The semi-direct splitting of the radical in a ring has been our concern in [8]. For a fairly wide class of rings the existence of a complement of the radical has been established. Nothing, however, was said concerning the extent to which such a complement is unique. For finite dimensional algebras which are separable modulo their radical this question was answered by A. Malcev ([13], th. 2, p. 42). He proved the following theorem:

Let \( A \) be a finite dimensional algebra and suppose \( \mathcal{R}(A) \) is separable. If \( A \) contains two semi-simple subalgebras \( H_1, H_2 \) such that

\[
A = H_1 + \mathcal{R} = H_2 + \mathcal{R} \quad \text{and} \quad H_i \cap \mathcal{R} = \{0\} \quad \text{for} \quad i = 1, 2,
\]

then there exists an element \( n \in \mathcal{R} \) such that \( (1 - n) H_i (1 - n)^{-1} = H_2 \); i.e., all complements of \( \mathcal{R} \) in \( A \) are conjugate under an inner automorphism generated by an element in the radical.

This uniqueness theorem was extended by C. Curtis ([5], th. 1 p. 80) to the case where \( \mathcal{R}(A) \) is complete in the \( R \)-adic topology and \( A/\mathcal{R} \) is finite dimensional, separable. Both authors use their hypotheses about the radical. Using the tools introduced in [7], we will be able to dispense with all the extra hypotheses on the structure of the radical; i.e., we will prove that any two complements of the radical in an algebra \( A \) which is finite dimensional, separable modulo the radical are conjugate under an inner automorphism generated by an element of the radical. Actually, we prove more. The semigroup method applied will turn out to be robust enough to extend the assertion to a wide class of linearly compact rings.

In order to have a brief expression, we will frequently say that the Malcev theorem is true for a ring \( A \), whenever it is true that any two complements of the radical in the category of topological rings are conjugate under an inner automorphism generated by an element in the radical. If we talk about com-

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plements in an algebra over a field $K$, we shall mean only such complements of the radical in the category of rings that are also subalgebras of the given algebra.

All the results on semigroups used in the following can be found in [4]. For the definition of summability in topological groups we refer to [3].

We start out with a lemma which will be crucial for everything that follows.

I Lemma. Let $A$ be a ring and $B$ an ideal contained in the radical of $A$. Let $e_1, e_2$ be orthogonal idempotents in $A/B$ and suppose there are idempotents $e_1, g_1, e_2, g_2$ such that $e_ie_j = g_ig_j = 0$ for $i \neq j$. Then there exist idempotents $f_i \in e_i$ for $i = 1, 2$ such that:

(1) $f_1 e_1$ and $f_2 e_2$

and $f_1 g_1$ and $f_2 g_2$, i.e., $f_1$ is $R$-equivalent to $e_1$ and $L$-equivalent to $g_1$ in the semigroup $e_1 + B$. Similarly, $f_2$ is $L$-equivalent to $e_2$ and $R$-equivalent to $g_2$ in $e_2 + B$. For definitions we refer to ([4], p. 47). Dually there exist idempotents $f_i' \in e_i$ for $i = 1, 2$ such that

\[ f_1' L e_1 \quad \text{and} \quad f_2' L e_2 \]

and $f_1' R g_1$ and $f_2' R g_2$.

(2) $f_1 f_2 = f_2 f_1 = 0$.

(3) $(e_1 + f_2) e_i (f_1 + e_2) = f_i$

and

\[ (g_1 + f_2) f_i (f_1 + g_2) = g_i. \]

These two statements together imply:

\[ (g_1 + f_2)(e_1 + f_2) e_i (f_1 + e_2)(f_1 + g_2) = g_i. \]

(4) If $1 = e_1 + e_2$, say, then $1 = g_1 + g_2$ and there exists $\bar{b}, b \in B$ such that

\[ (1 - b) = (g_1 + f_2)(e_1 + f_2) = (f_2 + g_1 e_1), \]

\[ (1 - \bar{b}) = (f_1 + e_2)(f_1 + g_2) = (f_1 + e_2 g_2) \]

and hence

\[ (1 - b) e_i (1 - \bar{b}) = g_i, \quad \text{for} \quad i = 1, 2, \]

where $\bar{b}$ is the quasi-inverse of $b$.

(5) If $1 = e_1 + e_2 = f_1 + f_2$ then it is not possible that $e_1 R f_1$ and $e_2 R f_2$ if $e_i \neq f_i$. 

