Applied Mathematics Letters 24 (2011) 1046–1050

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

# A fractional characteristic method for solving fractional partial differential equations

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ARTICLE INFO

*Article history:*
Received 22 August 2010
Received in revised form 18 January 2011
Accepted 25 January 2011

*Keywords:*
Modified Riemann–Liouville derivative
Fractional characteristics method
Fractional partial differential equations

ABSTRACT

The method of characteristics has played a very important role in mathematical physics. Previously, it has been employed to solve the initial value problem for partial differential equations of first order. In this work, we propose a new fractional characteristic method and use it to solve some fractional partial differential equations.



## 1. Introduction

In the past few centuries, many methods of mathematical physics have been developed for solving partial differential equations (PDEs) [1,2]; among these, the method of characteristics is a particularly efficient technique [3].

Fractional partial differential equations have attracted the interest of many researchers. Thus a question naturally arises: would it be possible to derive the exact solutions of partial differential equations (FPDEs) using a fractional method of characteristics? Recently, with a modified Riemann–Liouville derivative [4], it was obtained that

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha-1}(f(\xi)-f(0))d\xi, \quad n-1<\alpha<n. \tag{1} $$

Jumarie proposed a Lagrange characteristic method [5], which can solve FPDEs:

$$ a(x,t)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + b(x,t)\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = c(x,t), \quad 0<\alpha,\ \beta<1, \tag{2} $$

where $x \in [0, 1]$, $n-1 \le \alpha < n$ and $n \ge 1$. However, in most cases, the two fractional order parameters $\alpha$ and $\beta$ may not be equivalent. In this letter, we consider the case $\alpha \ne \beta$ and present a more general fractional method of characteristics.

## 2. Preliminaries

In recent years, in order to investigate the local behaviors of fractional models, several local versions of fractional derivatives have been proposed, i.e., the Kolwankar–Gangal local fractional derivative [6–8], Chen's fractal derivative [9,10], Cresson's derivative [11], Jumarie's modified Riemann–Liouville derivative [4] and Parvate's $F^{\alpha}$ derivative [12]; Jumarie's derivative is defined as

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \quad 0<\alpha<1 \tag{3} $$

where $f : R \to R$, $x \to f(x)$ denotes a continuous (but not necessarily first-order-differentiable) function. We can obtain the following properties:

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doi:10.1016/j.aml.2011.01.020

**Property 1.** *Let $f(x)$ satisfy the definition of the modified Riemann–Liouville derivative and $f(x)$ be a $k\alpha$th-order-differentiable function, where $k$ is arbitrary. The generalized Taylor series is given as* [4]

$$f(x+h)=\sum_{k=0}^{\infty}\frac{h^{\alpha k}}{(\alpha k)!}f^{(\alpha k)}(x),\quad 0<\alpha<1. \tag{4}$$

*Readers are to understand that the order $\alpha k$ used throughout the work is just notation; it is defined by $D^{\alpha k}f(x)=\underbrace{D^{\alpha}\cdots D^{\alpha}}_{k}f(x)$. Generally, we can see that*

$${}_0D_x^{\alpha}\,{}_0D_x^{\beta}\neq D^{\alpha+\beta}.$$

**Property 2.** *Assume that $f(x)$ denotes a continuous $R\to R$ function. We use the following equality for the integral w.r.t. $(\mathrm{d}x)^{\alpha}$* [4]*:*

$${}_0I_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}f(\xi)\mathrm{d}\xi=\frac{1}{\Gamma(\alpha+1)}\int_0^x f(\xi)(\mathrm{d}\xi)^{\alpha},\quad 0<\alpha\leq 1. \tag{5}$$

**Property 3.** *Some useful formulas:*

$$f^{(\alpha)}([x(t)])=\frac{\mathrm{d}f}{\mathrm{d}x}x^{(\alpha)}(t), \tag{6}$$

$${}_0D_x^{\alpha}x^{\beta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha}, \tag{7}$$

$$\int(\mathrm{d}x)^{\beta}=x^{\beta}, \tag{8}$$

$$\Gamma(1+\alpha)\mathrm{d}x=\mathrm{d}^{\alpha}x. \tag{9}$$

*The function $f(x)$ should be differentiable with respect to $x(t)$ and $x(t)$ is fractional differentiable in (6). The above results are employed in the following sections.*

