JOURNAL OF FUNCTIONAL ANALYSIS 33, 95–118 (1979)

Degenerate Series Representations of the Universal Covering Group of SU(2, 2)

BIRGIT SPEH

Department of Mathematics, The University of Chicago, Chicago Illinois 60637

Communicated by the Editors

Received February 1, 1978

We prove a reducibility criterion for certain families of representations induced from irreducible finite dimensional representations of the 11-dimensional parabolic subgroup of the universal covering group of $SU(2, 2)$. If an induced representation is reducible and can be considered as a representation of $SU(2, 2)$ as well, we compute the number of composition factors.

I. INTRODUCTION

Let $\tilde{G}$ be the universal covering group of $SU(2, 2)$, $\tilde{P}$ the 11-dimensional parabolic subgroup of $\tilde{G}$. The connected component $\tilde{P}^\circ$ of $\tilde{P}$ is a semidirect product of $SL(2, \mathbb{C})_{\mathbb{R}}$ and a solvable group S. The finite dimensional irreducible representations τ of $SL(2, \mathbb{C})_{\mathbb{R}}$ are parametrized by two nonnegative integers and we write $\tau(i, j)$, $i, j \in \mathbb{N}$. The characters χ of S are identified with their restriction to the one-dimensional center of S. The extensions of $\tau(i, j) \otimes \chi$ from $\tilde{P}^\circ$ to $\tilde{P}$ are parametrized by $[0, 2)$. Let $\tau(i, j, \lambda) \otimes \chi$, $\lambda \in [0, 2)$ be such an extension.

THEOREM 1. (a) *The representation* $\operatorname{ind}_{\tilde{P}}^{\tilde{G}} \tau(i, 0, \lambda) \otimes \chi$ *is reducible if*

$$\chi + \frac{i}{2} - \lambda = 0 \bmod 2$$

or

$$\chi + \frac{i}{2} + \lambda = 0 \bmod 2$$

(b) *The representation* $\operatorname{ind}_{\tilde{P}}^{\tilde{G}} \tau(0, i, \lambda) \otimes \chi$ *is reducible if*

$$\chi - \frac{i}{2} - \lambda = 0 \bmod 2$$

0022-1236/79/070095-24$02.00/0

or

$$\chi - \frac{i}{2} + \lambda = 0 \bmod 2 \quad \blacksquare$$

Here we identify χ with its differential at the identity and consider it to be a multiple of a short root. We normalize the induction so that if $\tau(0, 0, \lambda) \otimes \chi$ is unitary, then $\operatorname{ind}_{\mathscr{P}}^{G} \tau(0, 0, \lambda) \otimes \chi$ is unitary.

THEOREM 2. *The conditions of Theorem* 1 *are also necessary if* $\lambda = 0$ *or* $\lambda = 1$ *and* χ *arbitrary or if* $|\operatorname{Re} \chi| > i/2$ *and* λ *arbitrary.* $\blacksquare$

Since our techniques for proving Theorem 2 for $|\operatorname{Re} \chi| \leqslant i/2$ depend heavily on a classification of irreducible representations of $SU(2, 2)$, we could prove Theorem 2 only under the restriction $\lambda = 0, 1$, in which case the induced representation is actually a representation of $SU(2, 2)$. We conjecture however that the conditions of Theorem 1 are necessary in general.

THEOREM 3. (a) *Let* $\lambda = 0, 1$, *and assume* $\chi + i/2 - \lambda = 0$ *mod* 2.

If $|\chi| > \frac{i}{2} + 1$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(i, 0, \lambda) \otimes \chi$ *has* 6 *composition factors.*

If $|\chi| = \frac{i}{2} + 1$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(i, 0, \lambda) \otimes \chi$ *has* 6 *composition factors.*

If $|\chi| = \frac{i}{2}$, $i \neq 0$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(i, 0, \lambda) \otimes \chi$ *has* 5 *composition factors.*

If $\chi = i = 0$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(0, 0, 0) \otimes \chi$ *has* 3 *composition factors.*

If $|\chi| < \frac{i}{2}$, $i \neq 0$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(i, 0, \lambda) \otimes \chi$ *has* 7 *composition factors.*

(b) *Let* $\lambda \in [0, 2)$ *and assume* $|\operatorname{Re} \chi > 1/2$. *If* $\chi + i/2 - \lambda = 0$ mod 2 *and* $\chi + i/2 + \lambda \neq 0$, *or* $\chi + i/2 - \lambda \neq 0$ mod 2 *and* $\chi + i/2 + \lambda = 0$, $\operatorname{ind}_{\mathscr{P}}^{G} \tau(i, 0, \lambda) \otimes \chi$ *has* 3 *composition factors.*

(c) *For* $\operatorname{ind}_{\mathscr{P}}^{G} \tau(0, i, \lambda) \otimes \chi$ *we get the dual statements.* $\blacksquare$

Some of the composition factors we construct, correspond to representations obtained by analytic continuation of holomorphic and antiholomorphic discrete series representations. These were already obtained by K. Gross, W. Holman and R. Kunze [1], by H. Jacobson and M. Vergne [5] and H. Rossi and M. Vergne [6]. In a sequel to this paper we will discuss the unitarity of certain composition factors and their extension properties.

The organization of the paper is as follows. After introducing notations and definitions in Chapter II we consider properties of the induced representations in Chapter III. Then using special functions we derive our results for

$\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{G}}(\tau(0, 0, \lambda) \otimes \chi$ in Chapter IV. Our previous consideration in Chapter III now allows us to reduce the proof of Theorem 2 and 3 to dealing with 3 special cases, which is done in Chapter V and VI.

II. Preliminaries and Definitions

Let L be a real vector space of dimension 6 with a quadratic form Q of signature 2, and let $e_1 \cdots e_6$ be a basis such that

$$Q\left(\sum_{i=1}^{6} x_i e_i , \sum_{i=1}^{6} x_i e_i\right) = x_1{}^2 + \cdots + x_4{}^2 - x_5{}^2 - x_6{}^2.$$

Let G be the connected subgroup of $SL(6, \mathbb{R})$, which conserves Q, i.e. G is the connected component of $SO(4, 2)$, and let $\mathfrak{g}$ be its Lie algebra. In general small German letters will denote a Lie algebra, capital letters the corresponding groups and A° will denote the connected component of the identity of the group A. As usually e_{ij} denotes the 6×6 matrix with 1 in the intersection of the ith row and jth column and zero otherwise.

Once and for all we choose an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$

$$\mathfrak{a} = \mathbb{R}(e_{16} + e_{61}) \oplus \mathbb{R}(e_{26} + e_{52})$$

and by H we denote $e_{16} + e_{61}$. The maximal compact subgroup K of G is isomorphic to $SO(4) \times SO(2)$ and straightforward checking shows that $\mathfrak{g}$ is generated by $\mathbb{R}H$ and $\mathfrak{k} = \text{Lie K}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. Unless otherwise stated, we will always assume that all parabolics contain $\mathfrak{n}$. The minimal parabolic subalgebra will be denoted by $\mathfrak{b}$ and the 11 dimensional subalgebra will be denoted by $\mathfrak{p}$. We have the Langlands decomposition

$$B = M_B A N$$
$$P = M_p A_p N_p$$

where

$$M_B \cong SO(2) \times \mathbb{Z}_2$$
$$M_p \cong SO(3, 1) \times \mathbb{Z}_2$$

and we may assume that $A_p = \{\exp t\, H_1 , t \in \mathbb{R}\}$.

The group $G_1 = SU(2, 2) = \{g \in GL(4, \mathbb{C}), g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a, b, c, d are complex matrices satisfying

$$ad^* - bc^* = 1$$
$$cd^* = dc^*$$
$$ab^* = ba^*\}$$

is a double covering of G. The subgroups of G_1 with Lie algebra $\mathfrak{m}_p$, $\mathfrak{m}_B$ are isomorphic to $SL_{\pm}(2, \mathbb{C})$, the group of complex matrices with determinant ± 1, and $SO(2) \times \mathbb{Z}_2$ respectively.

The universal covering group $\tilde{G}$ of G can be described as follows: $SU(2, 2)$ acts on the space $H(2)$ of hermitian 2×2 matrices (up to a set of measure zero) by

$$gx = (ax + b)(cx + d)^{-1}, \qquad x \in H(2),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_1$$

Define δ_g by

$$\det(cx + d) = |\det(cx + d)|\, e^{i\delta_g(x)}$$

This defines δ_g up to a summand of the form $2\pi n$ and we define $\delta_g{}^n$ by requiring

$$2\pi n \leqslant \delta_g{}^n < 2\pi(n + 1), \qquad n \in \mathbb{Z}.$$

Then

$$\tilde{G} = \{(g, \delta_g{}^n), g \in G_1, n \in \mathbb{Z}\}$$

with the multiplication

$$(g, \delta_g{}^n)(g', \delta_{g'}^m) = (gg', \delta_{gg'}^n + \delta_{g'}^m).$$

The covering $\tilde{P}$ of P is the set

$$\left\{\left(\begin{pmatrix} a & b \\ 0 & a^{*-1} \end{pmatrix}, 2n\pi\right), \det a^* > 0, n \in \mathbb{Z}\right\}$$

$$\cup \left\{\left(\begin{pmatrix} a & b \\ 0 & a^{*-1} \end{pmatrix}, (2n + 1)\pi\right), n \in \mathbb{Z}, \det a^* < 0\right\}.$$

Hence $\tilde{P}$ has $\mathbb{Z}$ connected components.