Proof (5). If \( f_2 \mathcal{R} e_2 \), then \( e_2 f_2 - f_2 \) thus \( e_1 f_2 = 0 \). Now \( f_1 \mathcal{R} e_1 \) implies \( e_1 f_1 = f_1 \) and thus \( e_1 = e_1 e_1 f_1 + e_1 f_2 = e_1 f_1 + e_1 f_2 = f_1 \). Similarly one gets: \( e_2 = f_2 \).

(1) As all idempotents of \( e_i \mathcal{R} e \), \( i = 1, 2 \) are contained in the completely simple minimal ideal \( K_i \) of \( e_i \mathcal{R} e \) by ([7], Th. 1), there are idempotents \( f_i \), \( i = 1, 2 \) such that \( e_1 f_1 \) and \( f_1 g_1 \) and \( e_2 f_2 \) and \( f_2 g_2 \) (see the "egg-box picture").

(2) If we choose the idempotents \( f_i \) in this manner, then \( f_1 f_2 = 0 = f_2 f_1 \), as we will show next: \( e_1 f_1 \) implies \( e_1 f_1 = f_1 \) and \( f_1 e_1 = e_1 \) hence \( e_2 f_1 = e_2 e_1 f_1 = 0 \), \( f_1 g_1 \) implies \( f_1 g_1 = f_1 \) and \( g_1 f_1 = g_1 \), hence \( 0 = f_1 g_2 \), \( e_2 f_2 \) implies \( e_2 f_2 = e_2 \) and \( f_2 e_2 = f_2 \), hence \( 0 = f_1 e_1 \), \( f_2 g_2 \) implies \( f_2 g_2 = g_2 \) and \( g_2 f_2 = f_2 \), hence \( 0 = g_1 f_2 \).

If we put this information together we obtain:

\[
\begin{align*}
f_2 f_1 & = (f_2 e_2) e_1 f_1 f_1 = 0, \\
f_1 f_2 & = (f_1 g_1) (g_2 f_2) = f_1 (g_1 g_2) f_2 = 0.
\end{align*}
\]

This establishes (2).

(3)

\[
\begin{align*}
(e_1 + f_2) e_1 f_1 e_2 & = (e_1 + f_2 e_1) e_1 f_1 = e_1 f_1 = f_1, \\
(e_1 + f_2) e_2 f_1 e_2 & = (f_2 e_2) e_2 f_1 = f_2 e_2 = f_2, \\
(g_1 + f_2) f_1 g_1 + g_2 & = g_1 f_1 + f_1 g_2 = g_1 f_1 = g_1, \\
(g_1 + f_2) f_2 g_1 + g_2 & = g_1 f_2 + f_2 g_2 = f_2 g_2 = g_2.
\end{align*}
\]

The rest is clear.

By a similar calculation one obtains:

\[
\begin{align*}
(f_1 + e_2) (f_1 + g_2) (g_1 + f_2) (e_1 + f_2) & = e_2 + e_1, \\
(g_1 + f_2) (e_1 + f_2) (f_1 + e_2) (f_1 + g_2) & = g_1 + g_2.
\end{align*}
\]

(4) The semigroup \( 1 \mathcal{R} B \) contains a unique idempotent, namely 1. Hence \( 1 = g_1 + g_2 \). Let \( p : A \to A/B \) be the natural projection map.
If \( e_1 + e_2 = g_1 + g_2 = 1 \), then \( p(f_2 + g_1 e_1) = 1 = p(f_1 + e_2 g_2) \). Hence \( (f_2 + g_1 e_1) = 1 - b \) and \( f_1 + e_2 g_2 = 1 - b \) with \( b, \bar{b} \in B \). Now we use (*) and together with the extra hypothesis \( e_1 + e_2 = g_1 + g_2 = 1 \) we obtain \((1 - b)(1 - \bar{b}) = (1 - b)(1 - b) = 1\). Hence \( b \cdot \bar{b} = b \cdot \bar{b} = 0 \).

2 Theorem. Let \( A \) be a complete ring with identity and left ideal neighborhoods of zero. Suppose \( B \) is a closed ideal contained in the radical of \( A \). Let \( \{e_i : i \in I\} \) and \( \{g_i : i \in I\} \) be two sets of pairwise orthogonal, summable idempotents such that \( 1 = \sum e_i = \sum g_i \) and \( e_i = g_i(B) \) for each \( i \in I \). Then there exists an element \( b \in B \) such that

\[
(1 - b) e_i(l - \bar{b}) = g_i,
\]

with \( \bar{b} \) the quasi-inverse of \( b \).