*The modified Riemann–Liouville derivative has been successfully applied in probability calculus* [13]*, fractional Laplace problems* [14]*, the fractional variational approach with several variables* [15]*, the fractional variational iteration method* [16]*, the fractional variational approach with natural boundary conditions* [17] *and the fractional Lie group method* [18]*.*

## 3. The new fractional characteristic method

It is well known that the method of characteristics has played a very important role in mathematical physics. Previously, the method of characteristics has been used to solve the initial value problem for general first-order cases. Consider the following first-order equation:

$$a(x,t)\frac{\partial u(x,t)}{\partial x}+b(x,t)\frac{\partial u(x,t)}{\partial t}=c(x,t). \tag{10}$$

The goal of the method of characteristics is to change coordinates from $(x,t)$ to a new coordinate system $(x_0,s)$ in which the PDE becomes an ordinary differential equation (ODE) along certain curves in the $x$–$t$ plane. The curves are called the characteristic curves, and read

$$\frac{\mathrm{d}u}{c(x,t)}=\mathrm{d}s, \tag{11}$$

$$\frac{\mathrm{d}x}{a(x,t)}=\mathrm{d}s, \tag{12}$$

$$\frac{\mathrm{d}t}{b(x,t)}=\mathrm{d}s. \tag{13}$$

With the modified Riemann–Liouville derivative, Jumarie even gave a Lagrange characteristic method [5], which can solve some classes of fractional partial differential equations. In this work, we present a more generalized fractional method of characteristics and use it to solve linear fractional partial equations. More generally, we extend this method to linear space–time fractional differential equations

$$a(x,t)\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}+b(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=c(x,t),\quad 0<\alpha,\ \beta<1. \tag{14}$$

We expand $u$ as a fractional Jumarie–Taylor series of multivariate functions [4]. As a result, we derive the total derivative in more generalized form:

$$\mathrm{d}u = \frac{\partial^\beta u(x,t)}{\Gamma(1+\beta)\partial x^\beta}(\mathrm{d}x)^\beta + \frac{\partial^\alpha u(x,t)}{\Gamma(1+\alpha)\partial t^\alpha}(\mathrm{d}t)^\alpha, \quad 0 < \alpha,\ \beta < 1. \tag{15}$$

The function $u$ here is $\beta$th-order differentiable with respect to $x$ and $\alpha$th-order differentiable with respect to $t$ in the sense of Jumarie's derivative.

Similarly, note that the generalized characteristic curves can be presented as

$$\frac{\mathrm{d}u}{\mathrm{d}s} = c(x,t), \tag{16}$$

$$\frac{(\mathrm{d}x)^\beta}{\Gamma(1+\beta)\mathrm{d}s} = a(x,t), \tag{17}$$

$$\frac{(\mathrm{d}t)^\alpha}{\Gamma(1+\alpha)\mathrm{d}s} = b(x,t). \tag{18}$$

Eqs. (16)–(18) can be reduced using Jumarie's Lagrange method of characteristics (see A.12 in Ref. [5]) if we assume that $\alpha = \beta$:

$$\frac{(\mathrm{d}x)^\alpha}{a(x,t)} = \frac{(\mathrm{d}t)^\alpha}{b(x,t)} = \frac{\Gamma(1+\alpha)\mathrm{d}u}{c(x,t)} = \frac{\mathrm{d}^\alpha u}{c(x,t)}. \tag{19}$$

Obviously, if $\alpha = 1$ in Eq. (19), we can get a characteristic curve of integer order for Eq. (10).

## 4. Applications

**Example 1.** As the first example, we consider space–time fractional equations for the transport equation in porous media:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + c\frac{\partial^\beta u(x,t)}{\partial x^\beta} = 0, \quad 0 < \alpha,\ \beta \le 1, \tag{20}$$

where $c$ is a constant. Assume Eq. (20) is subject to the initial value $u(x,0) = \phi(x)$.

