point itself, respectively. Define the following equivalence relation on \mathbb{R}^2 :

$$x \simeq y \Leftrightarrow N_x = N_y, \quad x, y \in \mathbb{R}^2.$$

Consider the quotient set $C = \mathbb{R}^2 / \simeq = \{[x] : x \in \mathbb{R}^2\}$ where $[x] = \{y \in \mathbb{R}^2 : x \simeq y\}$, and denote by π the natural quotient map from \mathbb{R}^2 onto C : $\pi(y) = [x]$ whenever $x \simeq y$. We endow C with the quotient topology τ induced by π , that is, define $M \in \tau \Leftrightarrow \pi^{-1}(M)$ is open in the Euclidean plane \mathbb{R}^2 , for any $M \subseteq C$. The following proposition was proved in [30].

Proposition 3. *(C, τ) is an Alexandroff T_0 space which is an open quotient space of \mathbb{R}^2 (π is an open map), where for any $x \in \mathbb{R}^2$, $\pi^{-1}(\{[x]\}) = int_{H_x}(P_x)$. Moreover, $C = \mathcal{W}_2 \cup \mathcal{W}_1 \cup \mathcal{W}_0$, where $\mathcal{W}_2 = \pi(\mathcal{W}) = \{[x] : |N_x| = 1\}$, $\mathcal{W}_1 = \{[x] : |N_x| = 2\}$, $\mathcal{W}_0 = \{[x] : |N_x| \geq 3\}$. Among all open quotients of \mathbb{R}^2 which contain the family \mathcal{W}_2 , C is minimal in the following sense: For any other open quotient C' of \mathbb{R}^2 which contains \mathcal{W}_2 , any continuous open map of C onto C' , which is injective on \mathcal{W}_2 , is a homeomorphism. The specialization order of C provides a two-dimensional cell complex (with order dimension equal to two).*

As a consequence (for details, see [30]), any minimal open quotient C of \mathbb{R}^2 is a topological Alexandroff T_0 space unique up to homeomorphisms, which, for any fixed family \mathcal{W} , is named *the digital space constructed from \mathcal{W}* in [18] and [30].

Any $[x] \in \mathcal{W}_2 = \pi(\mathcal{W})$ is an open cell in (C, τ) . For $[x] \in \mathcal{W}_1$, $N_x = \{U, V\}$ for some distinct $U, V \in \mathcal{W}$, and then $U([x]) = \{[x], [y] = \pi(V), [z] = \pi(W)\}$. For $[x] \in \mathcal{W}_0$, $[x] = \{x\}$, and $U([x]) = \{[x]\} \cup \{[y] \in \mathcal{W} \cup \mathcal{W}_1 : x \in cl_{\mathbb{R}^2}(\pi^{-1}(\{[y]\}))\}$. The specialization order \leq of (C, τ) coincides with the natural bounding relation between polyhedra: $[x] \leq [y] \Leftrightarrow [x] \in cl_C([y]) \Leftrightarrow \pi^{-1}(\{[x]\}) \subseteq cl_{\mathbb{R}^2}(\pi^{-1}(\{[y]\}))$, for $[x], [y] \in X$.

From now on, **the term cell complex and the notation C or (C, \leq, \dim) always will refer to the cell complex constructed from any polygonal tiling \mathcal{T}** , as just explained. It is clear that each element of \mathcal{W} (the interior of a polygon) projects on a 2-cell, each element of \mathcal{W}_1 (a line segment without its end points) is a 1-cell, and the elements of \mathcal{W}_0 (isolated points) are the 0-cells of C . In the rest of the paper, (C, τ) **will denote the corresponding Alexandroff T_0 space, and T is supposed to be a finite subcomplex of C and will be named object**. Note that T is not supposed to be closed or open.

5. The incidence graph of a cell complex from a polygonal tiling

For $x \in C$, recall that $st(x) = \{y \in C : x \leq y\}$ (open star), $cl(\{x\}) = \{y \in C : y \leq x\}$ (closure), and $in(x) = \{y \in C : x \leq y \text{ or } y \leq x\}$ (incidence set). Let us also denote $st^*(x) = st(x) \setminus \{x\}$, $cl^*(x) = cl(\{x\}) \setminus \{x\}$, $in^*(x) = in(x) \setminus \{x\}$. Clearly $in(x) = st(x) \cup cl(\{x\})$ for any $x \in C$; $in(x) = st(x)$ for any 0-cell x , and $in(x) = cl(\{x\})$ for any 2-cell x . For $M \subseteq C$, recall $fr(M) = \{y \in X : y \leq m \text{ for some } m \in M \text{ and } y \leq m \text{ for some } m \in M^c\}$ (frontier in C), and $of(M) = \{y \in X : m \leq y \text{ for some } m \in M \text{ and } m \leq y \text{ for some } m \in M^c\}$ (open frontier in C). It is evident that $fr(M)$ does not contain any 2-cell, and $of(M)$ does not contain any 0-cell.

The following theorem is of essential importance because it provides the possibility of defining a connectivity number and to characterize global simplicity by local simplicity, which will be the content of the next section.

Theorem 4. *For the cell complex C under consideration and any cell $p \in C$, the set of cells $in^*(p)$ can be ordered in a cyclic sequence $\langle c_1, c_2, \dots, c_k \rangle$ such that, within this sequence,*

- (i) *any two consecutive cells are incident, and any cell is incident with exactly two other cells,*
- (ii) *the cells are alternating a-cells and b-cells, where $(a, b) = (1, 2)$ for any 0-cell p , $(a, b) = (0, 2)$ for any 1-cell p , $(a, b) = (0, 1)$ for any 2-cell p .*
- (iii) *For any 1-cell p , the sequence has exactly four cells.*

Proof. Any $p \in \mathcal{W}_2$ (2-cell) is open in (C, τ) , then $in^*(p) = cl^*(p) = fr_C(\{p\})$ which is a set of 1- and 0-cells. $\pi^{-1}(fr_C(\{p\}))$ is the boundary of the tile $V = cl_{\mathbb{R}^2}(W)$ where $W = \pi^{-1}(\{p\})$, hence it is a simple closed (Jordan) curve which consists of t line segments and t vertices. Let us register the vertices v_i and line segments s_i (taken without their end points) while tracing this curve in the mathematically positive sense, ordered due to their appearance, to generate a cyclic sequence $S = \langle v_1, s_1, v_2, s_2, \dots, v_t, s_t \rangle$.

