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On the Cauchy Problem for the Nonlinear Biharmonic Equation*

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1. INTRODUCTION

In [8] the author derived an a priori inequality from which uniqueness and continuous dependence on the data were deduced for the solution of the Cauchy problem for the inhomogeneous biharmonic equation and for some weakly coupled linear and nonlinear systems. These systems may be considered to have evolved from certain fourth order elliptic partial differential equations whose principal part is the biharmonic operator. In these improperly posed problems, the solution, or in the case of systems, the noncoupling function, was assumed to be uniformly bounded. Thus it was shown that one need not impose any additional restriction on the solution function (as was done in [4] for the biharmonic equation) to obtain such results.

Improperly posed problems have received much attention recently (see the bibliography in [5]), as they have been found to represent problems in elasticity, geophysics, and the theory of potentials (see [4, 2, 7]). Here we consider the Cauchy problem for the nonlinear biharmonic equation

$$\Delta^2 v = h(x, v, v_{,i}, \Delta v, \Delta v_{,i}) \tag{1.1}$$

where Δ is the Laplace operator, the comma notation denotes partial differentiation with respect to x_i, h is assumed to satisfy a Lipschitz condition in its latter four arguments, and v is a C^4 function which is assumed to be uniformly bounded in some domain. We shall derive an a priori inequality in a manner different from [8], from which uniqueness and stability of the solution can be deduced. The development of this inequality, which has further use in the determination of pointwise bounds, entails some "special handling" due to the derivative terms which may be present.

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In Section 5 we remark on the application of the technique developed here to certain coupled systems of elliptic partial differential equations.

2. Formulation of the Problem

Let D be a domain in Euclidean n-space with boundary S (a Lyapunov boundary) and let Σ be that portion of the surface S on which Cauchy data is prescribed. We assume $\bar{\Sigma}$ is a C^1 surface. Let us denote by

$$f(x) = \alpha, \qquad 0 < \alpha \leqslant 1, \tag{2.1}$$

where $x = (x_1, x_2, ..., x_n)$, a family of (not necessarily closed) surfaces which intersect D and form, for each α, a closed region D_α whose boundary consists only of points of Σ and points on the surface $f(x) = \alpha$. Let Σ_α denote the portion of Σ and S_α denote the portion of the surface $f(x) = \alpha$ which forms the boundary of D_α. That this family is nonempty is clear from [8], where for $n > 2$ and r_0 and R_0 certain constants, we can write

$$f(x) = \left[1 - \left(\frac{r_0}{r}\right)^{n-2}\right] \Big/ \left[1 - \left(\frac{r_0}{R_0}\right)^{n-2}\right].$$

We shall assume that f is a C^2 function in $\bar{D}_1$ which satisfies

$$\begin{aligned} &\text{(i)} \quad \text{if } 0 < \lambda < \mu \leqslant 1, \text{ then } D_\lambda \subset D_\mu, \\ &\text{(ii)} \quad |\operatorname{grad} f| > \beta_0 > 0, \text{ in } D_1, \\ &\text{(iii)} \quad \Delta f \leqslant 0, \ |\Delta f| \leqslant c\beta_0^2, \text{ in } D_1, \end{aligned} \tag{2.2}$$

where c is a fixed constant. Furthermore, we assume that D_α, for $0 < \alpha \leqslant 1$, has nonzero volume and D_0 has zero volume.

Although in the following work several simplifications are possible if we consider regions D_α with the portion S_α of its boundary spherical (as in [8]), we choose the more general regions D_α determined by (2.1) because of their usefulness in the determination of pointwise bounds [6].

Instead of (1.1), we assume u and v satisfy

$$\begin{aligned} \Delta u &= h(x, v, v_{,i}, u, u_{,i}) \\ \Delta v &= u \end{aligned} \tag{2.3}$$

in D, where the notation and conditions were set forth earlier. On Σ we assume that u and v satisfy

$$\begin{aligned} &\int_\Sigma (v - v_0)^2 \, d\sigma \leqslant \pi_1, \qquad \int_\Sigma (v_{,i} - v_i)(v_{,i} - v_i) \, d\sigma \leqslant \pi_2, \\ &\int_\Sigma (u - u_0)^2 \, d\sigma \leqslant \pi_3, \qquad \int_\Sigma (u_{,i} - u_i)(u_{,i} - u_i) \, d\sigma \leqslant \pi_4, \end{aligned} \tag{2.4}$$

where $d\sigma$ is the element of surface on Σ, the repeated index denotes summation from 1 to n, the quantities v_0, v_i, u_0, u_i are the respective measured values of v, $v_{,i}$, u, $u_{,i}$, on Σ and π_1, π_2, π_3, and π_4 are known constants.