We will now construct certain irreducible finite dimensional representations of $\tilde{P}$. We have the Langlands decomposition

$$\tilde{P} = \tilde{M}_p \tilde{A}_p \tilde{N}_p$$

where

$$\tilde{M}_p = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, 2n\pi\right), \det a = 1, n \in \mathbb{Z}\right\}$$

$$\cup \left\{\left(\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, (2n + 1)\pi\right), n \in \mathbb{Z}, \det a = -1\right\}$$

$$\tilde{A}_p = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, 2n\pi\right), \det a > 0, a \text{ diagonal}, n \in \mathbb{Z}\right\}$$

$$\tilde{N}_p = \left\{\left(\begin{pmatrix} e & x \\ 0 & e \end{pmatrix}, 2n\pi\right), x \in H(2), n \in \mathbb{Z}, \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \text{ identity in } GL(4, \mathbb{C})\right\}.$$

Put $\bar{e} = e_{11} - e_{22} + e_{33} - e_{44}$ and let $\tilde{E}$ be the subgroup of $\tilde{G}$ covering $\{e, \bar{e}\}$. Then $\tilde{E} \simeq \mathbb{Z}$ and commutes with the Cartan subgroup of the connected component of the covering $\tilde{M}_p$. Hence each finite dimensional representation of $\tilde{M}_p$ is determined by a representation of $SL(2, \mathbb{C})_{\mathbb{R}}$ and a representation of $\tilde{E}$ on the highest weight space.

The irreducible representations of $\tilde{E} \simeq \mathbb{Z}$ are parametrized by $\lambda \in [0, 2)$ as follows

$$e_\lambda(z) = e^{i\lambda\pi z} \qquad z \in \mathbb{Z}.$$

Let β_1, β_2 be the simple positive roots of $(sL(2, \mathbb{C})_{\mathbb{R}}) \otimes \mathbb{C}$. Then we write $\tau(n_1/2\beta_1 + n_2/2\beta_2, \lambda)$ for the representation with highest weight $n_1/2\beta_1 + n_2/2\beta_2$ and the representation e_λ of $\tilde{E}$ on the highest weight space. If there is no possibility of confusion, we will often write $\tau(n_i, \lambda)$ instead of $\tau(n_i/2\beta_i, \lambda)$, $n_i \neq 0$, and $\tau(0, \lambda)$ for $\tau(0\beta_1 + 0\beta_2, \lambda)$. In the following we will only consider the representations $\tau(n_i, \lambda)$, $i = 1, 2$, $n \in \mathbb{N}$.

Define a character χ_μ of $\tilde{A}_p\tilde{N}_p$ by

$$\chi_\mu(an, \delta_{un}) = a^\mu$$

where $\exp(tH)^\mu = e^{t\mu(H)}$ for μ in the complex dual $(\mathfrak{a}_p \otimes \mathbb{C})'$ of $\mathfrak{a}_p$. Then $\tau(n_i, \lambda) \otimes \chi_\mu$ is an irreducible representation of $\tilde{P}$. For $\lambda = 0, 1$ and $n = 0 \bmod 2$, it is a representation of P as well.

Put $\pi(n_i, \lambda, \mu) = \mathrm{ind}_{\tilde{P}}^{\tilde{G}} \tau(n_i, \lambda) \otimes \chi_\mu$. The induced representation is defined in such a way, that it acts on the left and is unitary if $\tau(n_i, \lambda) \otimes \chi_\mu$ is unitary. The representations $\pi(n_i, \lambda, \mu)$ are *degenerate series representations* of $\tilde{G}$.

We parametrize the representations of $\tilde{B} = \tilde{M}_B\tilde{A}_B\tilde{N}_B$ as follows. Since $\tilde{M}_B \simeq S^1 \times \mathbb{Z}$, the unitary dual $\hat{\tilde{M}}_B$ is isomorphic to $\mathbb{Z} \times [0, 2)$. If $(n, \lambda) \in \mathbb{Z} \times [0, 2)$ write $\rho_{n,\lambda}$ for the corresponding character of $\tilde{M}_B$. For ν in the dual $\mathfrak{a}_{\mathbb{C}}'$ of $\mathfrak{a} \otimes \mathbb{C}$ we define a character χ_ν of AN by $\chi_\nu(an) = a^\nu$. Here $a \in A$, $n \in N$ and $(\exp tX)^\nu = e^{t\nu(X)}$ for $X \in \mathfrak{a}$. Put $U(n, \lambda, \mu) = \mathrm{ind}_{\tilde{B}}^{\tilde{G}} \rho_{n,\lambda} \otimes \chi_\nu$. The representations $U(n, \lambda, \nu)$ are the *principal series representations* of $\tilde{G}$.

We write $\mathfrak{g}_o'$ for the dual of a subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$.

Let $h = (\mathfrak{a} \oplus \mathfrak{m}_B) \otimes \mathbb{C}$. Then h is a Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. In the root system Σ of $(\mathfrak{g} \otimes \mathbb{C}, h)$ choose a set Σ^+ of positive roots compatible with the ordering of the restricted roots Δ in $\mathfrak{a}'$ (determined through the choice of $\mathfrak{n}$). The highest weight of a finite dimensional representation of $\tilde{G}$ is a pair (ν_1, ν_2), where $\nu_1 \in \mathfrak{m}_B'$, $\nu_2 \in \mathfrak{a}'$ and $i\nu_1 + \nu_2$ is dominant and integral with respect to Σ^+.

III. Some Remarks about Degenerate Series Representations

A. *Embedding of degenerate series representations in principal series*

Let α_1, α_2, α_3 be the simple roots in Σ^+. Then α_2, $\frac{1}{2}(\alpha_1 + \alpha_3)$ are simple restricted roots and $\mathfrak{a}_p' = \mathbb{R}\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$. We identify $(\mathfrak{a}_p \otimes \mathbb{C})'$ with $\mathbb{C}$ by $\mu = \mu\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$, $(\mathfrak{a}_B \otimes \mathbb{C})'$ with $\mathbb{C}^2$ by $\nu = \nu = \nu_1\alpha_1 + \nu_2\frac{1}{2}(\alpha_1 + \alpha_2)$ and $\mathfrak{m}_B'$ with $\mathbb{R}$ by $\mathfrak{m}_B' = Ri\frac{1}{4}(\alpha_1 - \alpha_3)$. The differentials of characters of $\exp_{G_1}\mathfrak{m}$ are thus identified with $\mathbb{Z}$ and those of $\exp_G\mathfrak{m}$ are identified with $2\mathbb{Z}$. We choose α_1, α_3 as the positive roots in $SL(2, \mathbb{C})_{\mathbb{R}}$. Unless otherwise stated, we write $\tau(n, \lambda)$ for $\tau(n, 0, \lambda)$. Since the situation is symmetric with respect to $\tau(n, 0, \lambda)$ and $\tau(0, n, \lambda)$, all formulas will be derived only for $\tau(n, \lambda) = \tau(n, 0, \lambda)$.

We have $\tilde{P} \supset \tilde{B}$, thus

$$\begin{aligned} U(n, \lambda, \nu) &= \operatorname{ind}_{\tilde{B}}^{\tilde{G}} \rho_{n,\lambda} \otimes \chi_\nu \\ &= \operatorname{ind}_{\tilde{P}}^{\tilde{G}} \operatorname{ind}_{\tilde{B}}^{\tilde{P}} \rho_{n,\lambda} \otimes \chi_\nu . \end{aligned}$$

But

$$\operatorname{ind}_{\tilde{B}}^{\tilde{P}} \rho_{n,\lambda} \otimes \chi_\nu = \operatorname{ind}_{\tilde{B}\cap\tilde{M}_p}^{\tilde{M}_p}(\rho_{n,\lambda} \otimes \chi_\nu)_{|\tilde{B}\cap\tilde{M}_p} \otimes \tau_{\nu|\tilde{A}_p\tilde{N}_p} .$$

Now $\tilde{B}_1 \cap \tilde{M}_{p_1}$ is a Borel subgroup of $\tilde{M}_p$ and $\operatorname{ind}_{\tilde{B}\cap\tilde{M}_p}^{\tilde{M}_p}(\rho_{n,\lambda} \otimes \chi_\nu)_{|\tilde{B}\cap\tilde{M}_p}$ is a principal series representation. It follows [12].