Consider now one arbitrary line segment s_j of S . Since the tiling is not required to be edge-to-edge, there is no guarantee for s_j to provide a unique 1-cell of C . Nevertheless, V intersects only a finite number of other tiles (since \mathcal{T} is locally finite and $fr_{\mathbb{R}^2}(V)$ is compact). Hence s_j intersects only a finite number j_k of tiles distinct from V , say V_1, V_2, \dots, V_{j_k} . Due to the construction of C , $\pi(s_j)$ consists of the images, under π , of r line segments of the form $int_{s_j}(V \cap V_m)$ ($1 \leq m \leq j_k$) and $r - 1$ vertices given as the intersection of at least three tiles of the set $\{V, V_1, V_2, \dots, V_{j_k}\}$ where $1 \leq r \leq j_k$. Now, substitute s_j in the sequence S by the sequence of r line segments and $r - 1$ vertices just generated, but due to the ordering as they appear during the tracing of the Jordan curve $fr_{\mathbb{R}^2}(V)$.

Performing such a substitution for each s_j in S , we obtain a new cyclic sequence $S' = \langle v'_1, s'_1, v'_2, s'_2, \dots, v'_d, s'_d \rangle$, $d \leq |\{V' \in \mathcal{T} : V \cap V' \neq \emptyset, V \neq V'\}|$ of vertices v'_i and line segments s'_j (without end points). In consequence, $\langle c_1 = \pi(v'_1), c_2 = \pi(s'_1), \dots, c_{2d-1} = \pi(v'_d), c_{2d} = \pi(s'_d) \rangle$ is a cyclic sequence of alternating 0- and 1-cells, which by construction, satisfy properties (i)–(ii) of the theorem.

Now let $p \in \mathcal{W}_0$ (0-cell), then $p = [x] = \{x\}$ for some $x \in \mathbb{R}^2$ which lies in the intersection of at least three distinct tiles. Now $in^*(p) = st^*(p)$ which consists of 1- and 2-cells whose preimages (under π) are all open polygons of the finite set N_x and all line segments which belong to the intersection of exactly two elements of N_x and contain in its closure the point x . Based on the fact that $x \in int_{\mathbb{R}^2}(\cup\{cl(W) : W \in N_x\})$ (Lemma 4.2 of [30]), let us consider an open disc centered at x and contained in $int_{\mathbb{R}^2}(\cup\{cl(W) : W \in N_x\})$. Clearly the intersection of this disc with each $W \in N_x$ is an open sector. These (at least three) sectors, together with the line segments bounded by x , whose part inside the disc is contained in the intersection of the closures of exactly two sectors, can be ordered in a cyclic sequence of alternating sectors and line segments, whose images under π provide the sequence of alternating 2- and 1-cells, satisfying (i)–(ii) of the theorem.

For any $p \in \mathcal{W}_1$ (1-cell), by Proposition 3, there exist exactly two tiles V_1, V_2 with $W_1 = int_{\mathbb{R}^2}(V_1), W_2 = int_{\mathbb{R}^2}(V_2)$ such that $N_x = \{W_1, W_2\}$ for any $x \in \pi^{-1}(\{p\})$. In consequence, $in^*(p) \cap \mathcal{W}_2 = \{\pi(W_1), \pi(W_2)\}$. Now consider the two endpoints y_1, y_2 of the line segment $s = \pi^{-1}(\{p\})$ (these exist since s is bounded). Then $\pi(y_1), \pi(y_2) \in st^*(p) \subset in^*(p)$. If $\{z\}$ would be another 0-cell in $in^*(p)$, $z \neq y_1, z \neq y_2$, then $z \in \pi^{-1}(\{p\})$ (z is an “interior” point of s), but there would exist a third tile V_3 such that $int_{\mathbb{R}^2}(V_3) \in N_x$ which is a contradiction. In consequence $in^*(p) \cap \mathcal{W}_0 = \{\pi(y_1), \pi(y_2)\}$. Clearly p is not incident with any other 1-cell, implying $in^*(p) = \{\pi(W_1), \pi(W_2), \pi(y_1), \pi(y_2)\}$, which proves (iii) and also makes evident (ii) for any 1-cell. \square

Recall that any (undirected) graph (X, R) (X a set, R an irreflexive symmetric binary relation) has a representation in \mathbb{R}^2 , where the elements of X are drawn as points, and each pair of points $(x, y) \in R$ is represented by a straight line segment. We call the graph (X, R) a triangulation of the plane if it has a representation in \mathbb{R}^2 which is a polygonal edge-to-edge tiling whose tiles are all triangles. The degree of an element $x \in X$ is defined to be the number of $y \in X$ such that $(x, y) \in R$. We recall also (see [21]) that a subgraph (induced by) $M \subset X$ is defined as the graph given by all elements of M , and all pairs $(a, b) \in R$ such that $a, b \in M$. A cycle is defined to be a subgraph A in which each element is in relation with exactly two other elements of A . Evidently, any sequence of three elements $a, b, c \in X$ such that $(a, b), (b, c), (c, a) \in R$, is a cycle.