We set

$$V = v - \psi, \qquad U = u - \phi, \tag{2.5}$$

for approximating functions ψ and ϕ, where by (1.1) and (2.3), we assume

$$\phi = \Delta\psi. \tag{2.6}$$

From (2.3), we obtain

$$\Delta V = U, \qquad \Delta U = h(x, v, v_{,i}, u, u_{,i}) - h(x, \psi, \psi_{,i}, \phi, \phi_{,i}) + \mathscr{L}(\phi, \psi), \tag{2.7}$$

where

$$\mathscr{L}(\phi, \psi) = h(x, \psi, \psi_{,i}, \phi, \phi_{,i}) - \Delta\phi.$$

Thus

$$\begin{aligned} &|\Delta V| = |U|, \\ &|\Delta U| \leqslant L_1 |V| + L_2(V_{,i}V_{,i})^{1/2} + L_3 |U| + L_4(U_{,i}U_{,i})^{1/2} + |\mathscr{L}|, \end{aligned} \tag{2.8}$$

for L_1, L_2, L_3, and L_4, the appropriate Lipschitz constants.

Further, we introduce the notation

$$\begin{aligned} \epsilon_1 &= \int_\Sigma V^2\, d\sigma, \qquad \epsilon_2 = \int_\Sigma V_{,i}V_{,i}\, d\sigma, \\ \epsilon_3 &= \int_\Sigma U^2\, d\sigma, \qquad \epsilon_4 = \int_\Sigma U_{,i}U_{,i}\, d\sigma, \\ \epsilon_5 &= \iint_{D_1} [\mathscr{L}(\phi, \psi)]^2\, dx, \end{aligned} \tag{2.9}$$

where dx is the element of volume in D_1. Finally, since v is uniformly bounded, we assume

$$|V| \leqslant M$$

in D, for some prescribed constant M.

By means of the logarithmic convexity of a suitable functional, we shall derive the a priori estimate

$$\iint_{D_\alpha} (U^2 + V^2)\, dx \leqslant KM^{2(1-d)}[k_i\epsilon_i]^d, \tag{2.10}$$

where K and k_i are computable constants and d is a fixed number between 0 and 1. The uniqueness and stability results for the solution v in (2.3)–(2.4) then follow from (2.10). The form of the left side of (2.10) is chosen because of its usefulness in the more general problem discussed in Section 5 and in obtaining pointwise bounds.

3. Preliminary Results

Let us define the functional F by

$$F(\alpha) = \int_0^\alpha (\alpha - \eta) \iint_{D_\eta} [U_{,i}U_{,i} + U\Delta U + V_{,i}V_{,i} + V\Delta V]\, dx\, d\eta + k_i\epsilon_i \,, \tag{3.1}$$

where $0 \leqslant \alpha \leqslant 1$ and the k_i are constants to be determined. In the next section we shall show that F satisfies the differential inequality

$$FF'' - (F')^2 \geqslant - C_1FF' - C_2F^2 \tag{3.2}$$

for computable constants C_1 and C_2, where the prime denotes differentiation with respect to α. The desired a priori inequality follows by means of (3.2). Presently, we develop several estimates which are used to deduce (3.2).

By differentiation,

$$F' = \int_0^\alpha \iint_{D_\eta} [U_{,i}U_{,i} + U\Delta U + V_{,i}V_{,i} + V\Delta V]\, dx\, d\eta, \tag{3.3}$$

$$F'' = \iint_{D_\alpha} [U_{,i}U_{,i} + U\Delta U + V_{,i}V_{,i} + V\Delta V]\, dx. \tag{3.4}$$

Using Green's identity in (3.3) and recalling that $n_i = f_{,i} \,|\, \operatorname{grad} f \,|^{-1}$ on S_η, we can write

$$F' = \iint_{D_\alpha} (UU_{,i} + VV_{,i})f_{,i}\, dx + \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta. \tag{3.5}$$

Integration by parts now results in

$$F' = \frac{1}{2} \oint_{S_\alpha + \Sigma_\alpha} (U^2 + V^2) \frac{\partial f}{\partial n}\, d\sigma + \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta$$
$$- \frac{1}{2} \iint_{D_\alpha} (U^2 + V^2) f_{,ii}\, dx. \tag{3.6}$$