Lemma 3.1. *The representation* $\operatorname{ind}_{\tilde{M}_p\cap\tilde{B}}^{\tilde{M}_p} (\rho_{n,\lambda} \otimes \chi_\nu)|_{\tilde{B}\cap\tilde{M}_p}$ *contains the representation* $\tau(m, \lambda_o)$ *as invariant subspace if*

$$\lambda_o = \lambda,\ n = m,\ \nu_2 - \nu_1 = -1 - n/2. \quad \blacksquare$$

Hence $\tau(n, \lambda) \otimes \chi_\mu$ is a subrepresentation of $\operatorname{ind}_{\tilde{B}}^{\tilde{P}} \rho_{n,\lambda} \otimes \chi_\nu$ for $\nu = (\mu, -1 - n/2 + \mu)$, and thus

Lemma 3.2. $\pi(n, \lambda, \mu)$ *is a subrepresentation of* $U(n, \lambda, (\mu, -1 - n/2 + \mu))$. $\blacksquare$

B. *Equivalence and Duality*

If U_1, U_2 are finite dimensional contragredient representations of $\tilde{P}$, then $\operatorname{ind}_{\tilde{P}}^{\tilde{G}} U_1$ and $\operatorname{ind}_{\tilde{P}}^{\tilde{G}} U_2$ are contragredient. Hence, since $\tau((n, 0), \lambda) \otimes \chi_\mu$ and $\tau(0, n), \lambda) \otimes \chi_{-\mu}$ are contragredient, we have

Lemma 3.3. $\pi((n, 0), \lambda, \mu)$ *and* $\tau((0, n), \lambda, -\mu)$ *are contragredient.* $\blacksquare$

Let w_o be the shortest element in the Weyl group mapping $\alpha_1 + 2\alpha_2 + \alpha_3$ into its negative.

THEOREM 3.4. *There is an intertwining operator* $A(w_o, n, \lambda, \nu)$, *which maps* $U(n, \lambda, \nu)$ *into* $U(-n, \lambda, w_o\nu)$.

Proof. Although the original proof of [7] is under the assumption that G has finite center, we can use the reduction technique to reduce the construction of the intertwining operator to constructing an intertwining operator for principal series of $\widetilde{SL}(2, \mathbb{R})$. But this has been done in [8]. ▮

THEOREM 3.5. *There is an intertwining operator* $B(n, \lambda, \mu)$, *which maps* $\pi((n, 0), \lambda, \mu)$ *into* $\pi((0, n), \lambda, -\mu)$.

Proof. Consider $\pi((n, 0), \lambda, \mu)$ as a subrepresentation of $U(n, \lambda, (\mu, -1 - n/2 + \mu))$, and $\pi((0, n), \lambda, -\mu))$ as a subrepresentation of $U(-n, \lambda, (-1 - n/2 - \mu))$ respectively. Then

$$A\left(w_0, n, \lambda, \left(\mu, -1 - \frac{n}{2} + \mu\right)\right) U\left(n, \lambda, \left(\mu, -1 - \frac{n}{2} + \mu\right)\right)$$
$$\subset U\left(-n, \lambda, \left(-\mu, -1 - \frac{n}{2} - \mu\right)\right)$$

and

$$A(w_o, n, \lambda, (\mu, -1 - n + \mu))\pi((n, 0), \lambda, \mu) \subset \pi((0, n), \lambda, -\mu).$$

The operator $A(w_o, n, \lambda, (\mu, -1 - n + \mu))$ depends analytically on μ and hence so does its restriction to $\pi((n, 0), \lambda, \mu)$. We thus can find a normalization of the restricted operator in such a way, that it is nonzero for all μ. ▮

C. *A realization of* $\pi(0, 0, \mu)$ *and* $\pi(0, 1, \mu)$

Let $e \in L$ have the coordinates (1, 0, 0, 0, 0, 1). Then

$$K \cdot e = \{(x_1, ..., x_6) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 = x_5^2 + x_6^2\} \cong S^3 \times S^1.$$

The isotropy group at e is isomorphic to $SO(3)$ and will be denoted by $K(P)$. K operates on $\mathscr{L}^2(K/K(P))$ as well. Here we take the $\mathscr{L}^2$ with respect to the left invariant measure on $S^3 \times S^1$. This representation of K decomposes in a direct sum of finite dimensional representations and each representation of K with a $K(P)$ fixed vector occurs in this decomposition exactly once. We label these representations as follows:

Let Δ_1 be the Casimir of $SO(4)$, Δ_2 be the differential operator corresponding to the generator of $SO(2)$. Then

$$H(i, j) = \{f \in C^\infty(K/K(P), \Delta_1 f = -i(i + 1)f \quad \Delta_2 f = jf\}$$

is an irreducible invariant subspace and

$$\mathscr{L}^2(K/K(P)) = \bigoplus_{\substack{i \in N \setminus \{0\} \\ j \in \mathbb{Z}}} H(i,j).$$

For $x = (x_1, \dots, x_6) \in K \cdot e$ and $g \in G$ we define $g \cdot x = y = (y_1(gx_1), \dots, y_6(gx))$. The action of G on $K \cdot e$ is defined by

$$g \circ x = (y_1^2(gx)) + y_2^2(gx) + y_3^2(gx) + y_4^2(gx))^{1/2} \cdot gx.$$

This action is differentiable. For $\mu \in \mathbb{C}$ we define π_μ by

$$(\pi_\mu(g)f)(x) = \left(\sum_1^4 (y_i^2(g^{-1}x)\right)^{\mu/2} f(g^{-1} \circ x),$$

$$g \in G, \quad f \in \mathscr{L}^2(K \cdot e), \quad x \in K \cdot e.$$

One checks easily that this is indeed a representation.

CLAIM. *π_μ is the direct sum of two representations.*

Proof. Let $J: \mathscr{L}^2(K \cdot e) \to \mathscr{L}^2(K \cdot e)$ be defined by $(Jf)(x) = f(-x)$. Then we have

$$J\pi_\mu = \pi_\mu J.$$

Define

$$H_1 = \bigoplus_{\substack{i \in N \cup \{0\} \\ j \in \mathbb{Z} \\ i+j = 1 \bmod 2}} H(i,j)$$

and

$$H_2 = \bigoplus_{\substack{i \in N \cup \{0\} \\ j \in \mathbb{Z} \\ i+j = 0 \bmod 2}} H(i,j).$$

H_2 is the $+1$ Eigenspace of J and H_1 is the -1 Eigenspace of J. ∎

PROPOSITION 3.6. *π_μ restricted to H_1 or H_2 is equivalent to $\pi(0, 0, \mu)$, or $\pi(0, 1, \mu)$ respectively.*

Proof. Let $\tilde{x}$ be a representative of $x \in K \cdot e$ in K. Then

$$g\tilde{x} = k(g\tilde{x})m(g\tilde{x})a(g\tilde{x})n(g\tilde{x}),$$

where

$$k(g\tilde{x}) \in K,$$
$$m(g\tilde{x}) \in M_p^0,$$
$$a(g\tilde{x}) \in A_p,$$
$$n(g\tilde{x}) \in \tilde{K}_p.$$

Now e is the highest weight vector of a representation with highest weight $(0, \alpha_1 + 2\alpha_2 + \alpha_3)$ and thus

$$g\tilde{x} \cdot e = k(g\tilde{x}) \cdot a(g\tilde{x}) \cdot e.$$

Then $k(g\tilde{x})e \in S^3 \times S^1$ and $a(g\tilde{x})e = a(g\tilde{x})^{\alpha_1+2\alpha_3+\alpha_3}e = \sum_1^4 (y_i(gx)^2)^\mu e$. Hence $\pi_\mu = \mathrm{ind}_{P^0}^G \pi(0) \otimes \chi_\mu$, which is the direct sum of $\pi(0, 0, \mu)$ and $\pi(0, 1, \mu)$. ∎

D. *A different view of* $\pi(0, \lambda, \mu)$

Identify $S^3 \times (0, 2\pi)$ with an open subset of $S^3 \times S^1$ by

$$(x, \varphi) \to (x, e^{i\varphi})$$

and define for $X \in \mathfrak{g}$, $f \in C^\infty(S^3 \times (0, 2\pi))$ and $\mu \in \mathbb{C}$

$$(V_\mu(X)f)(x, \varphi) = \frac{d}{dt} \pi_\mu(\exp tX) f(x, \varphi)|_{t=0}.$$

V_μ is a representation of $\mathfrak{g}$ and hence of the enveloping algebra $U(\mathfrak{g})$.

We now consider the induced representation $\pi(0, \lambda, \mu)$ as acting on the sections of a bundle over $S^3 \times S^1$. Choosing a restriction to $S^3 \times (0, 2\pi)$ of the induced bundle, the C^∞ vectors of the representation form a V_μ-invariant subspace of $C^\infty(S^3 \times (0, 2\pi))$.

Put

$$H(i, j, \lambda) = \{f \in C^\infty(S^3 \times (0, 2\pi)), \text{ such that}$$
$$\Delta_1 f = -i(i + 1)f$$
$$\Delta_2 f = (\lambda + j)f\}$$

PROPOSITION 3.7. *The restriction of* V_μ *to* $H(\lambda) = \bigoplus_{i\in\mathbb{N}\, j\in\mathbb{Z}\, i+j=0 \bmod 2} H(i, j, \lambda)$ *is infinitesimal equivalent to the representation of the enveloping algebra on the space of K-finite vectors of* $\pi(0, \lambda, \mu)$. ∎

E. *Some remarks about tensor products with finite dimensional representations*

We say that a $(\mathfrak{g}, \tilde{K})$ module M is a *Harish-Chandra-module* (H.Ch.module), iff

(a) the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts nilpotent on M.

(b) considered as a $\tilde{K}$ module each irreducible representation of $\tilde{K}$ occurs with finite multiplicity and M is a direct sum of irreducible representations of $\tilde{K}$.