The last theorem implies the following fact:

Theorem 5. *The incidence graph of the cell complex from any polygonal tiling is a triangulation of the plane. Each 0-cell or 2-cell has degree ≥ 6 , each 1-cell has degree 4, each cycle of length 3 contains exactly one cell of each dimension 0, 1, 2.*

Proof. We will construct the desired representation in \mathbb{R}^2 of the incidence graph of C , using the construction of the cyclic sequences of cells in $in^*(p)$ as performed in the proof of the previous Theorem 4, and based on Proposition 3. For each cell $p \in C$, choose a unique point $\hat{p} \in \pi^{-1}(\{p\}) \subset \mathbb{R}^2$. For distinct $p_1, p_2 \in C$ it follows $\hat{p}_1 \neq \hat{p}_2$ since C is a decomposition of \mathbb{R}^2 . So, the set $\hat{C} = \{\hat{p} : p \in C\}$ is a set of points of \mathbb{R}^2 which represents C . For $p \in \mathcal{W}_0$, \hat{p} is an interior point of the (open) polygon $\pi^{-1}(\{p\})$; for $p \in \mathcal{W}_1$, \hat{p} is a point (distinct from the end points) of the line segment $\pi^{-1}(\{p\})$; for $p \in \mathcal{W}_2$, $p = \{x\}$ for some $x \in \mathbb{R}^2$ and $\hat{p} = x$. Now, for any distinct $p, q \in C$, connect the points \hat{p}, \hat{q} by a straight line segment $[\hat{p}, \hat{q}]$ whenever p, q are incident. We will show now that the representation of the incidence graph of C given by the vertex set \hat{C} and the straight line segments $[\hat{p}, \hat{q}]$, is an edge-to-edge triangular plane tiling.

(i) Let $p \in \mathcal{W}_2$. In the proof of Theorem 4, $in^*(p) = cl^*(p)$ was ordered in a cycle $\pi(S') = \langle c_1 = \pi(v'_1), c_2 = \pi(s'_1), \dots, c_{2d-1} = \pi(v'_d), c_{2d} = \pi(s'_d) \rangle$ of alternating 0-cells $\pi(v'_i)$ and 1-cells $\pi(s'_j)$, which represents one complete tracing of the boundary of the convex polygon $W = \pi^{-1}(\{p\})$. Clearly all pairs $[\hat{c}_i, \hat{c}_{i+1}]$, for $i = 1, 2, \dots, 2d - 1$, and the pair $[\hat{c}_{2d}, \hat{c}_1]$, are line segments of the representation. From the convexity of W it is clear that each line segment $[\hat{p}, \hat{c}_i] \setminus \{\hat{c}_i\}$ lies in W . From the construction of the cycle S' it follows that the set of triples $\{\{\hat{p}, \hat{c}_i, \hat{c}_{i+1}\}, i = 1, 2, \dots, 2d - 1\} \cup \{\{\hat{p}, \hat{c}_{2d}, \hat{c}_1\}\}$ form a set of triangles whose interiors are pairwise disjoint. As a consequence, the proposed representation, when (its application is) restricted to $cl_C(\{p\})$, is an edge-to-edge triangular tiling.

(ii) Very similar arguments can be used for the case $p \in \mathcal{W}_0$, where $p = [x] = \{x\}$ for some $x \in \mathbb{R}^2$. From the construction of the cycle of 1- and 2-cells of $in^*(p) = st^*(p)$ in the proof of Theorem 4, it follows that the proposed representation, when restricted to $st(p)$, is an edge-to-edge triangular tiling.

(iii) Consider now an arbitrary triangle of the representation, constructed in (i) or (ii), that is, which belongs to the restriction of the representation to a set $cl_C(\{p\})$ for some 2-cell p , or, to a set $st_C(\{p\})$ for some 0-cell p . Consider the first case (the other case could be treated analogously): let p be a 2-cell and consider the triangle $\Delta(\hat{p}, \hat{q}, \hat{r})$ of the representation. Since any two k -cells of C are not comparable, p, q, r have to be cells of pairwise distinct dimension (which confirms an affirmation

of the theorem), say, r a 1-cell and q a 0-cell. Then $q < r < p$, $q, r \in cl^*(p)$, $r, p \in st^*(q)$, which imply that the three line segments $[\hat{p}, \hat{q}] = [\hat{q}, \hat{p}]$, $[\hat{q}, \hat{r}] = [\hat{r}, \hat{q}]$, $[\hat{r}, \hat{p}] = [\hat{p}, \hat{r}]$ and hence the whole triangle $\Delta(\hat{p}, \hat{q}, \hat{r})$ belong to the restriction of the representation to $cl_C(\{p\})$ as well as to $st(q)$ (where q is a 2-cell). In consequence, each triangle reported in (i) coincides with a triangle reported in (ii). This implies that the whole incidence graph of C is connected.

(iv) To complete to prove that the whole incidence graph of C is a triangulation, we show that, given any triangle reported in (i) or in (ii), any side of this triangle belongs to another uniquely defined triangle reported in (i) or in (ii). Consider a triangle from (i) (the case of a triangle from (ii) could be treated analogously.): Let p be a 2-cell and consider $M = \Delta(\hat{p}, \hat{q}, \hat{r})$, where r is a 1-cell and q is a 0-cell, so, $q < r < p$. The line segment $[\hat{p}, \hat{q}]$, due to the construction in the proof of Theorem 4, is a side of exactly two triangles which belong to the representation restricted to $cl(\{p\})$, one of which coincides with M . Consider now the line segment $[\hat{q}, \hat{r}]$. A triangle distinct from M using the vertices \hat{q}, \hat{r} can only have a third vertex which is a 2-cell $s \neq p$, but this s is uniquely determined since the 1-cell r is incident with exactly two 2-cells. Then $q < r < s$ which implies that $\Delta(\hat{q}, \hat{r}, \hat{s})$ belongs to the representation restricted to $st(q)$. The analysis of the line segment $[\hat{p}, \hat{r}]$ is similar: A triangle distinct from M using the vertices \hat{p}, \hat{r} can only have a third vertex which is a 0-cell $t \neq q$, but this t is uniquely determined since the 1-cell r is incident with exactly two 0-cells. Then $t < r < p$ which implies that $\Delta(\hat{p}, \hat{r}, \hat{t})$ belongs to the representation restricted to $cl(\{p\})$.