Proof. For each \( i \) we consider the pairs

\[
e_i, \bar{e}_i = 1 - e_i; \quad g_i, \bar{g}_i = 1 - g_i.
\]

Then we construct idempotents \( f_i, \bar{f}_i \) as in Lemma 1 and get

\[
(f_i + g_i e_i) e_i(f_i + \bar{e}_i \bar{g}_i) = g_i.
\]

By Lemma 1 (4) \( f_i + g_i e_i = (1 - b_i) \) with some \( b_i \in B \). Since \( e_i \bar{e}_i = 0 \), we have

\[
(1 - b_i) e_i(f_i) = g_i, \quad \text{whence} \quad (e_i - b_i e_i) e_i(f_i) = g_i.
\]

By hypothesis the ring \( A \) has left ideal neighborhoods of zero and the idempotents \( e_i \) are summable, thus \( \{b_i e_i : i \in I\} \) is a summable family, too. By Lemma 1 (1) \( f_i \bar{g}_i g_i \) which implies \( f_i g_i = f_i \). By hypothesis the idempotents \( g_i \) are summable. Hence by the same reason as above the family of idempotents \( \{f_i : i \in I\} \) is a summable. Let \( (1 - b) = \sum \{e_i - b_i e_i : i \in I\} \) and \( a = \sum \{f_i : i \in I\} \), then \( b = \sum \{b_i e_i : i \in I\} \) is an element of \( B \), since \( B \) is closed by hypothesis. The idempotents \( e_i \) are pairwise orthogonal by assumption and \( e_i A f_i \) hence \( f_i = e_i f_i \), by Lemma 1(1). These remarks together with the continuity of multiplication in \( A \) imply

\[
(1 - b) e_i a = g_i.
\]

Again, by the continuity of multiplication

\[
(1 - b) a = (1 - b) \left( \sum e_i \right) a = \sum (1 - b) e_i a = \sum g_i = 1,
\]

\( a \) is a right inverse of \( 1 - b \) and therefore has the form \( a = 1 - \bar{b} \), with \( \bar{b} \in B \). Clearly \( \bar{b} \) is the right quasi-inverse of \( b \). But \( \bar{b} \in B \cap R(A) \). Hence \( b \)
ECKSTEIN has a twosided quasi-inverse, which means that $b$ is actually the quasi-inverse of $b$ and $a = 1 - b$ is also a left inverse of $(1 - b)$.

3 Remark. Let $A$ be a ring, $b \in R(A)$ and $b$ the quasi-inverse of $b$. If the ring $A$ does not have an identity the notation $(1 - b)x(1 - b)$ is still meaningful; one just has to interpret it as

$$x - bx - xb + bxb.$$

It is easily checked that the map

$$x \rightarrow x - bx - xb + bxb$$

is an automorphism of $A$. Whether or not $A$ contains an identity we will refer to the map

$$x \rightarrow (1 - b)x(1 - b) = x - bx - xb + bxb$$

as the inner automorphism of $A$ generated by the element $b \in R(A)$.

4 Corollary. Let $A$ be a complete ring with left ideal neighborhoods of zero. Suppose that $B$ is a closed ideal contained in the radical of $A$. Let $\{e_i : i \in I\}$ and $\{g_i : i \in I\}$ be two sets of pairwise orthogonal, summable idempotents such that $e_i = g_i(B)$ for $i \in I$. Then there exists an element $b \in B$ such that

$$(1 - b)e_i(1 - b) = g_i,$$

where $b$ is the quasi-inverse of $b$.

Proof. Let $e = \sum \{e_i : i \in I\}$ and $g = \sum \{g_i : i \in I\}$. If $A$ does not contain an identity, then we consider the ring $\tilde{A}$ obtained from $A$ by adjoining an identity $1$ such that $A$ is an open ideal in $\tilde{A}$. Then the ring $\tilde{A}$ and the idempotents $\{1 - e, e_i : i \in I\}$ and $\{1 - g, g_i : i \in I\}$ satisfy all the hypotheses of Theorem 2. Hence there exists a

$$b \in R(\tilde{A}) \rightarrow R(A)$$

such that

$$(1 - b)e(1 - b) = g \quad \text{and} \quad (1 - b)e_i(1 - b) = g_i,$$

where $b$ is the quasi-inverse of $b$.