Recall the definition of the Harish–Chandra homomorphism ψ of $Z(\mathfrak{g})$ into the Weyl group invariants in the symmetric algebra of $(\mathfrak{m}_B \oplus \mathfrak{a})' \otimes \mathbb{C}$ [11]. We will say that M has *central character* γ if $z \in Z(\mathfrak{g})$ acts by

$$(\psi(z)(\gamma)$$

and γ is in the positive closed Weyl chamber $\mathscr{C}^+$ (with respect to a fixed chosen ordering of the roots in $(\mathfrak{m} \oplus \mathfrak{a})' \otimes \mathbb{C}$.)

Let M be an H.Ch.module with central character γ and V_δ a finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight δ. The $(\mathfrak{g}, \tilde{K})$ module $M \otimes V_\delta$ has a summand with central character $\gamma + \delta$, which is again a H.Ch. module. Let $P_\gamma^{\gamma+\delta}$ be the projection on this summand and let $\psi_\gamma^{\gamma+\delta}$ be the functor

$$M \to P_\gamma^{\gamma+\delta}(M \otimes V_\delta).$$

Let $V_{-\delta}^*$ be the contragradient module to V_δ. Then $M \otimes V_{-\delta}^*$ has a direct summand with central character $\gamma - \delta$. Let $P_\gamma^{\gamma-\delta}$ be the projection on this submodule and let $\varphi_{\gamma-\delta}^\gamma$ be the functor

$$M \to P_\gamma^{\gamma-\delta}(M \otimes V_{-\delta}^*)$$

THEOREM 3.8. [13]. (a) *Let W_γ and $W_{\gamma+\delta}$ be the stabilizer in the Weyl group of $\gamma + \delta$ respectively. If $W_\gamma = W_{\gamma+\delta}$, then $\psi_\gamma^{\gamma+\delta}$ is an exact functor in the category of H.Ch.modules.*

(b) *$\varphi_{\gamma-\delta}^\gamma$ is a morphismus in the category of H.Ch.modules, if $W_{\gamma-\delta} \supset W_\gamma$.* ∎

If $W_{\gamma-\delta} \supset W_\gamma = \{id\}$ then $\varphi_{\gamma-\delta}^\gamma$ applied to an irreducible H.Ch.module is either irreducible or zero [10].

If π is an indecomposable representation of $\tilde{G}$, $\pi_{|\tilde{K}}$ a direct sum of irreducible representations of $\tilde{K}$ and $\mathrm{Hom}_{\tilde{K}}(\delta, \pi_{|\tilde{K}}) < \infty$ for all irreducible representations δ of $\tilde{K}$, then the representation of the enveloping algebra $U(\mathfrak{g})$ on the space of $\tilde{K}$-finite vectors of π is a Harish–Chandra module. Since by definition two representations of $\tilde{G}$ with the above properties are equivalent iff the corresponding H.Ch. modules are equivalent, we will not distinguish between a representation of $\tilde{G}$ and its H.Ch. module. A representation δ of $\tilde{K}$ is called a *K-type* of π if $\mathrm{Hom}_{\tilde{K}}(\delta, \pi_{|\tilde{K}}) \neq 0$.

We will often not distinguish between a representation of $\tilde{K}$ and its highest weight.

Straightforward computations show that the central character of $\pi(n, \lambda, \mu)$ has a nontrivial stabilizer iff

$$\mu = \frac{n}{2} + 1,$$

$$\mu = -\frac{n}{2} - 1,$$

$$\mu = \pm\frac{n}{2}.$$

In all these cases the stabilizer has order 2, unless $n = 0$. Then it has order 4.

By the considerations of Chapter 5 [9] and some elementary calculations follows:

PROPOSITION 3.9. *Let $\pi(n, \lambda, \mu)$ be a degenerate series representation, δ the highest weight of a finite dimensional representation.*

(a) *If* $\operatorname{Re}\mu < -n/2 - 1$, $\delta = \delta\alpha_a + 2\alpha_2 + (\delta + \tilde{\delta})\,\alpha_3$ *with* $0 \leqslant \tilde{\delta} \leqslant 2\delta$,

$$\psi_\nu^{\nu+\delta}\pi(n, \lambda, \mu) = \pi(n + \tilde{\delta}, \lambda(\tilde{\delta}), \mu - \delta)$$

(b) *If* $-n/2 - 1 \leqslant \operatorname{Re}\mu \leqslant -n/2$, $\delta = \delta\alpha_1 + 2\delta\alpha_2 + 3\delta\alpha_3$ *and* $(\mu, n) \neq (0, 0)$,

$$\psi_\mu^{\mu+\delta}\pi(n, \lambda, \mu) = \pi(n + 2\delta, \lambda(2\delta), \mu - \delta)$$

(c) *If* $-n/2 < \operatorname{Re}\mu < n/2$, $n \neq 0$, $\delta = (\tilde{\delta} - \delta)\,\alpha_1 - \delta\alpha_2 - 2\delta\alpha_3$, $0 \geqslant \tilde{\delta} > \delta/2$,

$$\psi_\mu^{\mu+\delta}\pi(n, \lambda, \mu) = \pi(n - \tilde{\delta}, \lambda(\tilde{\delta}), \mu + \delta)$$

Here
$$\lambda(\tilde{\delta}) = \lambda \bmod 2, \qquad \text{if } \tilde{\delta} \in \mathbb{Z}$$
$$\lambda(\tilde{\delta}) = \lambda + 1 \bmod 2, \qquad \text{if } \tilde{\delta} = \tfrac{1}{2} \bmod \mathbb{Z} \quad \blacksquare$$

IV. REDUCIBILITY AND COMPOSITION SERIES OF $\pi(n, \lambda, \mu)$. THE CASE $|\operatorname{Re}\mu| > n/2$

Let $e(\lambda, i, j)$ be the unique vector in $H(\lambda, i, j)$, which transforms according to $e_\lambda \times \tau(0)_{|K(P)}$. Since H commutes with $\tilde{P} \cap \tilde{K}$, we have

$$\pi(0, \lambda, \mu)(H)\, e(\lambda, i, j) = \sum_{\bar{i}\bar{j}} \gamma(\mu, i, j, \bar{i}, \bar{j})\, e(\lambda, \bar{i}, \bar{j})$$

where $\gamma(\mu, i, j, \bar{i}, \bar{j}) \in \mathbb{C}$.

By [12] reducibility of a representation is equivalent to reducibility of the corresponding H.Ch. module. Since each K-type of $\pi(n, \lambda, \mu)$ has multiplicity one, to prove irreducibility of $\pi(0, \lambda, \mu)$ it suffices to show that $\pi(0, \lambda, \mu)(H)$ acts transitively on the set $\{e(\lambda, i, j)\}$.

And conversely, since H and $\mathfrak{k}$ generate $\mathfrak{g}$ each H invariant subspace corresponds to an invariant subspace of $\pi(0, \lambda, \mu)$.

Using the theory of spherical harmonics [4], we write for $(x_5, x_6) \neq (1, 0)$

$$e(\lambda, i, j) = (x_6 + ix_5)^{j+\lambda} C_i(x_1),$$

where $C_i(\ )$ is the Gegenbauer polynomial of degree i.

LEMMA 4.1. [4].

$$C_l(x) = \frac{1}{2}(C_{l+1}(x) + C_{l-1}(x))$$

$$(1 - x^2)\frac{d}{dx} C_l(x) = \left(\frac{l}{2} + 1\right) C_{l+1}(x) - \frac{l}{2} C_{l-1}(x)$$

$$(1 - x^2)\, C_l(x) = \frac{1}{2} C_l(x) - \frac{1}{4} C_{l-2}(x) - \frac{1}{4} C_{l+2}(x) \quad \blacksquare$$

Now let

$$A(t) := \exp(-t\,H)$$

Then

$$\begin{aligned} V_\mu(H)\, e(\lambda, i, j) &= \frac{d}{dt}\Big[[(\cosh tx_1 + \sinh tx_6)^2 + x_2^2 + x_3^2 + x_4^2]^{\mu/2} \\ &\quad \times y\left(\frac{x_6}{[(\cosh tx_1 + \sinh tx_6)^2 + x_2^2 + x_3^2 + x_4^2]^{1/2}} + ix_5\right)^{j+\lambda} \\ &\quad \times C_i\left(\frac{x_1}{[(\cosh tx_1 + \sinh tx_6)^2 + x_2^2 + x_3^2 + x_4^2]^{1/2}}\right)\Big]\Big|_{t=0} \\ &= \mu x_1 x_6 (x_6 + ix_5)^{j+\lambda} C_i(x_1) + x_6(1 - x_1^2)\frac{d}{dx_1}(x_6 + ix_5)^{j+\lambda} C_i(x_1) \\ &\quad + x_1(1 - x_6^2)\frac{d}{dx_6}(x_6 + ix_5)^{j+\lambda} C_i(x_1). \end{aligned}$$

Using the previous lemma we get

$$\begin{aligned} V_\mu(H)\, e(\lambda, i, j) &= \left[\frac{\mu}{4} + \frac{1}{2}\left(\frac{i}{2} + 1\right) + \frac{j+\lambda}{4}\right] e(\lambda, i+1, j+1) \\ &\quad + \left[\frac{\mu}{4} - \frac{i}{4} + \frac{j+\lambda}{4}\right] e(\lambda, i-1, j+1) \\ &\quad + \left[\frac{\mu}{4} + \frac{1}{2}\left(\frac{i}{2} + 1\right) - \frac{j+\lambda}{4}\right] e(\lambda, i+1, j-1) \\ &\quad + \left[\frac{\mu}{4} - \frac{i}{4} - \frac{j+\lambda}{4}\right] e(\lambda, j-1, i-1). \end{aligned}$$

THEOREM 4.2. *$\pi(0, \lambda, \mu)$ is reducible iff*

$$\mu + \lambda = 0 \bmod 2$$

$$\mu - \lambda = 0 \bmod 2.$$

Proof. Only if the above conditions are satisfied, one of the coefficients in the above formula is zero. ∎

Let π be a representation of $\tilde{G}$ on a Banach space V. We call an irreducible representation π' a *composition factor* or *subquotient* of π, if there are closed π invariant spaces V_1, V_2 of V, $V_1 \subset V_2$ and π' is equivalent to the representation on V_2/V_1.