LEMMA 1. *For F defined by* (3.1), *we have*

$$(k_i - \theta_i)\,\epsilon_i + \frac{1}{2}\iint_{D_\alpha} \beta(U^2 + V^2)\,dx \leqslant F(\alpha)$$
$$\leqslant \frac{c+1}{2}\iint_{D_\alpha} \beta(U^2 + V^2)\,dx + (k_i + \theta_i)\,\epsilon_i\,, \tag{3.7}$$

where $\beta = f_{,i} f_{,i}$, the θ_i are computable constants, c is given by (2.2), *and the k_i are to be chosen so that $k_i > \theta_i$.*

From (3.6) we have

$$\begin{aligned}
F(\alpha) &= \int_0^\alpha F'(\eta)\,d\eta + k_i\epsilon_i \\
&= \int_0^\alpha \left\{\frac{1}{2}\int_{S_\eta}(U^2+V^2)\frac{\partial f}{\partial n}\,d\sigma + \frac{1}{2}\int_{\Sigma_\eta}(U^2+V^2)\frac{\partial f}{\partial n}\,d\sigma\right\}d\eta \\
&\quad + \int_0^\alpha\int_0^\eta\int_{\Sigma_\zeta}\left(U\frac{\partial U}{\partial n} + V\frac{\partial V}{\partial n}\right)d\sigma\,d\zeta\,d\eta \\
&\quad - \tfrac{1}{2}\int_0^\alpha\iint_{D_\eta}(U^2+V^2)f_{,ii}\,dx\,d\eta + k_i\epsilon_i \\
&= \frac{1}{2}\iint_{D_\alpha}\beta(U^2+V^2)\,dx + \frac{1}{2}\int_0^\alpha\int_{\Sigma_\eta}(U^2+V^2)\frac{\partial f}{\partial n}\,d\sigma\,d\eta \\
&\quad + \int_0^\alpha\int_0^\eta\int_{\Sigma_\zeta}\left(U\frac{\partial U}{\partial n} + V\frac{\partial V}{\partial n}\right)d\sigma\,d\zeta\,d\eta \\
&\quad - \tfrac{1}{2}\int_0^\alpha\iint_{D_\eta}(U^2+V^2)f_{,ii}\,dx\,d\eta + k_i\epsilon_i\,.
\end{aligned}$$

Consequently, by means of (2.2) and the arithmetic mean-geometric mean inequality (abbreviated A-G inequality), the result follows.

Before stating the next lemma we note that by means of (2.8) and the A-G inequality there exist constants a_1 and a_2 such that

$$\left|\iint_{D_\eta}[U\Delta U + V\Delta V]\,dx\right|$$
$$\leqslant a_1\iint_{D_\eta}\beta(U^2+V^2)\,dx + \tfrac{1}{2}\iint_{D_\eta}[U_{,i}U_{,i} + V_{,i}V_{,i}]\,dx + a_2\epsilon_5\,. \tag{3.8}$$

LEMMA 2. *For U and V defined by* (2.5) *and F by* (3.1), *we have*

$$\left| \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n} \right) d\sigma\, d\eta \right| \leqslant B_1 F, \tag{3.9}$$

$$\int_0^\alpha \iint_{D_\eta} [U_{,i} U_{,i} + V_{,i} V_{,i}]\, dx\, d\eta \leqslant 2F' + 2B_2 F, \tag{3.10}$$

$$\left| \int_0^\alpha \iint_{D_\eta} [U \Delta U + V \Delta V]\, dx\, d\eta \right| \leqslant F' + 2B_2 F, \tag{3.11}$$

for computable constants B_1 and B_2.

The first result here follows by the A-G inequality and (3.7). Since

$$\begin{aligned}\int_0^\alpha \iint_{D_\eta} [U_{,i} U_{,i} + V_{,i} V_{,i}]\, dx\, d\eta &= F' - \int_0^\alpha \iint_{D_\eta} [U \Delta U + V \Delta V]\, dx\, d\eta \\ &\leqslant F' + \left| \int_0^\alpha \iint_{D_\eta} [U \Delta U + V \Delta V]\, dx\, d\eta \right|,\end{aligned}$$

we have by (3.8)

$$\begin{aligned}&\int_0^\alpha \iint_{D_\eta} [U_{,i} U_{,i} + V_{,i} V_{,i}]\, dx\, d\eta \\ &\qquad \leqslant 2F' + 2 \int_0^\alpha \left\{ a_1 \iint_{D_\eta} \beta (U^2 + V^2)\, dx + a_2 \epsilon_5 \right\} d\eta.\end{aligned}$$