Theorem 5. Let $A$ be a complete ring with left ideal neighborhoods of zero. Suppose there is an isomorphism $f : A/R \rightarrow \prod \{A_i : i \in I\}$ of $A/R$ onto the direct product of rings $A_i$ with identities $\{e_i : i \in I\}$. Assume the radical in $A$ is closed and splits semi-directly. Let $H_1$ and $H_2$ be two complements of $R$ in the category of topological rings, then there exist two sets $\{e_i : i \in I\}, \{f_i : i \in I\}$ of pairwise orthogonal, summable idempotents in $H_1, H_2$ respectively, which map onto the
idempotents $f^{-1}(e_i)$ of $A/R$ under the canonical projection $h : A \to A/R$.
Furthermore the Malcev theorem is true for $A$ if it is true for the rings $f_iAf_i$.

Proof. By assumption the restriction $fh_i$, $i = 1, 2$ of the map $fh$ to the subrings $H_1$, $H_2$ respectively, are isomorphisms. We identify $\tilde{e}_i$ with its canonical image in $\prod \{ A_i : i \in I \}$. Clearly the idempotents $\{ \tilde{e}_i : i \in I \}$ are orthogonal, summable and add up to the identity of $\prod A_i$. Hence there exist sets $\{ e_i = h_1^{-1}f^{-1}(\tilde{e}_i) : i \in I \}$, $\{ f_i = h_2^{-1}f^{-1}(\tilde{e}_i) : i \in I \}$ respectively, of pairwise orthogonal, summable idempotents of $H_1$, $H_2$ respectively, and $h(e_i) = h_1(e_i) = h_2(f_i) = h(f_i) = f^{-1}(\tilde{e}_i)$. Denote by $\sum \{ e_iH_1e_i : i \in I \}$ the subring of $H_1$ generated by the subrings $e_iH_1e_i$. Since $H_1$ is isomorphic to $A = \prod \{ A_i : i \in I \} = \prod \{ e_iA_i.e_i : i \in I \}$ the subring $\sum \{ e_iH_1e_i : i \in I \}$ is dense in $H_1$. The subring $\sum \{ f_iH_2f_i : i \in I \}$ is defined in the same way. By the same reason as above it is dense in $H_2$. To prove the last assertion of the theorem it suffices by the principle of extending identities to exhibit an element $b \in R(A)$ such that

$$(1 - b) \left( \sum e_iH_1e_i \right)(1 - b) = \left( \sum f_iH_2f_i \right).$$

and $b$ is the quasi-inverse of $b$. In order to show this it suffices to exhibit an element $b_0 \in R$ such that

$$(1 - b_0) e_iH_1e_i(1 - b_0) = f_iH_2f_i \quad \text{for all } i \in I,$$

and $b_0$ is the quasi-inverse of $b_0$. Let $e = \sum \{ e_i : i \in I \}$ and $f = \sum \{ f_i : i \in I \}$, then by Corollary 4 there exists an element $b_0 \in R(A)$ such that

$$(1 - b_0) e(1 - b_0) = f \quad \text{and} \quad (1 - b_0) e_i(1 - b_0) = f_i \quad \text{for all } i \in I.$$

Hence

$$(1 - b_0) e_iH_1e_i(1 - b_0) = f_iA_i \quad \text{for all } i \in I.$$

In particular

$$(1 - b_0) e_iH_1e_i(1 - b_0) - f_i(1 - b_0) H_1(1 - b_0) f_i \subseteq f_iAf_i.$$

The radical $R(f_iAf_i)$ is $f_iAf_i \cap R = f_iRf_i$ and

$$f_iAf_i = f_iH_2f_i - f_iRf_i = f_i(1 - b_0) H_1(1 - b_0) f_i + f_iRf_i.$$ 

Furthermore $f_iH_2f_i \cap f_iRf_i = \{ 0 \}$, since $f_iH_2f_i \subseteq H_2$, $f_iRf_i \subseteq R$ and $H_2 \cap R = \{ 0 \}$. Also $f_i(1 - b_0) H_1(1 - b_0) f_i \cap f_iRf_i = \{ 0 \}$, since $H_2 \cap R = \{ 0 \}$.

We assumed that $A = H_1 + R = H_2 + R$ splits in the category of topological rings. Hence

$$f_iAf_i = f_iH_2f_i + f_iRf_i = f_i(1 - b_0) H_1(1 - b_0) f_i + f_iRf_i.$$

splits in the category of topological rings. By hypothesis there exists an element $b_i \in f_iRf_i$ such that