THEOREM 4.3. (a) *Let $\mu - \lambda = 0 \bmod 2$ and $\mu + \lambda = 0 \bmod 2$.*

If $|\mu| > 1$, $\pi(0, \lambda, \mu)$ has 6 composition factors
If $|\mu| = 1$, $\pi(0, \lambda, \mu)$ has 5 composition factors
If $\mu = 0$, $\pi(0, 0, 0)$ has 3 composition factors.

(b) *If $\mu - \lambda = 0 \bmod 2$ and $\mu + \lambda \neq 0 \bmod 2$ or if $\mu + \lambda = 0 \bmod 2$ and $\mu - \lambda \neq 0 \bmod 2$, then $\pi(0, \lambda, \mu)$ has 3 composition factors.* ∎

Our considerations about tensoring with finite dimensional representations imply:

THEOREM 4.4. *Let $|\operatorname{Re} \mu| > n/2$.*

(a) *$\pi(n, \lambda, \mu)$ is reducible iff*

$$\mu + \frac{n}{2} - \lambda = 0 \bmod 2$$

or

$$\mu + \frac{n}{2} + \lambda = 0 \bmod 2.$$

(b) *$\pi((0, n), \lambda, \mu)$ is reducible iff*

$$\mu - \frac{n}{2} - \lambda = 0 \bmod 2$$

or

$$\mu - \frac{n}{2} + \lambda = 0 \bmod 2.$$

(c) *Let $\mu + n/2 - \lambda = 0 \bmod 2$ and $\mu + n/2 + \lambda = 0 \bmod 2$.*

If $|\mu| = n/2 + 1$, then $\pi(n, \lambda, \mu)$ has 5 composition factors.
If $|\mu| > n/2 + 1$, then $\pi(n, \lambda, \mu)$ has 6 composition factors.

(d) *If* $\mu + n/2 - \lambda = 0 \bmod 2$ *and* $\mu + \lambda + n/2 \neq 0 \bmod 2$, *or if* $\mu + n/2 - \lambda \neq 0 \bmod 2$ *and* $\mu + n/2 + \lambda = 0 \bmod 2$, *then* $\pi(n, \lambda, \mu)$ *has* 3 *composition factors.*

(e) *For* $\pi((0, n), \lambda, \mu)$ *the dual statements to c, d are true.*

In the next chapter we will study the representations $\pi((0, n), \lambda, \mu))$ and $\pi((n, 0), \lambda, \mu)$ for $|\operatorname{Re} \mu| \leqslant n/2$.

Proof of Theorem 4.3. (a) Let $\mu = \lambda = 0$. Then

$$V_1 = \bigoplus_{\substack{j+i\leqslant -1\\ i+j=0 \bmod 2}} H(i, j, 0)$$

$$V_2 = \bigoplus_{\substack{j-i>1\\ i+j=0 \bmod 2}} H(i, j, 0)$$

$$V_3 = \bigoplus_{\substack{i+j>-1\\ i-j<1\\ i+j=0 \bmod 2}} H(i, j, 0)$$

are invariant subspaces of $\pi(0, 0, 0)$ and $V_1 \oplus V_2 \oplus V_3 = H(0)$.

(b) Let $\mu = -1$, $\lambda = 1$. Then

$$V_1 = \bigoplus_{\substack{i-j=0\\ i+j=0 \bmod 2}} H(i, j, 1)$$

and

$$V_2 = \bigoplus_{\substack{i+j=2\\ i+j=0 \bmod 2}} H(i, j, 1)$$

are invariant subspaces of $H(1)$ under $\pi(0, 1, -1)$. Furthermore $H(1)/V_1 \oplus V_2$ is a direct sum of 3 invariant subspaces

$$V_3 = \bigoplus_{\substack{i+j>2\\ i-j<0\\ i+j=0 \bmod 2}} H(i, j, 1)$$

$$V_4 = \bigoplus_{\substack{i+j<-2\\ i+j=0 \bmod 2}} H(i, j, 1)$$

$$V_5 = \bigoplus_{\substack{i-j>0\\ i+j=0 \bmod 2}} H(i, j, 1).$$

(c) To complete the proof of the first part of Theorem 2, by IIIE it suffices to consider the case $\mu = -2$, $\lambda = 0$. Here $V_0 = H(0, 0, 0)$ is an invariant subspace of $H(0)$ under $\pi(0, 1, -1)$. Furthermore $H(0)/V_0$ has two invariant irreducible subspaces

$$V_1 = \bigoplus_{\substack{(i,j)\neq(0,0)\\ -3<i+j\leqslant 0\\ i+j=0 \bmod 2}} H(i,j,0)$$

and

$$V_2 = \bigoplus_{\substack{(i,j)\neq(0,0)\\ -3<i-j\leqslant 0\\ i+j=0 \bmod 2}} H(i,j,0).$$

The space $(H(0)/V_0)/V_1 \oplus V_2$ is the direct sum of 3 irreducible invariant subspaces

$$V_3 = \bigoplus_{\substack{i+j\leqslant -3\\ i+j=0 \bmod 2}} H(i,j,0)$$

$$V_4 = \bigoplus_{\substack{i-j\leqslant -3\\ i+j=0 \bmod 2}} H(i,j,0)$$

$$V_5 = \bigoplus_{\substack{i+j>0\\ i-j>0\\ i+j=0 \bmod 2}} H(i,j,0).$$

This proves the first part of Theorem 2.

(d) The last assertion is proved similarly. ∎

V. Reducibility of $\pi(n, \lambda, \mu)$. The Case $|\operatorname{Re}\mu| < n/2$

We will first prove that the representation $\pi(n, \lambda, \mu)$ is reducible for a certain set of parameters by producing a nontrivial composition factor, and then use some results about intertwining operators to show that the representation is irreducible otherwise.

Recall some results about $\widetilde{SL}(2, \mathbb{R})$, the universal covering group of $SL(2, \mathbb{R})$. The irreducible representations of $\widetilde{SL}(2, \mathbb{R})$ are constructed as follows [8]. Let $\tilde{D}$ be the Borel subgroup of $\widetilde{SL}(2, \mathbb{R})$. Then

$$\tilde{D} \cong \left\{\begin{pmatrix} a & x \\ 0 & a^{-1}\end{pmatrix}, a \in \mathbb{R}^*, x \in \mathbb{R}\right\} \times \mathbb{Z}$$

All irreducible finite dimensional representations of $\tilde{D}$ are parametrized by a character χ_ν of $(\tilde{D})^\circ$ defined by $\chi_\nu\left(\left(\begin{smallmatrix} a & n \\ 0 & a^{-1}\end{smallmatrix}\right)\right) = a^\nu$, and a character e_λ, $\lambda \in [0, \lambda)$, as defined in Chapter II. Put $\tilde{C} = \widetilde{SL}(2, \mathbb{R})$.

THEOREM 5.1 [8]. (a) $\operatorname{ind}_{\tilde{B}}^{\tilde{G}} \chi_\gamma \otimes e_\lambda$ *is reducible,* iff $\gamma + \lambda = 1 \mod 2$ *or* $\gamma - \lambda = 1 \mod 2$.

(b) *If* $\gamma + \lambda = 1 \mod 2$ *and* $\gamma \in \mathbb{Z}\backslash 0$, *then* $\operatorname{ind}_{\tilde{B}}^{\tilde{G}}(\chi_\gamma \otimes e)_\lambda$ *has* 3 *composition factors, the finite dimensional representation* F_γ, *the holomorphic discrete series representation* D_γ^+, *and the antiholomorphic discrete series representation* D_γ^-.

(c) *If* $\gamma + \lambda = 1 \mod 2$, $\gamma \notin \mathbb{Z}\backslash 0$, *then* $\operatorname{ind}_{\tilde{B}}^{\tilde{G}} \chi_\gamma \otimes e_\lambda$ *has a holomorphic continuation of discrete series representation* D_γ^+ *as composition factor and another infinite dimensional composition factor* I_γ^+. *If* $\gamma - \tilde{\lambda} = 1 \mod 2$, $\gamma \notin \mathbb{Z}\backslash 0$, *then* $\operatorname{ind}_{\tilde{B}}^{\tilde{G}} \chi_\gamma \otimes e_\lambda$ *has an analytically continued antiholomorphic discrete series representation* D_γ^- *as composition factor and an infinite dimensional composition factor* I_γ^-.