(v) In the incidence graph of C , each 2-cell has degree ≥ 6 since each polygon has at least three vertices and three sides. Each 1-cell has degree four, by Theorem 4. Each 0-cell has degree ≥ 6 since it is incident with at least three 2-cells by Proposition 3 and then also with at least three 1-cells. \square

6. Simplicity, local simplicity, and connectivity number

In analogy to usual definitions (see [24,22]) of simple points and end points in adjacency graphs, we define the following:

Definition 6. Let $p \in T$. The cell p is named **globally simple** if T has the same number of connected components as $T \setminus \{p\}$, and, T^c has the same number of connected components as $T^c \cup \{p\}$. The cell p is named **end cell** if it is incident with exactly one cell $q \in T$, $q \neq p$.

Kovalevsky proposed in [14] and [15] two other definitions of simplicity, which are locally determined, and will be reported in the following:

Definition 7. Let $p \in T$. The cell p is named **(locally) simple** if $p \in fr(T)$ and $st^*(p) \cap T$ and $st^*(p) \cap T^c$ both are non-empty and connected, or, if $p \in of(T)$, and $cl^*(p) \cap T$ and $cl^*(p) \cap T^c$ both are non-empty and connected. The cell p is named **in-simple** or **incidence-simple** if $in^*(p) \cap T$ and $in^*(p) \cap T^c$ both are non-empty and connected.

A locally simple cell was named simple in [14], and an in-simple cell was named IS-simple in [15]. These two concepts of local simplicity are in general not equivalent. The following relation between them is easily derived:

Lemma 8. Let $p \in (fr(T) \cup of(T)) \cap T$.

- (i) If p is a 1-cell and $p \in fr(T)$, then p is simple if and only if one of the two 2-cells of $in^*(p)$ lies in T and the other in T^c ; p is not simple if and only if both 2-cells of $in^*(p)$ belong to T^c .
- (ii) If p is a 1-cell, and $p \in of(T)$, then p is simple if and only if one of the two 0-cells of $in^*(p)$ lies in T and the other in T^c ; p is not simple if and only if both 0-cells of $in^*(p)$ belong to T^c ;
- (iii) If p is a 0-cell or a 2-cell or a non-end 1-cell, then p is simple if and only if p is in-simple.
- (iv) Any end 1-cell is in-simple, but its local simplicity can fail.

Proof. Let $p \in (fr(T) \cup of(T)) \cap T$.

(i) and its dual version (ii) are clear.

(iii) If p is a 0-cell then $p \in fr(T)$ and $in^*(p) = st^*(p)$; analogously, if p is a 2-cell then $p \in of(T)$ and $in^*(p) = cl^*(p)$. In both cases, using (i), clearly p is simple if and only if p is in-simple.

Now let p be a non-end 1-cell. From Theorem 4, $in^*(p)$ can be written as a cyclic sequence (c_1, c_2, c_3, c_4) , where, without restriction of generality, c_1, c_3 are 2-cells, and c_2, c_4 are 0-cells. Suppose that $p \in fr(T)$, then from (i) we can assume $c_3 \in T^c$, and simplicity of p implies $c_1 \in T$ (or $c_1 \in T$), which implies that p is in-simple, independently from the values (1 or 0) of c_2, c_4 . Conversely, if p is in-simple, because $p \in T$, $c_3 \in T^c$, the hypothesis $c_1 \in T^c$, based on the fact that $in^*(p)$ intersects T in at least two cells, would imply $c_2, c_4 \in T$, contradicting that p is in-simple. In consequence, $c_1 \in T$ and hence p is simple. The proof for the case $p \in of(T)$ would be completely analogous.

(iv) If p is an end 1-cell, then $in^*(p) \cap T = \{q\}$. Evidently, then p is in-simple. Nevertheless, taking into account (i), if $p \in fr(T)$ and q is not a 2-cell (or, analogously, if $p \in of(T)$, and q is not a 0-cell), then p is not locally simple. \square

We note in passing that the definition of simplicity in [15] said “in case that $p \in fr(T)$, if...” instead of our version “if $p \in fr(T)$, and...”. The original definition from [15] would cause contradictions in the context of our paper where we do not suppose the object T to be closed (this supposition is implicit in [15]), and then, there can exist (end 1-) cells in $fr(T) \cap of(T) \cap T$ which would be simple, considered as belonging to $fr(T)$ but not simple, when considered as belonging to $of(T)$; or vice versa.

It is easy to see that any globally simple or end cell of T belongs to $(fr(T) \cup of(T)) \cap T$, so, only cells of this set are candidates to be deleted in a thinning algorithm. In order to achieve a reasonable efficiency of practical thinning algorithms, the global simplicity of a cell has to be decided by local properties, see [24,22]. For thinning on adjacency graphs, for example, template matching techniques and “connectivity numbers” are popular. For a thinning algorithm to be theoretically well founded and preserving topology, it is necessary that these local properties provide a confident base for the decision for a cell to be deleted. The fact that local simplicity (which is used in the algorithm) implies global simplicity, would guarantee to delete not too many cells; whereas the equivalence between the two simplicities would also guarantee to delete not too few cells.

Based on [Theorem 4](#) we now define a connectivity number, which will be used for deducing a characterization of global simplicity by local simplicity, but also provides a key for an efficient implementation of Kovalevsky’s algorithm.