$$(f_i - b_i)(1 - b_0) H_1(1 - b_0)(f_i - b_i) = f_iH_2f_i.$$
for all \( i \in I \), where \( b_i \) is the quasi-inverse of \( b_i \). By hypothesis \( A \) has left ideal neighborhoods of zero, hence the summability of the set of idempotents \( \{ f_i : i \in I \} \) implies the summability of the sets \( \{ f_i - b_i - (f_i - b_i) f_i : i \in I \} \) and \( \{ f_i - b_i = (f_i - b_i) f_i : i \in I \} \). We recall that \( f = \sum f_i \) and \( f_i f_j = 0 \) for \( i \neq j \). Let \( f - b' = \sum (f_i - b_i : i \in I) \) and \( f - b'' = \sum (f_i - b_i : i \in I) \), then \( b', b'' \in R \) since \( R \) is closed and \( b' \) is the quasi-inverse of \( b'' \). We define \( (f - b) = (f - b')(1 - b_0) \) and \( (f - b) = (1 - b_0)(f - b') \). Then we obtain by construction:

\[
(1 - b) e_i H e_i (1 - b) = (1 - b')(1 - b_0) e_i H e_i (1 - b_0)(1 - b'') = (f - b') f_i (1 - b_0) H_i (1 - b_0) f_i (f - b'') = f_i H_2 f_i
\]

This finishes the proof by our initial remark.

6 Definition. Let \( A \) be a complete, topological ring with identity and left ideal neighborhoods of zero. Then a family \( \{ e_{ij} : i, j \in I \} \) of elements of \( A \) is called a system of matrix units iff

1. \( e_i e_{kl} = \delta_{jk} e_{il} \) and
2. The subfamily \( \{ e_{ij} : j \in I \} \) is summable and \( 1 = \sum e_{ij} \).

We usually will write \( e_i \) instead of \( e_{ii} \).

7. Let \( A \) be a ring with identity, left ideal neighborhoods of zero and a set \( \{ e_{ij} : i, j \in I \} \) of matrix units. Then all the subrings \( e_{ij} A e_{kl} \) are isomorphic. Specifically \( e_{ij} A e_{kl} = e_{ij} A e_i \) for all \( i, j, k \in I \).

8. Let the hypothesis be as in (7). Then the rings

\[
B^1 = \{ b \in A : b = \sum e_{kl} a e_{lk} : k \in I, a \in A \}
\]

and

\[
B^2 = \{ b \in A : b = \sum e_{kl} a e_{lk} : k \in I, a \in A \}
\]

are equal.

Proof. Clearly \( B^1 \subset B \) (after (7)). As \( e_{il} A e_{jl} = e_{il} A e_{lj} \), it follows that for each \( e_{il} A e_{jl} \subset e_{il} A e_{lj} \) there exists an \( a' \in A \) such that \( e_{il} A e_{jl} = e_{il} a' e_{lj} \). Let \( b \in B \), say

\[
b = \sum e_{kl} a e_{lk} \quad : \quad k \in I,
\]

then

\[
b = \sum e_{kl} (e_{il} a e_{lj}) e_{lk} \quad : \quad k \in I = \sum e_{kl} a' e_{lk} : k \in I \in B^1.
\]

Similarly one shows \( B^2 = B \).
Let $A = H + R(A)$ be a semi-direct product of a ring $H$ and its radical. Then we will refer to any set of matrix units of $A$ contained in $H$, as a system of matrix units associated with $H$.

Let the hypotheses be as in (8) and let $\Delta : A \to B$ be the diagonal map defined by $\Delta(a) = \sum \{e_i a e_i : i \in I\}$. Then

(a) $B$ is a closed subring of $A$.

(b) $B \simeq e_i A e_i$ for all $k$.

(c) Suppose the radical $R(A)$ is closed, then the radical $R(B)$ of $B$ is closed and contained in $R(A)$.

(d) If $A = H + R(A)$ is a splitting in the category of topological rings and if $\{e_{ij} : i, j \in I\}$ is a system of matrix units associated with the complement $H$, then the radical in $B$ splits and $\Delta(e_i H e_i)$ is a complement of $\Delta(e_i R e_i)$ the radical of $B$.

(e) The ring $B$ commutes with all the matrix units $\{e_{ij} : i, j \in I\}$.

Proof.

\[
\sum \{e_i e_j a e_i : k \in I\} e_{ij} = e_i a e_i
\]

(a) $A$ contains the ring $\sum e_i A e_i$ which is isomorphic to the ring $\prod e_i A e_i$ [8, 1]. Clearly $B \subseteq \sum e_i A e_i$, hence the map $\Delta$ is continuous. Furthermore $\Delta^2 = \Delta$, thus $\Delta(A) = B$ is the kernel of the continuous map $(1 - \Delta)$, which means $B$ is closed.