THEOREM 5.2 [8]. *The lowest K-type of* D_γ^+, D_γ^- *has the weight* $1 + \gamma$, *or* $-1 - \gamma$ *respectively.* ∎

We now return to $\widetilde{SU}(2, 2)$. Let $P_1 \supset B$ be a parabolic not conjugate to P. Then

$$P_1 \cong SL(2, \mathbb{R}) \times S^1 \times \mathbb{R}^+ \times N_1$$

and

$$\begin{aligned} \tilde{P}_1 &\cong \widetilde{SL}(2, \mathbb{R}) \times S^1 \times \mathbb{R}^+ \times N_1 \subset \widetilde{SU}(2, 2) \\ &\cong \widetilde{SL}(2, \mathbb{R}) \times S^1 \times A_1 N_1 . \end{aligned}$$

LEMMA 5.3. *Let* $\nu = (\nu_1, \nu_2)$. *The representation* $U(n, \lambda, \nu)$ *has*

$$\operatorname{ind}_{\tilde{P}_1}^{\tilde{G}} D^+_{\nu_1\, 2-\nu_2/4} \otimes \rho_n \otimes \chi_\nu|_{A_1N_1} \qquad \textit{or} \qquad \operatorname{ind}_{\tilde{P}_1}^{\tilde{G}} D^-_{\nu_1/2-\nu_2/4} \otimes \rho_n \otimes \chi_\nu|_{A_1N_1}$$

as composition factors, if

$$\tilde{\lambda} + \frac{\nu_1}{2} - \frac{\nu_2}{4} = 1 \mod 2,$$

or if

$$\tilde{\lambda} - \frac{\nu_1}{2} + \frac{\nu_2}{4} = 1 \mod 2 \quad \textit{respectively}.$$

If $\nu_1/2 - \nu_2/4$ *is positive, then these representations occur as sub-representations.* ∎

This lemma follows by a step-by-step induction argument as in IIIA.

PROPOSITION 5.4. *Assume* $\mu \in \mathbb{R}$, $(\mu + 1 + n/2) \geq 0$ *and* $\mu + n/2 - \lambda = 0$ mod 2. *The invariant subspaces*

$$\pi(n, \lambda, \mu) \qquad \textit{and} \qquad \operatorname{ind}_{\tilde{P}_1}^{\tilde{G}} D^+_{\mu+1+n/2} \otimes \rho_n \otimes \chi_{(\mu, -1-n/2+\mu)}|_{A_1N_1}$$

of $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ *have a nontrivial intersection.*

Proof. The strategy of the proof is as follows. We will construct a K-type in $U(n, \lambda, (\mu, -1 - n/2 + \mu)$, which occurs with multiplicity one. Then we will show, that this K-type is contained in $\mathrm{ind}_{\bar{P}_1}^{G} D_{\mu+1+n/2} \otimes \rho_{-n} \otimes \chi_{(\mu,-1-n/2+\mu)}|_{AN}$ and in $\pi(n, \lambda, \mu)$. Hence it is contained in the intersection of both subspaces.

We assume α_1 and α_3 are compact roots. Let H_{α_i}, $i = 1, 2, 3$ be the coroots to α_i. Then

$$H_{\alpha_2} = -\tfrac{1}{2}H_{\alpha_1} + \tfrac{1}{2}H_{\alpha_3} + H_\gamma .$$

Here H_γ is the generator of the abelian summand of k, normalized such that

$$H_\gamma = \tfrac{1}{2}e_{56} - \tfrac{1}{2}{}_{65} .$$

We consider H_{α_2} as a generator of the Lie algebra of the maximal compact subgroup of the $SL(2, \mathbb{R})$-factor of P_1.

CLAIM. *If a K-type with highest weight $(n/2\alpha_1, \gamma)$, $\gamma \in \mathbb{R}$, occurs in $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ with nontrivial multiplicity, then it has multiplicity one.*

Proof. The generator of $\mathfrak{m}_B$ is $(-H_{\alpha_1} + H_{\alpha_3})$. In its spectrum under the representation with highest weight $(n/2\ \alpha_1, \gamma)$ the eigenvalue $n/2$ has multiplicity one. By Frobenius reciprocity the assertion follows. ∎

CLAIM. *If a K-type with highest weight $(n/2\ \alpha, \gamma)$, $\gamma \in \mathbb{R}$, occurs in $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ with nontrivial multiplicity, then it is a K-type of the subrepresentation $\pi(n, \lambda, \mu)$.*

Proof. The representation of $SO(4)$ with highest weight $(n/2)\,\alpha_1$ remains irreducible when restricted to the diagonal subgroup $SO(3)$. By Frobenius reciprocity the assertion follows. ∎

Let us now assume $\gamma = \lambda + m$, $m \in \mathbb{Z}$. Then H_{α_2} has on the lowest weight space of the representation with highest weight $((n/2)\,\alpha_1, \gamma)$ the Eigenvalue

$$e(H_{\alpha_2}) = \frac{n}{4} + \gamma(H_\gamma) = \frac{n}{4} + \frac{\lambda}{2} + \frac{m}{2} .$$

All other Eigenvalues of H_{α_2} on this K-type are smaller than $e(H_{\alpha_2})$.

By Frobenius reciprocity and 5.2, a K type with highest weight $((n/2)\alpha, \lambda + m)$ is a K-type of

$$\mathrm{ind}_{\bar{P}}^{G} D^+_{(\mu+1+n/2)} \otimes \rho_n \otimes \chi_{(\mu,-1-n/2+\mu)}|_{A_1N_1} ,$$

if there is at least one Eigenvalue of H_{α_2} of the form $\frac{1}{2}(\mu + 1 + n/2) + \frac{1}{2} = \lambda/2 + d/2$, d an odd integer.

Choose $m = d - n$. Then the K-type with highest weight $((n/2)\,\alpha_1, \tilde{\lambda} + m)$ is a K-type of

$$\operatorname{ind}_{P_1}^{G} D^{+}_{(\mu+1+n/2)} \otimes \rho_n \otimes \chi_{(\mu,-1-n/2+\mu)}|_{A_1N_1}$$

and hence of $U(n, \lambda, (\mu, -1 - n/2 + \mu))$. By the first claim it has multiplicity one in $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ and by the second claim it is also a K-type of $\pi(n, \lambda, \mu)$. Thus this K-type is contained in the intersection of both invariant subspaces. ∎

The same considerations can be used to prove

PROPOSITION 5.5. *Assume* $\mu + 1 + n/2 \geqslant 0$ *and* $\mu + n/2 + \lambda = 0 \bmod 2$. *The invariant subspaces* $\pi(n, \lambda, \mu)$ *and*

$$\operatorname{ind}_{P_1}^{G_1} D^{-}_{\mu+1+n/2} \otimes \rho_{-n} \otimes \chi_{(\mu,-1-n/2+\mu)}|_{A_1N_1}$$

of $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ *have a nontrivial intersection.* ∎

THEOREM 5.6. *Let* $\lambda = 0,1$. *The representation* $\pi(n, \lambda, \mu)$ *is reducible*, iff

$$\mu + \frac{n}{2} - \lambda = 0 \bmod 2 \quad ∎$$

To prove this theorem we will use a result of R. Langlands for linear groups [3]. Let $U(n, \lambda, \nu)$, $\lambda = 0, 1$, be a principal series representation. Assume ν is dominant with respect to the positive root system Δ^+ determined by $\mathfrak{n}$. There is an intertwining operator $A(n, \lambda, \nu)$ from $U(n, \lambda, \nu)$ to $U(w'n, \lambda, w'\nu)$, where w' is the longest element in the Weyl group W.

THEOREM 5.7 (Langlands [3]). *The image of* $A(n, \lambda, \nu)$ *is irreducible if* ν *is strictly dominant, otherwise it is a direct sum of irreducible if* ν *is strictly dominant, otherwise it is a direct sum of irreducible representations.* ∎

Proof of Theorem 5.6. Let $U(n, \lambda, \nu)$ be a principal series representation, $\lambda = 0, 1$, ν arbitrary. For each root $\alpha \in \Delta$, define $D_\alpha(n, \lambda) \subset \mathfrak{a}'_{\mathbb{C}}$ as follows. If α is a long root, then

$$D_\alpha(n, \lambda) = \left\{\nu \in \mathfrak{a}'_{\mathbb{C}} \,\middle|\, \frac{(\nu, \alpha)}{2} = 1 \bmod 2 \text{ if } \lambda + n = 0 \bmod 2 \right.$$
$$\left. \text{and } \frac{(\nu, \alpha)}{2} = 0 \bmod 2 \text{ if } \lambda + n = 1 \bmod 2 \right\}.$$

If α is a short root, define

$$D_\alpha(\lambda, n) = \left\{\nu \in \mathfrak{a}'_{\mathbb{C}} \,\middle|\, \left|\frac{(\alpha, \nu)}{2}\right| = |\, n \,| + m,\ m \geqslant 2 \text{ and } m = 0 \bmod 2 \right\}.$$

By [8], $U(n, \lambda, \nu)$ is reducible iff $\nu \in \bigcup_{\alpha \in \Delta} D_\alpha(\lambda, n)$. Now assume $\nu \in D_\alpha(\lambda, n)$ for exactly one $\alpha \in \Delta^+$. Let $w \in W$ such that $w\nu$ is dominant. Then by 3.8 of [10] the image of $A(wn, \lambda, w\nu)$ is a degenerate series representation $\pi_\alpha(\lambda, n, \nu)$.