Thus we obtain the second result by means of (3.7). Again, by (3.8)

$$\begin{aligned}\left| \int_0^\alpha \iint_{D_\eta} [U \Delta U + V \Delta V]\, dx\, d\eta \right| &\leqslant \int_0^\alpha \left\{ a_1 \iint_{D_\eta} \beta (U^2 + V^2)\, dx + a_2 \epsilon_5 \right\} d\eta \\ &\quad + \tfrac{1}{2} \int_0^\alpha \iint_{D_\eta} [U_{,i} U_{,i} + V_{,i} V_{,i}]\, dx\, d\eta,\end{aligned}$$

so that the third result follows by (3.10).

LEMMA 3. *If F is defined by* (3.1), *then*

$$| F' | \leqslant 3F' + 4B_2 F, \tag{3.12}$$

where B_2 is the computable constant of the previous lemma.

This is an immediate consequence of (3.10) and (3.11).

LEMMA 4. *For U and V defined by* (2.5) *and F by* (3.1),

$$\left| \iint_{D_\alpha} \{U_{,i}U_{,i} + V_{,i}V_{,i} - 2\beta^{-1}[(U_{,i}f_{,i})^2 + (V_{,i}f_{,i})^2]\} \, dx \right| \leqslant B_3F' + B_4F \tag{3.13}$$

for computable constants B_3 and B_4.

As a special case of the identity in Lemma 2 in [6], we have

$$\begin{aligned}
&\iint_{D_\alpha} \{U_{,i}U_{,i} + V_{,i}V_{,i} - 2\beta^{-1}[(U_{,i}f_{,i})^2 + (V_{,i}f_{,i})^2]\} \, dx \\
&\quad = \int_0^\alpha \int_{\Sigma_\eta} \beta^{-1} \left\{ 2f_{,i} \left[U_{,i} \frac{\partial U}{\partial n} + V_{,i} \frac{\partial V}{\partial n} \right] - \frac{\partial f}{\partial n} [U_{,i}U_{,i} + V_{,i}V_{,i}] \right\} d\sigma \, d\eta \\
&\quad + \int_0^\alpha \iint_{D_\eta} \{(\beta^{-1}f_{,i})_{,i} (U_{,k}U_{,k} + V_{,k}V_{,k}) \\
&\quad - 2(\beta^{-1}f_{,i})_{,k} (U_{,i}U_{,k} + V_{,i}V_{,k})\} \, dx \, d\eta \\
&\quad - 2 \int_0^\alpha \iint_{D_\eta} \beta^{-1}f_{,i}(U_{,i}\Delta U + V_{,i}\Delta V) \, dx \, d\eta.
\end{aligned}$$

Using the A-G inequality, (2.8) and Lemmas 1 and 2, we arrive at (3.13).

We recall that in (2.3)–(2.4) only v was assumed to be uniformly bounded in D. Now we avoid imposing any additional requirement on v or on its derivatives, as in [4], by establishing that integrals of U^2 over compact subsets of D_1 are bounded in terms of M^2. This result is important in the final analysis of the convexity argument. Thus we prove

THEOREM I. *If U is defined by* (2.5), *then $\iint_{D_{\bar\alpha}} U^2 \, dx \leqslant C_0M^2$ for C_0 a computable constant and $\bar\alpha$ between* 0 *and* 1.

We define the function τ as

$$\tau(x) = \begin{cases} 1 & \text{in} \quad \bar D_{\bar\alpha} \\ \dfrac{1 - f(x)}{1 - \bar\alpha} & \text{in} \quad \bar D_1 - \bar D_{\bar\alpha} \end{cases},$$

where $\tau_{,i}\tau_{,i} \leqslant M_1$ and $| \tau_{,ii} | \leqslant M_2$ for constants M_1 and M_2, so that

$$\iint_{D_{\bar\alpha}} U^2 \, dx \leqslant \iint_{D_1} \tau^4 U^2 \, dx. \tag{3.14}$$