(b) First we show: $e_i A e_i = B e_i$. Clearly $B e_i = \{(\sum e_i e_i a e_i) e_i = e_i a e_i : a \in A\} - e_i A e_i$. The mapping $B \to B e_i$, defined by $b \to b e_i$, is an isomorphism.

(c) The inverse of the isomorphism of item (b) is the diagonal map $\Delta : e_i A e_i \to B$, defined by $\Delta(a) = \sum \{e_i e_i a e_i : k \in I\}$. Clearly $\Delta$ maps the radical $e_i R e_i$ of $e_i A e_i$ onto the radical of $B$. By hypothesis $R(A)$ is closed in $A$, hence $e_i R e_i = e_i A e_i \cap R = R(e_i A e_i)$ is closed in $e_i A e_i$.

(d) It suffices to show that the splitting of the radical of $A$ implies the splitting of the radical of $e_i A e_i$, where $e_i$ belongs to an associated system of matrix units. Let $A = H + R$ and $e_i^2$. $e_i \in H$, then $e_i A e_i = e_i H e_i + e_i R e_i$. $e_i H e_i$ is a subring of $e_i A e_i$ and $e_i H e_i \cap e_i R e_i = \{0\}$, since $e_i H e_i \subseteq H$, $e_i R e_i \subseteq R$ and $R \cap H = \{0\}$.

11 Theorem. Let $A$ be a complete, topological ring with identity and left ideal neighborhoods of zero. Suppose that the radical $R(A)$ of $A$ is closed and
splits. Furthermore assume that \( A \) contains a set of matrix units \( \{g_{ij} : i, j \in I\} \). Then the Malcev theorem holds for \( A \), whenever it is true for at least one of the rings \( g_i^{-1}g_i \).

**Proof.** If \( A \) contains a set of matrix units, then \( A/R \) contains a set \( \{g_{ij} : i, j \in I\} \) of matrix units. Let \( H_1, H_2 \) be two different complements of the radical \( R(A) \) in the category of topological rings: \( A = H_1 + R = H_2 + R \). Then, due to the splitting, \( H_1 \), respectively \( H_2 \), contains a set of matrix units \( \{e_{ij} : i, j \in I\}, \{f_{ij} : i, j \in I\} \), respectively, mapping onto \( \{g_{ij} : i, j \in I\} \) under the canonical map. By (4) there exists an \( n_1 \in R(A) \) such that

\[
(1 - n_1) e_i (1 - n_1) = f_i \quad \text{for all } i \in I,
\]

where \( n_1 \) is the quasi-inverse of \( n_1 \). As \( A \) has left ideal neighborhoods and the set \( \{e_{ij} = e_i e_j : i \in I\} \) is summable, the set \( \{f_{ij} (1 - n_1) e_{ij} : i \in I\} \) is summable in \( A \). We define

\[
u = \sum \{ f_{ij} (1 - n_1) e_{ij} : i \in I\} = 1 - n
\]

with some \( n \in R(A) \). The last equality holds, since \( u = \overline{1}(R) \). The inverse of \( u \) is equal to \( u^{-1} = \sum \{e_{ij} (1 - n_1) f_{ij} : i \in I\} \), since

\[
e_{ij} (1 - n_1) f_{ij} f_{ij} (1 - n_1) e_{ij} = e_{ij} (1 - n_1) f_{ij} (1 - n_1) e_{ij} = e_{ij} e_{ij} e_{ij} = e_{ij}.
\]

Similarly one shows \( f_{ij} (1 - n_1) e_{ij} (1 - n_1) f_{ij} = f_{ij} \). As \( u^{-1} = \overline{1}(R) \) and \( uu^{-1} = u^{-1}u = 1 \), we know \( u^{-1} = (1 - n) \), where \( n \) is the quasi-inverse of \( n \). Next we observe:

\[
u(e_{ij}) u^{-1} = (1 - n) e_{ij} (1 - n) = f_{ij};
\]

for

\[
\left( \sum f_{kl} (1 - n_1) e_{1k} \right) e_{il} \left( \sum e_{kl} (1 - n_1) f_{1k} \right) = f_{il} (1 - n_1) e_{il} e_{ij} (1 - n_1) f_{ij} = f_{il} f_{ij} = f_{ij}.
\]

Clearly \( H_3 = (1 - n) H_1 (1 - n) \) is also a complement of \( R \) in \( A \), i.e., \( A = H_3 + R \). By construction \( H_3 \) has the same matrix units \( \{f_{ij} : i, j \in I\} \) as \( H_2 \).