The representation $\pi_\alpha(\lambda, n, \nu)$ is characterized by the following property. Let w_α be the Weyl group element with maps α into a simple root and define a parabolic subgroup $P_\alpha = M_\alpha A_\alpha N_\alpha$ by the condition $\mathfrak{a}_{\alpha\mathbb{C}} = \text{kern } w_\alpha \alpha$. Then $\text{ind}_B^{P_\alpha}(w_\alpha^{-1}\rho_{n,\lambda} \otimes \chi_{w_\alpha^{-1}\nu})$ has a finite dimensional composition factor $\tau_\alpha(n, \lambda, \nu)$ and $\pi_\alpha(\lambda, n, \nu) = \text{ind}_{P_\alpha}^G \tau_\alpha(n, \lambda, \nu)$.

Consider $\pi(n, \lambda, \mu)$ with $\lambda = 0, 1, \mu + n/2 - \lambda \neq 0 \bmod 2$ and $| \text{Re}\, \mu | \leqslant n/2$ as a subrepresentation of $U(n, \lambda, (\mu, -1 - n/2 + \mu))$ as in 3.2. But $(\mu, -1 - n/2 + \mu) \in D_\alpha(n, \lambda)$ for exactly one root $\alpha \in \Delta^+$ and α is simple. Hence by 3.1 and 3.2 $\pi(n, \lambda, \mu) = \pi_\alpha(n, \lambda, (\mu, -1 - n/2 + \mu))$. Thus by 5.7 $\pi(n, \lambda, \mu)$ is irreducible. If $| \text{Re}\, \mu | > n/2$, 4.4 implies the theorem.

Remark. Comparing our results with those of Jacobson, Vergne [4] and Gross, Holman, Kunze [1], it is easy to show that the representation constructed in Proposition 5.4 as the intersection of 2 invariant subspaces is equivalent to the representation constructed by analytic continuation of holomorphic discrete series.

VI. Composition Series for $\pi(n, \lambda, \mu)$. The Case $| \text{Re}\, \mu | \leq n/2$, $\lambda = 0, 1$

Recall the classification of irreducible representations of G as worked out in [11]. Let $\mathfrak{t}_C$ be a Cartan subalgebra of $k \otimes \mathbb{C}$, μ the highest weight of an irreducible representation of K and $\langle , \rangle$ the restriction of the Killing form of $\mathfrak{g}_\mathbb{C}$ to $\mathfrak{t}_\mathbb{C}$, ρ_C half the sum of the positive compact roots. Define

$$\| \mu \| = \langle \mu + 2\rho_C , \mu + 2\rho_C \rangle.$$

A K-type μ is called *lowest* K-type of representation π, if μ is minimal with respect to this norm among all K-types of π. A subquotient of a representation, which contains a lowest K-type, is called *bottom quotient.*

The irreducible representations of G fall into three disjoint classes:

(a) discrete series representations,

(b) bottom subquotients of principal series representations,

(c) bottom subquotients of representations induced from a discrete series representation of P_1.

Using the results on reducibility, duality and tensoring with finite dimensional representations we deduce that it suffices to compute the composition series for $\pi(2, 0, 1)$ and $\pi(2, 1, 0)$.

We now proceed as follows. We list all irreducible representations with these central characters and using K-type considerations and intertwining operators we compute the multiplicity of the representations in the Jordan–Holder series of $\pi(2, 0, 1)$ and $\pi(2, 1, 0)$.

Denote the positive roots in $\mathfrak{t}_{\mathbb{C}}$ by β_1 and β_2. We assume that β_1 is the Cayley transform of α_1, and β_2 of α_2 respectively.

Case 1: $\pi(2, 0, 1)$

(a) The central character is singular, hence there are no discrete series representations with this central character.

(b) Bottom quotients of the principal series:

I. $U(-2, 0, -\frac{3}{2}(\alpha_1 + \alpha_3) - 2\alpha_2)$

II. $U(-2, 1, -\frac{3}{2}(\alpha_1 + \alpha_3) - 2\alpha_2)$

III. $U(-4, 1, -\alpha_1 - \alpha_2 - \alpha_3)$

IV. $U(-4, 0, -\alpha_1 - \alpha_2 - \alpha_3)$.

By 3.2, the bottom quotient of I is the bottom quotient of $\pi(2, 0, 1)$.

The lowest K-types of II have highest weight $(\beta_1, 0)$ and $(\beta_2, 0)$. It is left to the reader to check that they are not K-types of $\pi(2, 0, 1)$.

By [9], IV is irreducible and some of its K-types have multiplicities larger than one. Hence IV cannot be a subquotient of $\pi(2, 0, 1)$.

The lowest K-type of III has highest weight $(2(\beta_1 + \beta_2), 0)$ and multiplicity one in $\pi(2, 0, 1)$. Hence to prove that III is not a composition of $\pi(2, 0, 1)$, it suffices to prove the

Claim. *The lowest K-type of* III *is a K-type of the bottom subquotient of* $\pi(2, 0, 1)$.

Proof of the Claim. The multiplicity of $(2(\beta_1 + \beta_2), 0)$ in $U(-2, 0, -\frac{3}{2}(\alpha_1 + \alpha_3) - 2\alpha_2)$ is two. By 5.7, to prove the claim it suffices to show that one of the K-types is contained in the kernel of the corresponding intertwining operator. By 3.14 of [10], it suffices to show that this K-type is contained exactly once in the direct sum of the kernels of the factors of this operator. By 3.8 of [10], the kernels of the factors have the same composition factors as $U(-4, 1, -\alpha_1 - \alpha_2 - \alpha_3)$, $\mathrm{ind}_{P_1}^{G} D_2^{+} \otimes \rho_1 \otimes \chi_{(1,1)}|_{A_1N_1}$ and $\mathrm{ind}_{P_1}^{G} D_2^{-} \otimes \rho_1 \otimes \chi_{(1,1)}|_{A_1N_1}$. An easy checking shows that $(2(\beta_1 + \beta_2), 0)$ has multiplicity one in $U(-4, 1, -\alpha_1 - \alpha_2 - \alpha_3)$ and multiplicity zero in the other two representations. This proves the claim.

(c) Bottom quotients of generalized principal series representations:

I. $\mathrm{ind}_{P_1}^{G} D_2^{+} \otimes \rho_2 \otimes \chi_{(0,0)}|_{A_1N_1}$

II. $\operatorname{ind}_{P_1}^G D_2^- \otimes \rho_2 \otimes \chi_{(0,0)}|_{A_1N_1}$

III. $\operatorname{ind}_{P_1}^G D_3^+ \otimes \rho_1 \otimes \chi_{(1,-\frac{1}{2})}|_{A_1N_1}$

IV. $\operatorname{ind}_{P_1}^G D_3^- \otimes \rho_1 \otimes \chi_{(1,-\frac{1}{2})}|_{A_1N_1}$

V. $\operatorname{ind}_{P_1}^G D_1^+ \otimes \rho_1 \otimes \chi_{(1,-\frac{1}{2})}|_{A_1N_1}$

VI. $\operatorname{ind}_{P_1}^G D_1^- \otimes \rho_1 \otimes \chi_{(1,\frac{1}{2})}|_{A_1N_1}$.

By [8] I and II are irreducible and some of their K-types have multiplicities larger than one. Hence I and II cannot be subquotients of $\pi(2, 0, 1)$.

It is left to the reader to verify that the subquotients of $\pi(2, 0, 1)$ constructed in 5.4 and 5.5 are the bottom subquotients of III and IV.

Exactly as in the proof of 5.4 and 5.5, one verifies that the bottom quotients V and VI are subquotients of $\pi(2, 0, 1)$.

Since all K-types of $\pi(2, 0, 1)$ have multiplicity one, each composition factor occurs with multiplicity one in the composition series of $\pi(2, 0, 1)$. Hence we proved

PROPOSITION 6.1. *$\pi(2, 0, 1)$ has 5 composition factors.*

COROLLARY 6.2. *Put $\delta(n) = n \bmod 2$. The representations $\pi(n, \delta(n), n/2)$ and $\pi(n, \delta(n), -n/2)$ have 5 composition factors.*

Case 2. $\pi(2, 1, 0)$

(a) The central character is regular. Thus by [12], there are 6 inequivalent discrete series representations with this central character. Since discrete series representations for a fixed central character are uniquely characterized by their minimal K-types [2], we identify the discrete series representations with their minimal K-types.

The minimal K-types of discrete series representations with this central character are: $(0, 4)$, $(0, -4)$, $(2\beta_1, 0)$, $(2\beta_2, 0)$, $(\beta_1 + \beta_2, 1)$ and $(\beta_1 + \beta_2, -1)$.