Definition 9. Let $p \in T$ and let $\langle c_1, \dots, c_k \rangle$ be a cyclic sequence of the cells of $in^*(p)$, which satisfies all properties of [Theorem 4](#). Denote by v_i the value (1 or 0) of the cell c_i in the binary image represented by C , for $i = 1, \dots, k = |in^*(p)|$. Define the **connectivity number** of p as $cn(p) = \sum_{i=0}^{|in^*(p)|} |v_i - v_{(i+1)}|$, where the sum $(i + 1)$ is calculated modulo $|in^*(p)|$.

The connectivity number is the number of changes from value 1 to value 0 or vice versa, in the set $in^*(p)$, and its definition depends on [Theorem 4](#). It is easy to see that $cn(p)$ is independent of the particular selection of the cyclic sequence (two distinct such sequences only distinguish by a shifting on the cycle), and we have the following properties:

Lemma 10. (i) For any non-isolated cell $p \in T$ (that is, $in(p)^* \cap T \neq \emptyset$), $cn(p)$ is strictly positive if and only if $p \in fr(T) \cup of(T)$.
(ii) For $p \in fr(T) \cup of(T)$, the number of connected components of $in(p)^* \cap T$ equals the number of connected components of $in(p)^* \cap T^c$, and both are equal to $\frac{1}{2} cn(p)$. Hence, $cn(p)$ is an even number.

Proof. (i) Let $p \in T$. $cn(p) > 0$ implies $p \in fr(T) \cup of(T)$, since, if $p \notin fr(T) \cup of(T)$, then $st(p)$ and $cl(\{p\})$ are subsets of T , hence $in(p)^* \subseteq T$ which implies $cn(p) = 0$.

Now let $p \in fr(T) \cup of(T)$. Consider $p \in fr(T)$, so, $st(p)$ intersects T^c . Taking into account that p is non-isolated, $in(p)^*$ intersects both T and T^c , implying $cn(p) > 0$. The case $p \in of(T)$ can be treated analogously, taking $cl(\{p\})$ instead of $st(p)$.

(ii) We have $cn(p) > 0$. Recall that $in(p)^*$ is a cycle in the incidence graph of C . We find then an even number l of components A_j , where $2 \leq l \leq |in(p)^*|$, which can be ordered, following the cycle of cells in $in(p)^*(1)$, as $\{A_1, \dots, A_l\}$, such that, without restricting generality, A_1, A_3, \dots, A_{l-1} are the components of $in(p)^* \cap T$, whereas A_2, A_4, \dots, A_l are the components of $in(p)^* \cap T^c$. It is then evident that $in(p)^* \cap T$ has $\frac{1}{2} cn(p)$ components, and that the same is true for $in(p)^* \cap T^c$. \square

The connectivity number will be used to characterize global simplicity, based on the following property:

Proposition 11. If $p \in (fr(T) \cup of(T)) \cap T$ is (locally) simple then p is globally simple.

Proof. Let $p \in fr(T)$, then p is a 0- or 1-cell (the proof for the case $p \in of(T)$ is analogous). Assuming that p is not globally simple, we prove that $st^*(p) \cap T$ or $st^*(p) \cap T^c$ is not connected: Based on [Definition 6](#), we have to study the following two suppositions:

(1) $T \setminus \{p\}$ has strictly more components than T .

(2) $T^c \cup \{p\}$ has strictly less components than T^c ($T^c \cup \{p\}$ cannot have strictly more components than T^c since $p \in fr(T)$).

Consider supposition (1) (the proof under (2) is similar). If p is a 0-cell, then there exists $q_1, q_2 \in T \setminus \{p\}$ such that there is no q_1q_2 -path in $T \setminus \{p\}$, but there exists a q_1q_2 -path w in T . Hence $w = \{q_1 = \gamma_1, \dots, \gamma_{i-1}, p, \gamma_{i+1}, \dots, \gamma_n = q_2\}$, where $2 \leq i \leq n - 1$ and $\gamma_{i-1}, \gamma_{i+1}$ are distinct. Supposing that $st^*(p) \cap T$ is connected, there is a $\gamma_{i-1}, \gamma_{i+1}$ -path z in $st^*(p) \cap T$, implying that $\{q_1 = \gamma_1, \dots, \gamma_{i-1}\} \cup z \cup \{\gamma_{i+1}, \dots, \gamma_n = q_2\}$ is a q_1q_2 -path in $T \setminus \{p\}$ which contradicts our supposition. In consequence, $st^*(p) \cap T$ is not connected.

If p is a 1-cell, let $\{p_1, p_2\} = cl^*(p)$, and $\{c_1, c_2\} = st^*(p)$. Because $p \in fr(T) \cap T$, $st^*(p) \cap T^c \neq \emptyset$. Furthermore, by studying the incidence set of p , it is not difficult to prove that $st^*(p) \cap T = \emptyset$, which implies by [Theorem 4](#) that $st^*(p) \cap T^c = \{c_1, c_2\}$ which is not connected. \square

The following proposition applies the previous one as well as a digital Jordan curve theorem proved in [21]. Recall the graph theoretic concepts from Section 5. Moreover, an arbitrary graph (X, R) is called *locally Hamiltonian* if for each $p \in X$, the set $\{q \in X : q \neq p, (p, q) \in R\}$ is a cycle (see [21]). A *triangle* is defined to be a cycle of three elements (and three pairs). Recall that the incidence graph $G = (C, R)$ of the cell complex C is given by the set of all cells of C together with the incidence relation R . Then (C, R) is locally Hamiltonian if for each $p \in C$, $in^*(p)$ is a cycle.

Theorem 12. For any $p \in (fr(T) \cup of(T)) \cap T$, p is globally simple if and only if $cn(p) = 2$.