We define a closed subring \( B \) of \( A \) as in (8)

\[
B = \left\{ b \in A : b = \sum \{ k_{kl} a f_{1k}, a \in A \} \right\}.
\]

As was shown in (10), the radical of \( B \) splits, and \( H_3' = \Delta(f_{11} H_{10} f_{11}) \), \( H_2' = \Delta(f_{1} H_{2} f_{1}) \), respectively, are complements of \( R(B) \) in \( B \). By hypothesis, by (7) and (10(b)) there exists an \( m \in R(B) \) such that

\[
(1 - m) H_3' (1 - m) = H_2' \quad (*)
\]
where $\bar{m}$ is the quasi-inverse of $m$. We introduce a new topology $\mathcal{I}$ on $A$ which, however, in general will not be a ring topology. A neighborhood basis of zero will be given by the right ideals

$$U(F) = \{a \in A : f_i a = 0, \text{ for } i \in F\},$$

where $F$ runs over all finite subsets of the index set $I$. As $1 = \sum \{f_i : i \in I\}$ we have $\cap\{U(F) : F \subseteq I\} = \{0\}$. The element $m$ which generates the inner automorphism of $(*)$ commutes with all $f_i$ by the construction of $B$. Hence $(1 - m) U(F)(1 - \bar{m}) \subseteq U(F)$ for all finite subsets $F \subseteq I$, and the automorphism generated by $m$ is continuous w.r.t. $\mathcal{I}$.

Let

$$H_k = \left\{ \sum \{h_{ij} f_{ij} : i, j \in I\} : h_{ij} \in H_k^e \quad \text{and} \quad h_{ij} = 0 \quad \text{for almost all } i \right\}$$

for $k = 2, 3$, then we claim $H_k$ is dense in $H_k$ for $k = 2, 3$ w.r.t. the topology $\mathcal{I}$. To show this, we take e.g. $h \in H_2$, and let $h_{ij} = \sum \{f_k h f_{jk} : k \in I\}$ and

$$h' = \sum \{h_{ij} f_{ij} : j \in I\},$$

where $h_{ij} \neq 0$ only for $i \in F$, for some finite subset $F$ of $I$; then $h' = \sum \{f_j h f_{ij} : j \in I\}$, where $f_j h f_{ij} \neq 0$ only for $i \in F$. Now $h' \in H_2$ and $f_i(h h') = f_i h - f_i \sum f_j = f_i h - h = 0$. Hence $h' \in h + U(F)$ and $H_2$ is dense in $H_2$. As $m$ commutes with all $f_{ij}$, $(1 - m) H_2 (1 - \bar{m}) = H_2$. By the continuity and the density of $H_k$ in $H_k$ for $k = 2, 3$, we conclude $(1 - m) H_2 (1 - \bar{m}) = H_2$. Using again that $m$, $\bar{m}$ commute with $\{f_{ij} : i, j \in I\}$ we finally get

$$(1 - m)(1 - n) H_2 (1 - m)(1 - \bar{m}) = H_2.$$  

12 COROLLARY. Let $A$ be a complete topological ring with identity and left ideal neighborhoods of zero which is an algebra over a field $K$. Suppose the radical of $A$ is closed and splits. If $A$ contains a set of matrix units $\{g_{ij} : i, j \in I\}$ such that $g_{1} A g_{1} / g_{1} R g_{1} \simeq K$, then the Malcev theorem is true for $A$.

Proof. The ring $g_{1} A g_{1}$ is an algebra over $K$ and has an identity, namely $g_{1}$. Hence we may assume $K \subseteq g_{1} A g_{1}$. By our assumption $K$ is a complement of $g_{1} R g_{1}$ in $g_{1} A g_{1}$. Since $A$ has left ideal neighborhoods of zero, $g_{1} A g_{1}$ has left ideal neighborhoods of zero hence so has $g_{1} A g_{1} / g_{1} R g_{1} \simeq K$, which means $g_{1} A g_{1} / g_{1} R g_{1}$ is discrete. Hence $K$ is a complement of the radical $R(g_{1} A g_{1})$ in the category of topological rings. Each algebra complement of $R(g_{1} A g_{1})$ in $g_{1} A g_{1}$ contains the identity $g_{1}$, hence $K$. Thus it is equal to $K$. This, however, means $g_{1} R g_{1}$ has only one complement in $g_{1} A g_{1}$ which is also a subalgebra and the hypothesis of theorem (11) is trivially satisfied.

13 LEMMA. Let $S$ be a system of $m$ linear equations in $n$ unknowns, $0 < n$. 

m ∈ ℤ, with coefficients in a field F. Suppose S has a solution in a field L, which is an extension of F, then S also has a solution in F.