Now by (2.7) and integration by parts, we have

$$\iint_{D_1} \tau^4 U^2\,dx = \int_{\Sigma_1} \tau^4 U \frac{\partial V}{\partial n}\,d\sigma - \iint_{D_1} \tau^4 U_{,i} V_{,i}\,dx - \iint_{D_1} \tau^4{}_{,i} U V_{,i}\,dx,$$

so that utilizing the A-G inequality with positive constants γ_1 and γ_2, we obtain

$$\begin{aligned}\iint_{D_1} \tau^4 U^2\,dx \leqslant \int_{\Sigma_1} \tau^4 U \frac{\partial V}{\partial n}\,d\sigma &+ \frac{1}{2}\gamma_1 \iint_{D_1} \tau^6 U_{,i} U_{,i}\,dx \\ &+ \tfrac{1}{2}\gamma_1^{-1} \iint_{D_1} \tau^2 V_{,i} V_{,i}\,dx \\ &+ 2M_1\gamma_2 \iint_{D_1} \tau^4 U^2\,dx + 2\gamma_2^{-1} \iint_{D_1} \tau^2 V_{,i} V_{,i}\,dx.\end{aligned} \tag{3.15}$$

We now integrate the second integral on the right by parts and use (2.8) and the A-G inequality with constants γ_3 and γ_4. Thus

$$\begin{aligned}&\iint_{D_1} \tau^6 U_{,i} U_{,i}\,dx \\ &= \int_{\Sigma_1} \tau^6 U \frac{\partial U}{\partial n}\,d\sigma - \iint_{D_1} \tau^6 U \Delta U\,dx - \iint_{D_1} \tau^6{}_{,i} U U_{,i}\,dx \\ &\leqslant \int_{\Sigma_1} \tau^6 U \frac{\partial U}{\partial n}\,d\sigma + (\tfrac{3}{2} + L_3 + \tfrac{1}{2}\gamma_4^{-1} + 3M_1\gamma_3^{-1}) \iint_{D_1} \tau^4 U^2\,dx \\ &\quad + \tfrac{1}{2}L_1^2 \iint_{D_1} \tau^8 V^2\,dx + \tfrac{1}{2}L_2^2 \iint_{D_1} \tau^8 V_{,i} V_{,i}\,dx + \tfrac{1}{2} \iint_{D_1} \tau^8 \mathscr{L}^2\,dx \\ &\quad + (\tfrac{1}{2}L_4^2\gamma_4 + 3\gamma_3) \iint_{D_1} \tau^6 U_{,i} U_{,i}\,dx.\end{aligned}$$

By choosing $\gamma_3 = (12)^{-1}$ and $\gamma_4 = 2^{-1}L_4^{-2}$, collecting terms, and substituting in (3.15), we obtain

$$\begin{aligned}\iint_{D_1} \tau^4 U^2\,dx \leqslant \int_{\Sigma_1} \tau^4 U \frac{\partial V}{\partial n}\,d\sigma &+ \gamma_1 \int_{\Sigma_1} \tau^6 U \frac{\partial U}{\partial n}\,d\sigma \\ &+ [\gamma_1(\tfrac{3}{2} + L_3 + L_4^2 + 36M_1) + 2M_1\gamma_2] \iint_{D_1} \tau^4 U^2\,dx \\ &+ \tfrac{1}{2}L_1^2\gamma_1 \iint_{D_1} \tau^8 V^2\,dx + (\tfrac{1}{2}L_2^2\gamma_1 + \tfrac{1}{2}\gamma_1^{-1} + 2\gamma_2^{-1}) \\ &\times \iint_{D_1} \tau^2 V_{,i} V_{,i}\,dx + \tfrac{1}{2}\gamma_1 \iint_{D_1} \tau^8 \mathscr{L}^2\,dx.\end{aligned} \tag{3.16}$$

We now integrate by parts twice so that

$$\iint_{D_1} \tau^2 V_{,i} V_{,i}\, dx = \int_{\Sigma_1} \tau^2 V \frac{\partial V}{\partial n}\, d\sigma - \iint_{D_1} \tau^2 V \Delta V\, dx - \frac{1}{2} \int_{\Sigma_1} \frac{\partial \tau^2}{\partial n} V^2\, d\sigma + \frac{1}{2} \iint_{D_1} \tau^2{}_{,ii} V^2\, dx.$$

Use of the A-G inequality with a constant γ_5 results in

$$\iint_{D_1} \tau^2 V_{,i} V_{,i}\, dx \leqslant \int_{\Sigma_1} \tau^2 V \frac{\partial V}{\partial n}\, d\sigma - \int_{\Sigma_1} \tau \frac{\partial \tau}{\partial n} V^2\, d\sigma + \tfrac{1}{2}\gamma_5 \iint_{D_1} \tau^4 U^2\, dx + (\tfrac{1}{2}\gamma_5^{-1} + M_1 + M_2) \iint_{D_1} V^2\, dx.$$