Proof. Let $p \in fr(T)$, then p is a 0- or 1-cell (the proof for $p \in of(T)$ would be analogous). Suppose first that $cn(p) = 2$. To show that p is globally simple, it is sufficient by [Proposition 11](#) to prove that $st^* \cap T$ and $st^* \cap T^c$ both are non-empty and connected. If p is a 0-cell then $in^*(p) = st^*(p)$, and [Lemma 10](#) implies that each of the sets $st^*(p) \cap T$ and $st^*(p) \cap T^c$ consists of exactly one component. If p is a 1-cell then $st^*(p) \cap T = \emptyset$, or $st^*(p) \cap T$ has exactly one element. In both situations p is globally simple.

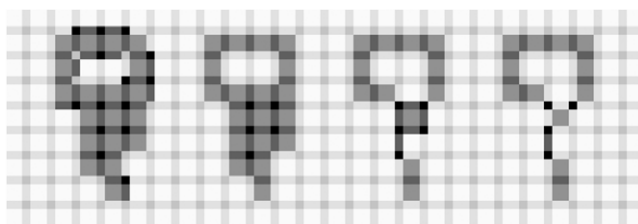


Fig. 1. Example of the application of Kovalevsky's algorithm to a subcomplex (of the quadratic cell complex) which is neither closed nor open. From left to the right: Original subcomplex (object), Result after the first step of the first iteration (simple non-end cells from the frontier were deleted), Result after the second step of the first iteration (simple non-end cells from the open frontier were deleted), Result after the first step of the second iteration which is the Kovalevsky skeleton.

Suppose now that $cn(p) \neq 2$, and let us prove that then p is not globally simple. By Lemma 10 and using $p \in fr(T)$, we can assume that $cn(p) \geq 4$. Hence each of the two sets $in^*(p) \cap T$ and $in^*(p)$ has at least two components. Choose a cyclic sequence $\langle c_0, c_1, \dots, c_k \rangle$ of $in^*(p)$ such that the cells c_α, c_β belong to distinct components of $in^*(p) \cap T^c$, and c_γ, c_δ belong to distinct components of $in^*(p) \cap T$, and $\alpha < \gamma < \beta < \delta$. If c_α, c_β belong to distinct components of T^c then p is not globally simple because c_α, c_β belong to the same component of $T^c \cup \{p\}$. Suppose now that c_α, c_β belong to the same component of T^c . If c_γ, c_δ belong to distinct components of $T \setminus \{p\}$ then they belong to the same component of T , hence again, p is not globally simple.

Now consider that c_γ, c_δ are cells of the same component of $T \setminus \{p\}$. In the following we will show that this is not possible (together with, that c_α, c_β belong to the same component of T^c). Suppose this situation. Since the incidence graph of C is connected by Theorem 5 and T induces a finite subgraph, there is a (finite) c_γ, c_δ -path $w = \{a_1 = c_\gamma, a_2, \dots, a_k = c_\delta\}$ in the subgraph induced by $T \setminus \{p\}$. Then clearly $w' = w \cup \{p\}$ is a closed path which does not contain c_α, c_β . Since the incidence graph is by Theorem 5 a triangulation, it is easy to reduce the closed (cyclic) path $w' = \langle a_0 = p, a_1, \dots, a_k \rangle$ into a cycle $v = \langle a_0 = p, a'_1, a'_2, \dots, a'_k \rangle$ (where each element is incident only with its predecessor and its successor in v), $v \subseteq w'$. (The idea to reduce w' into v is as follows: Perform one complete tracing of w' , and for $i = 0, 1, \dots, k$, if the pairs (a_i, a_{i+1}) and (a_{i+1}, a_{i+2}) belong to the same triangle, then eliminate a_{i+1} from w' ; calculate the indices modulo $k + 1$.) Note that the existence of c_α, c_β implies that v contains p and contains strictly more than three elements. The cycle v satisfies the suppositions of Theorem 1 of [21] (to be a graph-theoretical Jordan curve): it is a cycle of length strictly greater than three, in a triangulation which clearly is a connected planar locally Hamiltonian graph. Hence the subgraph induced by $C \setminus v$ has exactly two components in C . It is evident that c_α, c_β lie in distinct components of $C \setminus v$. Hence, any c_α, c_β -path must intersect $v \subset T$, which contradicts that c_α, c_β lie in the same component of T^c .

In consequence, $cn(p) \neq 2$ implies that p is not globally simple which completes the proof. \square

Theorem 12, Lemmas 10 and 8 imply the following equivalences.

- Corollary 13.** (i) For any $p \in T$, p is globally simple if and only if p is in-simple.
 (ii) If $p \in (fr(T) \cup of(T)) \cap T$ is a 0-cell or 2-cell or non-end 1-cell, then p is globally simple if and only if p is (locally) simple.

7. Kovalevsky's thinning algorithm

Recall that each cell of the object T has value 1, and each cell of T^c has value 0. To delete a cell of T means that its value is changed from 1 to 0. Thus a cell belongs after its deletion to T^c . We quote from [14] and [15] the following thinning algorithm:

Definition 14 (Kovalevsky's Algorithm). Let T be an object in the cell complex C . Each iteration consists in the following two steps:

Step 1: Detect and delete all cells from $fr(T) \cap T$, which are simple and non-end. Count the number of cells which are deleted in this step, and denote it by a . Let T be the remaining object.

Step 2: Detect and delete all cells from $of(T) \cap T$, which are simple and non-end. Count the number of cells which are eliminated in this step, and denote it by b . Let T be the remaining object.

In the case that $a + b \neq 0$, perform the next iteration starting with step 1; in the case $a + b = 0$, the algorithm is finished, and the actual remaining object T is considered the result of the algorithm and will be called the **Kovalevsky skeleton**.

Kovalevsky's proposal of thinning algorithm is general and can be applied to any object within any cell complex. The unique example in [14,15,17] involves a closed object within the quadratic cell complex. The simplicity concept used in [14] is equivalent to our "locally simple" concept, whereas the simplicity concept used by Kovalevsky in [15] and [17] corresponds to our "in-simple" concept. Our propositions and theorems of the previous section prove that cells which are "simple" or "IS-simple" according to Kovalevsky's definitions are also globally simple. The Fig. 1 shows an example of the application of Kovalevsky's algorithm to a subcomplex which is neither closed nor open.