The generalized characteristic curves satisfy

$$\frac{\mathrm{d}u}{\mathrm{d}s} = 0, \tag{21}$$

$$(\mathrm{d}x)^\beta = c\Gamma(1+\beta)\mathrm{d}s, \tag{22}$$

$$(\mathrm{d}t)^\alpha = \Gamma(1+\alpha)\mathrm{d}s. \tag{23}$$

As a result, we can obtain

$$\frac{x^\beta}{\Gamma(1+\beta)} = cs + C_1, \tag{24}$$

$$\frac{t^\alpha}{\Gamma(1+\alpha)} = s + C_2, \tag{25}$$

$$u = C_3 \tag{26}$$

where $C_1$, $C_2$ and $C_3$ are integral constants. Eliminating the parameter $s$, we find that the fractional curves are $\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)} = x_0$, and $u$ is a constant along the fractional curves.

We can directly derive the exact solution of Eq. (19) in the following form:

$$u = f\left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right), \tag{27}$$

where $f\left(\frac{x^\beta}{\Gamma(1+\beta)}\right) = \phi(x)$. By setting $u(x,t) = f\left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)$, we get an exact solution for the initial value problem here.

**Example 2.** We investigate a more complicated line fractional differential equation:

$$\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{x^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=0,\quad 0<\alpha,\beta\leq 1. \tag{28}$$

We can obtain the generalized curve equations

$$\frac{\mathrm{d}u}{\mathrm{d}s}=0, \tag{29}$$

$$\frac{(\mathrm{d}x)^{\beta}}{\Gamma(1+\beta)\mathrm{d}s}=\frac{x^{\beta}}{\Gamma(1+\beta)}, \tag{30}$$

$$\frac{(\mathrm{d}t)^{\alpha}}{\Gamma(1+\alpha)\mathrm{d}s}=\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{31}$$

We assume that the function $u$ is differentiable w.r.t. $\frac{x^{\beta}}{\Gamma(1+\alpha)}$. We can readily obtain the following solution through Eq. (8):

$$\int\frac{(\mathrm{d}x)^{\beta}}{x^{\beta}}=\ln\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\right). \tag{32}$$

As a result, direct calculation leads to

$$\frac{x^{\beta}}{\Gamma(1+\beta)}=c_1\mathrm{e}^{s}, \tag{33}$$

$$\frac{t^{\alpha}}{\Gamma(1+\alpha)}=c_2\mathrm{e}^{s}, \tag{34}$$

$$u=c_3. \tag{35}$$

Then we find Eq. (28) that has the following solutions:

$$u=f\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\Big/\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right), \tag{36}$$

where the function $f$ is arbitrary.

Assume that $X=\frac{x^{\beta}}{\Gamma(1+\beta)}/\frac{t^{\alpha}}{\Gamma(1+\alpha)}$. We note that

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=f_X X_t^{(\alpha)}=-f_X\frac{x^{\beta}}{\Gamma(1+\beta)}\Big/\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}, \tag{37}$$

and

$$\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=f_X X_x^{(\beta)}=f_X\Big/\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{38}$$

As a result, we can check that

$$\frac{t^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{x^{\beta}}{\Gamma(1+\beta)}\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=f_X\frac{x^{\beta}}{\Gamma(1+\beta)}\Big/\frac{t^{\alpha}}{\Gamma(1+\alpha)}-f_X\frac{x^{\beta}}{\Gamma(1+\beta)}\Big/\frac{t^{\alpha}}{\Gamma(1+\alpha)}=0.$$

## 5. Summary

The classical method of characteristics is an efficient technique for solving partial differential equations. In this study, the fractional method of characteristics is considered for application to some classes of fractional partial differential equations, and two examples are given to illustrate its efficiency. Besides this, the method presented provides an efficient tool for solving fractional symmetry equations in the Lie group method [18].

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