Consequently, substituting this estimate into (3.16), choosing

$$\gamma_1 = [6(\tfrac{3}{2} + L_3 + L_4^2 + 36M_1)]^{-1}, \qquad \gamma_2 = (12M_1)^{-1},$$
$$\gamma_5 = [3(\tfrac{1}{2} L_2^2 \gamma_1 + \tfrac{1}{2}\gamma_1^{-1} + 2\gamma_2^{-1})]^{-1}$$

and collecting terms, we arrive at

$$\begin{aligned}\iint_{D_1} \tau^4 U^2\, dx \leqslant{}& 2 \int_{\Sigma_1} \left(\tau^4 U \frac{\partial V}{\partial n} + \gamma_1 \tau^6 U \frac{\partial U}{\partial n}\right) d\sigma \\ &+ (L_2^2 \gamma_1 + \gamma_1^{-1} + 4\gamma_2^{-1}) \int_{\Sigma_1} \left(\tau^2 V \frac{\partial V}{\partial n} - \tau \frac{\partial \tau}{\partial n} V^2\right) d\sigma \\ &+ [L_1^2 \gamma_1 + (L_2^2 \gamma_1 + \gamma_1^{-1} + 4\gamma_2^{-1})(\tfrac{1}{2}\gamma_5^{-1} + M_1 + M_2)] \\ &\times \iint_{D_1} V^2\, dx + \gamma_1 \iint_{D_1} \tau^8 \mathscr{L}^2\, dx.\end{aligned}$$

Finally, since the ϵ_i, which occur as a result of the A-G inequality, can be bounded in terms of M^2, we find that the conclusion of the theorem follows.

4. A Priori Estimate

We shall now show that the functional F satisfies the differential inequality (3.2). First we note that by means of (3.5)

$$\begin{aligned}(F')^2 &= F' \iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx + F' \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta \\ &\leqslant \left(\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right)^2 + 2F' \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta.\end{aligned}$$

However, due to the derivative terms in (2.8) and the region of integration D_α in (3.4), we are unable to manipulate the term

$$F \iint_{D_\alpha} (U\Delta U + V\Delta V)\, dx$$

which occurs on the left side of (3.2) so as to dominate appropriate portions of the right side. Consequently, we add and subtract the same expression which, when combined with this term, will enable us to verify that F satisfies (3.2). Thus, we form

$$\begin{aligned}(F')^2 \leqslant &\iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} \beta^{-1}[(U_{,i} f_{,i})^2 + (V_{,i} f_{,i})^2]\, dx \\ &+ \left\{\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right\}^2 \\ &+ 2F' \int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta \\ &- \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} \beta^{-1}[(U_{,i} f_{,i})^2 + (V_{,i} f_{,i})^2]\, dx.\end{aligned} \tag{4.1}$$

Using (3.7), (3.4), and (4.1), we see that

$$\begin{aligned}FF'' - (F')^2 \geqslant &\tfrac{1}{2} \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} (U_{,i} U_{,i} + V_{,i} V_{,i})\, dx \\ &+ F \iint_{D_\alpha} (U\Delta U + V\Delta V)\, dx \\ &- \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} \beta^{-1}[(U_{,i} f_{,i})^2 + (V_{,i} f_{,i})^2]\, dx \\ &- \left\{\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right\}^2 \\ &- 2\,|F'|\left|\int_0^\alpha \int_{\Sigma_\eta} \left(U \frac{\partial U}{\partial n} + V \frac{\partial V}{\partial n}\right) d\sigma\, d\eta\right| \\ &+ \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} \beta^{-1}[(U_{,i} f_{,i})^2 + (V_{,i} f_{,i})^2]\, dx,\end{aligned}$$

that is,

$$FF'' - (F')^2 \geqslant \tfrac{1}{2} \iint_{D_\alpha} \beta(U^2 + V^2)\, dx$$
$$\times \iint_{D_\alpha} \{U_{,i}U_{,i} + V_{,i}V_{,i} - 2\beta^{-1}[(U_{,i}f_{,i})^2 + (V_{,i}f_{,i})^2]\}\, dx$$
$$- 2\,|F'|\,\left|\int_0^\alpha\int_{\Sigma_\eta} \left(U\frac{\partial U}{\partial n} + V\frac{\partial V}{\partial n}\right) d\sigma\, d\eta\right| + J, \qquad (4.2)$$

where

$$I = \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \iint_{D_\alpha} \beta^{-1}[(U_{,i}f_{,i})^2 + (V_{,i}f_{,i})^2]\, dx$$
$$- \left\{\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right\}^2, \qquad (4.3)$$

$$J = I - F \iint_{D_\alpha} (|\, U\Delta U\,| + |\, V\Delta V\,|)\, dx. \qquad (4.4)$$