Kovalevsky's algorithm can be performed both by a sequential or by a parallel implementation. In a sequential implementation, step 1 of the algorithm (and, similarly, step 2) works in the following manner: First, determine $fr(T) \cap T = \{m_1, m_2, \dots, m_k\}$. Then, for $i = 1, 2, \dots, k$, in the case that m_i is recognized to be in-simple and non-end in T , delete m_i immediately from T , that is, $T := T \setminus \{m_i\}$, before proceeding to check the next element m_{i+1} . In consequence, the property of the cell m_{i+1} to be in-simple and non-end can be influenced by whether the cell m_i was deleted before or still exists. It is evident that the sequential deletion of in-simple (equivalently, globally simple) elements always preserves topology. Hence, in particular, the performance of a sequential implementation of Kovalevsky's algorithm, applied to a non-empty connected object, produces a non-empty connected Kovalevsky skeleton.

The situation is distinct for the **parallel implementation of Kovalevsky's algorithm**. In this case, step 1 (and, similarly, step 2) works as follows: First, determine $fr(T) \cap T = \{m_1, m_2, \dots, m_k\}$. Then, for $i = 1, 2, \dots, k$, in the case that m_i is recognized to be in-simple and non-end in T , it is marked but not (yet) deleted. When all cells of $\{m_1, m_2, \dots, m_k\}$ have been checked, then all marked cells are deleted, that is, $T := T \setminus \{m \in fr(T) \cap T : \text{min-simple and non-end in } T\}$. In consequence, the fact that some cell m_i is marked as to be deleted later, does not have any influence on whether m_{i+1} is simple in T or not. In other words, the in-simplicity of a cell is checked always with respect to the object which equals the actual remaining whole object at the beginning of the step.

It is well-known from the literature about thinning that the resulting skeletons obtained from sequential and from parallel implementations of the same thinning algorithm can be quite distinct, and that it is far from trivial to find out whether a parallel implementation of a thinning algorithm preserves topology. We will show in the next section that the parallel implementation of Kovalevsky's algorithm preserves topology.

8. Parallel thinning due to Kovalevsky's algorithm

Based on the equivalence between global simplicity and in-simplicity by [Corollary 13](#), we will use in-simplicity in this section. Recall from [Definition 14](#) that each iteration of Kovalevsky's algorithm consists of two steps. The following theorem will imply that Kovalevsky's algorithm can be parallelized.

Theorem 15. *Let T be a non-empty object in the cell complex C . Denote by T_k the remaining object after having applied k steps of the parallel implementation of Kovalevsky's algorithm to the input object T , for $k \geq 0$, where $T_0 = T$. Then, for all $k \geq 1$, the number of connected components of T_k is equal to the number of components of T , and also, the number of connected components of T_k^c equals the number of components of T^c .*

Proof. We present the main idea of the proof, given by induction on k . Let us call two objects equivalent if they have the same number of connected components, and if the numbers of connected components of their complements also coincide. In the induction base it is proved that T_1 is equivalent to T , and then, under the induction hypothesis that T_k is equivalent to T , it is proved that T_{k+1} is equivalent to T . Both the induction base and the induction step proof, are based on the following reasoning:

Let R be an object which is equivalent to T , and whose in-simple non-end cells r_1, \dots, r_n are cells of its frontier [or, analogously, of its open frontier], which are arbitrarily ordered and have been detected in a parallel manner. The latter means that each of these cells was detected to be in-simple and non-end, as a cell of the whole object R . It is proved using $(n - 1)$ steps that $R_1 = R \setminus \{r_1, \dots, r_n\}$ is equivalent to T . The l -th step consists in proving that r_{l+1}, \dots, r_n are simple in $R \setminus \{r_1, \dots, r_l\}$. That $R \setminus \{r_1, \dots, r_l\}$ is equivalent to T , is obtained in the $(l - 1)$ -th step, for $l > 1$. In the case $l = 1$, we apply that R is equivalent to T . In the $(n - 1)$ -th step, it is proved that the cell r_n is in-simple in the object $R \setminus \{r_1, \dots, r_{n-1}\}$, so, it is proved that R_1 is equivalent to T . Observe that R_1 is the result of having applied to the object R one step of the parallel implementation of Kovalevsky's algorithm, which corresponds to the treating of cells of the frontier [or, analogously, of the open frontier]. \square

Corollary 16. *The parallel implementation of Kovalevsky's algorithm preserves topology. In particular, the Kovalevsky skeleton of any non-empty object is non-empty, and, the Kovalevsky skeleton of any connected object is connected.*

For illustration, the [Figs. 2 and 3](#) show results of the application of the parallel implementation of Kovalevsky's algorithm to objects modelled on the quadratic cell complex. Both figures contain on the left an object modelled as subcomplex and its Kovalevsky skeleton inscribed, and on the right a standard (pixel set) skeleton generated by a thinning procedure included in the Digital Image Analysis System DIAS [5], which essentially is the Eckhardt/Maderlechner skeleton [6]. In these examples, the exactness of the Kovalevsky skeleton with respect to topology preservation can be observed, as well as certain similarity with the Eckhardt/Maderlechner skeleton.

Note that in this paper, the concept of parallel implementation has the theoretical meaning explained above and is not necessarily related to the use of parallel processors. In the sequential implementation, during tracing the (open) frontier of the object, each cell detected to be simple non-end, is deleted immediately. But applying the parallel implementation, such a cell is only marked, and after having checked the whole (open) frontier, in each iteration we need an additional tracing of the (open) frontier to delete all marked cells, when using a standard computer.