The representations with minimal K-types $(0, 4)$, $(0, -4)$ are not subquotients of $\pi(2, 1, 0)$, since $\pi(2, 1, 0)$ has only K-types of dimension 3 or larger.

Using Blattner's formula [2] we derive that multiplicity of the K-types $(2\beta_1 + 2\beta_2, 2)$ and $(2\beta_1 + 2\beta_2, -2)$ in the discrete series representation with minimal K-type $(\beta_1 + \beta_2, 1)$, and $(\beta_1 + \beta_2, -1)$ respectively, is two. Thus these discrete series representations are not subquotients of $\pi(2, 1, 0)$.

Again by Blattner's formula [2], the multiplicity of the K-types $(2\beta_1 + \beta_2, 0)$ and $(\beta_1 + 2\beta_2, 0)$ in the discrete series representation with minimal K-type $(2\beta_1, 0)$ and $(2\beta_2, 0)$ respectively, are two. Thus these representations are not subquotients of $\pi(2, 1, 0)$.

(b) Bottom quotients of principal series representations:

$$\begin{array}{rl} \text{I.} & U(0, 0, -\tfrac{3}{2}(\alpha_1 + \alpha_3) - \alpha_2) \\ \text{II.} & U(0, 1, -\tfrac{3}{2}(\alpha_1 + \alpha_3) - \alpha_2) \\ \text{III.} & U(2, 0, -(\alpha_1 + \alpha_3) - 2\alpha_2) \\ \text{IV.} & U(2, 1, -(\alpha_1 + \alpha_3) - 2\alpha_2) \\ \text{V.} & U(4, 0, -\tfrac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2) \\ \text{VI.} & U(4, 1, -\tfrac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2). \end{array}$$

The bottom quotients of I, II and III are not composition factors of $\pi(2, 1, 0)$, since their lowest K-types are not K-types of $\pi(2, 1, 0)$.

By 3.2, the bottom quotient of IV is the bottom subquotient of $\pi(2, 1, 0)$. By [10], VI is irreducible and hence at least one of its K-types has multiplicities larger than one. Hence IV cannot be a subquotient of $\pi(2, 1, 0)$.

As in case 1, we are left with one possible contribution of a principal series representation. By the same argumentation as in that case, we deduce that the bottom quotient of V is not a composition factor of $\pi(2, 1, 0)$.

(c) Bottom quotients of generalized principal series representations:

$$\begin{array}{rl} \text{I.} & \operatorname{ind}_{P_1}^G D_1^{+(-)} \otimes \rho_0 \otimes \chi_{-\rho}|_{A_1N_1} \\ \text{II.} & \operatorname{ind}_{P_1}^G D_3^{+(-)} \otimes \rho_0 \otimes \chi_{(0,-1)}|_{A_1N_1} \\ \text{III.} & \operatorname{ind}_{P_1}^G D_2^{+(-)} \otimes \rho_1 \otimes \chi_{(2,1)}|_{A_1N_1} \\ \text{IV.} & \operatorname{ind}_{P_1}^G D_2^{+(-)} \otimes \rho_{-1} \otimes \chi_{(2,1)}|_{A_1N_1} \\ \text{V.} & \operatorname{ind}_{P_1}^G D_1^{+(-)} \otimes \rho_2 \otimes \chi_{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3)}|_{A_1N_1} \\ \text{VI.} & \operatorname{ind}_{P_1}^G D_1^{+(-)} \otimes \rho_{-2} \otimes \chi_{\frac{1}{2}(\alpha_1+\alpha_2+\alpha_3)}|_{A_1N_1}\,. \end{array}$$

The bottom quotient of I is a subquotient of $\pi(0, 0, 2)$ by 5.4. Thus the proof of 4.3 implies that it has a one-dimensional K-type. Since $\pi(2, 1, 0)$ has no one dimensional K-types, I is not a subquotient of $\pi(2, 1, 0)$.

It is left to the reader to verify that the subquotients of $\pi(2, 0, 1)$ constructed in 5.4 and 5.5 are the bottom subquotients of III and IV.

The lowest K-type of II is $(\beta_1 + \beta_2, -2)$. This K-type has multiplicity 1 in $\pi(2, 1, 0)$.

CLAIM. *The bottom quotient of* II *is a compositionfactor of* $\pi(2, 1, 0)$.

Proof. We first show that the bottom quotient of II is a composition factor of $U(2, 1, -(\alpha_1 + \alpha_3) - 2\alpha_2)$. By 5.3 and 2.8 of [10], the bottom quotient of

$\mathrm{ind}_{P_1}^G D_3^{+(-)} \otimes \rho_0 \otimes \chi_{(0,1)|A_1N_1}$ is a composition factor of $U(0, 0, -\rho)$. By 3.2, $U(2, 1, -(\alpha_1 + \alpha_3) - 2\alpha_2) \cong U(0, 0, -\rho)/\pi(0, 0, 2)$. The proof of 4.3 shows that no subquotient of $\pi(0, 0, 2)$ has lowest K-type $(\beta_1 + \beta_2, -2)$. Hence the bottom quotient of II is a composition factor of $U(2, 1, -(\alpha_1 + \alpha_3) - 2\alpha_2)$. By 3.2, $U(4, 0, -\frac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2) \cong U(2, 1, -(\alpha_1 + \alpha_3) - 2\alpha_2)/\pi(2, 1, 0)$. Hence to prove that the bottom quotient of II is a subquotient of $\pi(2, 1, 0)$, it suffices to prove that it contains a K-type, which is not a K-type of $U(4, 0, -\frac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2)$. The K-type $(\frac{1}{2}(\beta_1 + \beta_2), 3)$ is a K-type of II and not a K-type of $U(4, 0, -\frac{1}{2}(\alpha_1 + \alpha_3) - \alpha_2)$, hence it is a K-type of a common composition factor π of II and $\pi(2, 1, 0)$. Since none of the lowest K-types of the already constructed composition factors is a K-type of II and none of the lowest K-types of V and VI, π is the bottom quotient of $\mathrm{ind}_{P_1}^G D_3^{+(-)} \otimes \rho_0 \otimes \chi_{(0,1)|A_1N_1}$. ∎

The lowest K-types of V and VI are $(\frac{1}{2}\beta_1 + \frac{3}{2}\beta_2, \pm 1)$ and $(\frac{3}{2}\beta_1 + \frac{1}{2}\beta_2, \pm 1)$. These K-types have multiplicity one in $\pi(2, 1, 0)$ and in the representations III and IV. Hence to prove that the bottom subquotients of V and VI are not composition factors of $\pi(2, 1, 0)$, it suffices to show that they are K-types of the bottom quotients of III and IV. Using the same argumentation as in the proof of the claim in case 1b, this follows.

Thus we proved:

PROPOSITION 6.3. *The representation* $\pi(2, 1, 0)$ *has* 7 *composition factors.*

PROPOSITION 6.4. *The representation* $\pi(n, \lambda, \mu)$, $\mu + n/2 - \lambda = 0 \bmod 2$, $|\mu| < n/2$ *has* 7 *composition factors.*

ACKNOWLEDGMENT

The author wants to thank Professor I. Segal for introducing her to this problem and the participants of Prof. Segal's seminar H. Jacobson, B. Ørsted and M. Vergne for many long and helpful discussions.

REFERENCES

1. K. I. GROSS, W. J. HOLMAN III, AND R. A. KUNZE, The generalized gamma function, new Hardy spaces and representations of holomorphic type for the conformal group, *Bull. Amer. Math. Soc.* **83**, 3 (1977), 412–415.
2. H. HECHT AND W. SCHMID, A proof of Blattner's conjecture, *Invent. Math.* **31** (1975), 129–154.
3. R. LANGLANDS, On the classification of irreducible reprzsentations of real algebraic groups, Institute for Advanced Study, 1973, mimeographed notes.
4. W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer–Verlag, New York/Berlin, 1966.

5. H. Plesner-Jacobson and M. Vergne, Wave and Dirac operators and representations of the conformal group, *J. Functional Analysis* **24** (1977), 52–106.
6. H. Rossi and M. Vergne, Analytic continuation of holomorphic discrete series of a semisimple Lie group, *Acta Math.* **136** (1976), 1–59.
7. G. Schiffman, Intégrales d'entrelacement et fonctions de Whittaker, *Bull. Soc. Math. France* **99** (1971), 3–72.
8. P. Sally, Analytic continuation of the irreducible unitary representations of the universal covering group of $SL(2, \mathbb{R})$, *Mem. Amer. Math. Soc.* **69** (1967).
9. B. Speh and D. Vogan, A reducibility criterion for generalized principal series, *Proc. Nat. Acad. Sci. USA* **74** (1977), 5252.
10. B. Speh and D. Vogan, Reducibility of generalized principal series representations, preprint.
11. D. Vogan, "Lie Algebra Cohomology and the Representations of Semisimple Lie Groups," Thesis, M. I. T., 1976.
12. G. Warner, "Harmonic Analysis on Semisimple Lie Groups, I. II.," Springer-Verlag, New York/Berlin, 1972.
13. G. Zuckerman, Tensorproducts of finite and infinite dimensional representations of semisimple Lie groups, *Ann. Math.* **106** (1977), 295–309.

Printed in Belgium