We shall show that

$$J \geqslant - B_5F^2 - B_6FF' \qquad (4.5)$$

for computable constants B_5 and B_6 and thus overcome the difficulty previously mentioned.

We apply Schwarz's inequality, (2.8) and the A-G inequality to get

$$\left\{\iint_{D_\alpha} (|\, U\Delta U\,| + |\, V\Delta V\,|)\, dx\right\}^2$$
$$\leqslant \iint_{D_\alpha} (U^2 + V^2)\, dx \iint_{D_\alpha} [(\Delta U)^2 + (\Delta V)^2]\, dx$$
$$\leqslant \iint_{D_\alpha} \beta(U^2 + V^2)\, dx \left\{b_1 \iint_{D_\alpha} \beta(U^2 + V^2)\, dx\right.$$
$$\left. + b_2 \iint_{D_\alpha} (U_{,i}U_{,i} + V_{,i}V_{,i})\, dx + b_3\epsilon_5\right\}$$

for positive constants b_1, b_2, and b_3. Further, by (4.3) and (3.7), we have

$$\left\{\iint_{D_\alpha} (|\, U\Delta U\,| + |\, V\Delta V\,|)\, dx\right\}^2$$
$$\leqslant 2F\left\{b_1 \iint_{D_\alpha} \beta(U^2 + V^2)\, dx\right.$$
$$\left. + b_2 \iint_{D_\alpha} \{U_{,i}U_{,i} + V_{,i}V_{,i} - 2\beta^{-1}[(U_{,i}f_{,i})^2 + (V_{,i}f_{,i})^2]\}\, dx + b_3\epsilon_5\right\}$$
$$+ 2b_2\left\{I + \left[\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right]^2\right\}.$$

Since by (3.5) and A-G inequality

$$\left\{\iint_{D_\alpha} (UU_{,i} + VV_{,i}) f_{,i}\, dx\right\}^2 \leqslant \{b_4\epsilon_1 + b_5\epsilon_2 + b_6\epsilon_3 + b_7\epsilon_4 + |F'|\}^2$$

for positive constants b_4, b_5, b_6, and b_7, we obtain

$$\begin{aligned}&\left\{\iint_{D_\alpha} (|U\Delta U| + |V\Delta V|)\, dx\right\}^2 \\ &\quad\leqslant 2b_2 I + 2F\{b_8 F + b_2(B_3F' + B_4F)\} + 2b_2(b_9F + |F'|)^2,\end{aligned} \tag{4.6}$$

where we have used Lemma 1 and 4 with computable constants b_8 and b_9.

We now combine (4.6) with (4.4) and use the A-G inequality on the term involving braces so that

$J \geqslant I - F[2b_2 I + F^2 + \{b_8F + b_2(B_3F' + B_4F)\}^2 + 2b_2(b_9F + |F'|)^2]^{1/2}$. By

the elementary inequality

$$(c_1^2 + c_2^2 + c_3^2 + c_4^2)^{1/2} \leqslant |c_1| + |c_2| + |c_3| + |c_4|,$$

we arrive at

$$\begin{aligned}J \geqslant I - F(2b_2I)^{1/2} - F^2 - F[b_8F + b_2(B_3F' + B_4F)] \\ - (2b_2)^{1/2} F(b_9F + |F'|).\end{aligned}$$

Finally, applying the inequality

$$c_1^2 - 2c_1c_2 \geqslant -c_2^2$$

to the first two terms on the right, we obtain

$$\begin{aligned}J \geqslant -F\{(1 + \tfrac{1}{2} b_2) F + [b_8F + b_2(B_3F' + B_4F)] \\ + (2b_2)^{1/2} (b_9F + |F'|)\}.\end{aligned}$$

Clearly, then, there are computable constants B_5 and B_6 so that (4.5) is satisfied.