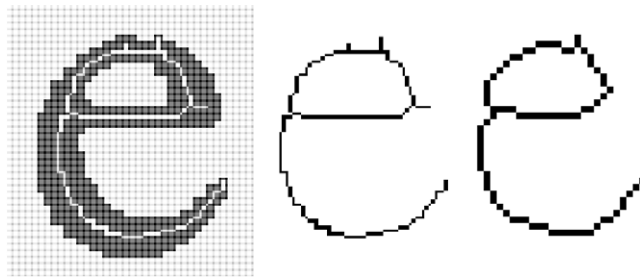


Fig. 2. On the left: an object subcomplex and its Kovalevsky skeleton inscribed; in the middle: the Kovalevsky skeleton which is a subcomplex; on the right: the Eckardt/Maderlechner skeleton of the object which is a pixel set.

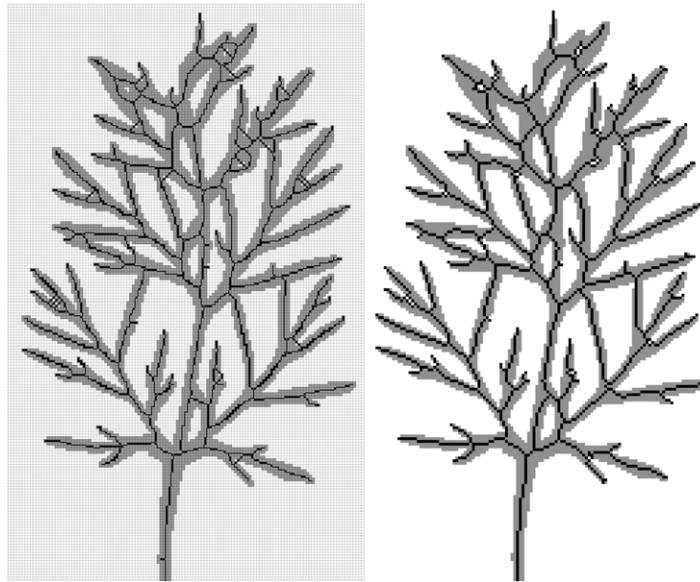


Fig. 3. On the left: an object subcomplex and its Kovalevsky skeleton inscribed (black); on the right: the Eckardt/Maderlechner skeleton (black) of the object.

9. Comments and concluding remarks

In this paper we provided a theoretical foundation of Kovalevsky's thinning algorithm, if it is applied to a 2D binary image modelled by a cell complex constructed from any polygonal tiling. We proved that local simplicity is sufficient for global simplicity, and we characterized global simplicity by a connectivity number which is locally computed. This characterization indicates that incidence-simplicity is equivalent to global simplicity. For the corresponding proof, we applied a digital Jordan theorem from [21] valid in any connected planar locally Hamiltonian graph. We proved in detail in this paper that the incidence graph constructed from any polygonal tiling has these properties, showing that it is a triangulation of the plane, with some additional special properties. This latter result is of interest in digital topology independently from thinning.

Moreover, we presented the idea of a proof that the parallel realization of the algorithm preserves topology. The reasoning in this proof has similarity with the discussion in [9], where the simplicity of sets is studied. We will further investigate the possibility to prove our theorem based on results of [9].

Both parallel and sequential implementations of Kovalevsky's thinning algorithm use local tests for deciding whether a cell should be deleted or not. This implies that the time complexity of the algorithm is $O(n)$ where n is the number of elements of the object, which is standard for thinning algorithms [22]. Although in practice, the skeleton can have much less elements than the original object, there exists the worst case such that each element of the object belongs to the skeleton. In Kovalevsky's algorithm, for each object cell to be deleted or not, the corresponding local test studies the open star or the closure of this cell, which for example in the case of the quadratic cell complex is a set of at most eight other cells. Practical running time of the algorithm can be speeded up by involving at the beginning of each iteration a rapid boundary tracing algorithm (see for example Section 11.2 of [17]) in order to determine the (open) frontier. The number of necessary iterations depends on the "thickness" of the object.

With respect to the space complexity of Kovalevsky's algorithm, we comment that storage of the whole object is needed. Whereas for other thinning algorithms, the object is a pixel set, our object is a subcomplex of 2-, 1-, and 0-cells. Under use

of the quadratic cell complex, if our subcomplex is constructed from a pixel set with m elements by interpretation of the pixels as 2-cells and generation of the lower-dimensional cells as additional structure (as proposed in [17]), then we need to store approximately $4m$ object cells (each pixel = square generates two 1-cells and a vertex 0-cell, and the whole object will need some 1- and 0-cells more to complete the frontier if we wish the object to be a closed subcomplex).

Future projects include, besides generalizations of our theoretical studies to weaker conditions on the initial tiling and to higher dimensional images, the computational analysis of the algorithm, the study of topological and geometrical properties of the skeletons, the comparison of Kovalevsky skeletons with those obtained by other thinning algorithms which have been proposed in the literature and are based on the adjacency graph model, and the study of the quality and usefulness of Kovalevsky skeletons related to practical applications (including preprocessing and postprocessing for achieving certain quality). We mention that Kovalevsky's thinning algorithm was implemented on the quadratic cell complex and a triangular cell complex using the connectivity number [20], and for the hexagonal cell complex [27]. Both sequential and parallel implementations for the quadratic cell complex based on template matching were performed in [32], where also a first idea for converting any Kovalevsky skeleton (subcomplex) into a sub-adjacency-graph was developed.

To see that topological properties of the Kovalevsky skeleton are interesting, we mention the observation from [29] that despite its irreducibility (it does not contain any simple non-end cell), the skeleton of a (two-dimensional) object can have order (and hence topological) dimension equal to two, which contradicts the intuitive idea, that a skeleton should be "thin" or "curve-like". Similar properties were observed for skeletons on adjacency graphs [19]. This fact could be interesting in the context of defining digital curves.

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