Returning now to (4.2) and using (3.7), (3.9), (3.12), (3.13), and (4.5), we find that

$$FF'' - (F')^2 \geqslant -C_1FF' - C_2F^2$$

where

$$C_1 = 6B_1 + B_3 + B_6 \qquad \text{and} \qquad C_2 = 8B_1B_2 + B_4 + B_5.$$

Since a solution of (3.2) vanishes identically, if it vanishes for one value of α in [0, 1] (see [3]), we assume $F(\alpha) > 0$ on [0, 1]. To complete the derivation of the a priori inequality, we let

$$\rho = \exp(-C_1\alpha)$$

and note that

$$\frac{d^2}{d\rho^2}[\log F\rho^{-C_2/C_1^2}] = \frac{C_2F^2 + FF'' + C_1FF' - (F')^2}{F^2C_1^2\rho^2} \geqslant 0,$$

where prime denotes differentiation with respect to α. As the function

$$\mathscr{F}(\rho) = \log F\rho^{-C_2/C_1^2}$$

is convex, we have by Jensen's inequality [1], for $0 \leqslant \alpha \leqslant \bar{\alpha}$,

$$F(\alpha) \leqslant K_0[F(0)]^d[F(\bar{\alpha})]^{1-d},$$

where

$$d = \frac{\rho - \bar{\rho}}{1 - \bar{\rho}}, \qquad \bar{\rho} = \exp(-C_1\bar{\alpha}),$$

$$K_0 = \exp\frac{[-C_2\alpha + (1-d)C_2\bar{\alpha}]}{C_1}.$$

Consequently, by (3.1), (3.7), and Theorem I, we have

THEOREM II. *If U and V are defined by* (2.5) *and ϵ_i by* (2.9), *then*

$$\iint_{D_\alpha}(U^2 + V^2)\,dx \leqslant KM^{2(1-d)}[k_i\epsilon_i]^d,$$

where K, M, and k_i are computable constants and d is a fixed number between 0 *and* 1.

5. REMARKS

Instead of considering the Cauchy problem for (1.1), one might study the same problem for the system

$$\begin{aligned} \Delta u &= h(x, v, v_{,i}, u, u_{,i}) \\ \Delta v &= g(x)\,u, \end{aligned} \tag{5.1}$$

where in D, u and v are C^2 functions, with only v assumed to be uniformly bounded, and g is a C^2 function satisfying $|g(x)| > g_0 > 0$. More generally, we may replace the Laplace operator Δ in (5.1) by a uniformly elliptic operator L.

In these cases, we no longer assume (2.6) and, consequently, introduce the additional notation

$$\epsilon_6 = \iint_{D_1} [g\phi - \Delta\psi]^2 \, dx.$$

We then have obvious changes for ΔV in (2.7) and (2.8) and consequent changes in the results which depended on them. However, none of these changes render the results invalid; they only require a modification of coefficients and the addition of terms involving ϵ_6. Thus in these problems we obtain an inequality of the form (2.10) and deduce simultaneously that the solutions u and v of (5.1)–(2.4) are unique and depend continuously on the Cauchy data.

Further, similar results seem to be possible for more general coupled elliptic systems. This will be considered in a subsequent paper.

References

1. E. F. Beckenbach and R. E. Bellman, "Inequalities," Springer, Berlin-Göttingen-Heidelberg, 1965.
2. M. M. Lavrentiev, "Some Improperly Posed Problems of Mathematical Physics," Springer-Verlag, New York, 1967.
3. H. Levine, Convexity and differential inequalities in Hilbert space, Doctoral dissertation, Cornell University, 1969.
3. L. E. Payne, Pointwise bounds in the Cauchy problem of elastic plates, *J. Res. Nat. Bur. Standards Sect. B* **65** (1961), 157–163.
5. L. E. Payne, "On Some Non Well-Posed Problems for Partial Differential Equations," Proc. Symp. Numerical Solutions Nonlinear Diff. Eq., University of Wisconsin, 239–263, John Wiley and Sons, 1966.
6. L. E. Payne, On a priori bounds in the Cauchy problem for elliptic equations, *SIAM J. Math. Anal.* **1** (1970), 82–89.
7. C. Pucci, "Some Topics in Parabolic and Elliptic Equations," Lecture Ser. No. 36, Inst. Fl. Dyn. Appl. Math., College Park, Md., 1958.
8. P. W. Schaefer, On the Cauchy problem for an elliptic system, *Arch. Rational Mech. Anal.* **20** (1965), 391–412.