We will omit the proof of this lemma.

14 Theorem. Let A be an algebra over a field F. Suppose the radical of A splits semi-directly and A ⊗ R(A) is finite dimensional, separable over F. Then the Malcev theorem is true for A.

Proof. We consider A with respect to the discrete topology. By Theorem 5 we may assume that A has an identity and that A ⊗ R is simple, i.e.,

\[ A = S + R, \]

where S is a simple ring. As the identity of A is contained in S we may assume F is contained in the center Z of S. By assumption Z is a finite, separable extension of F ([1], Th. 21, p. 44). Let \( Z = F(x) \) and \( L = F(y_1, \ldots, y_n) \) a root field of the irreducible polynomial of the primitive element x. Then

\[ S \otimes_F L = \sum \left\{ \bigoplus S_i \otimes_{F(y_i)} L : i = 1, \ldots, n \right\} \]

and each summand \( S_i \otimes_{F(y_i)} L \) is central simple over L. The stem field \( F(y_i) \) is the center of the algebra S, which is isomorphic to S over F. By ([2], Corollary 2a, p. 85, Th. 2, p. 87) we have:

\[ A \otimes_F L = \sum \left\{ \bigoplus S_i \otimes_{F(y_i)} L : i = 1, \ldots, n \right\} + R \otimes_F L. \]

Let E be a splitting field of finite degree over L, then we obtain

\[ A \otimes_F E = \sum \left\{ \bigoplus S_i \otimes_{F(y_i)} E : i = 1, \ldots, n \right\} + R \otimes_F E. \]

and each of the algebras \( S_i \otimes_{F(y_i)} E \) is a matrix ring over E. In order to show that the Malcev theorem is true for the algebra \( A \otimes_F E \), we may assume that

\[ A \otimes_F E = S_1 \otimes_{F(y_1)} E + R \otimes_F E \]

by Theorem 5. Now we are ready to apply corollary 12, and conclude that the Malcev theorem is true for the \( E \) - algebra \( A \otimes_F E \). The following lemma will finish the proof.

15 Lemma. Let A be an algebra with identity over a field F and suppose that the radical in A splits semi-directly. Let \( H_1, H_2 \) be two complements of the radical in A

\[ A = H_1 + R = H_2 + R. \]

Suppose there exists a field extension E of F such that \( \overline{A} = (A \otimes_F E) \) is semi-simple and finite dimensional over E. Furthermore assume that the center
of each simple factor $A_i$ of $A$ is $\bar{e}_i(1 \otimes E)$ where $\bar{e}_i$ is the identity of $A_i$. Then the complements $H_i$, $i = 1, 2$, are conjugate under an inner automorphism generated by an element in the radical $R(A)$, whenever the algebras $H_i \otimes F E$, $i = 1, 2$, are conjugate by an inner automorphism generated by an element of $R \otimes F E$.

Proof. By hypothesis the canonical projection $f : A \to A/R$ has two right inverses $g_i$, such that $g_i(A_i/R) = H_i$, for $i = 1, 2$. The dimension of $H_i$, $i = 1, 2$, is finite over $F$, since $\dim_F H_i = \dim_E (H_i \otimes E) = \dim_E \bar{A} < \infty$ by assumption. Let $\{a_i : i = 1, \ldots, n_i\}$ be a basis of $H_i$ over $F$, then the map $g_2f$ maps $H_i$ isomorphically onto $H_2$. Hence $b_i = g_2f(a_i)$, $i = 1, \ldots, n$, is a basis for $H_2$ over $F$. Let $\{c_j : j \in J\}$ be a basis of $R$ over $F$, then there exist elements $v_j \in F$ such that

$$b_i = a_i + \sum \{v_j c_j : j \in J(i)\} \quad \text{for} \quad i = 1, \ldots, n,$$

and $J(i)$ a finite subset of $J$. The lemma will be proved if we can show that the following system of equations

$$\begin{align*}
(1 - x) a_i &= b_i (1 - x) \\
(1 - x) v_j &= b_j (1 - x)
\end{align*}$$

has a solution in $R$.

We will denote again by $a_i$, $b_i$, $c_j$ the bases over $E$ in $A \otimes F E$ corresponding to the bases introduced above. By hypothesis there exists an inner automorphism $h_m$ of $A \otimes E$ generated by an element $m \in R \otimes F E$ such that $h_m(H_1 \otimes E) = H_2 \otimes E$. The inverse of $h_m$ is denoted by $h_m$. We consider the following diagram:

$$
\begin{array}{ccc}
R \otimes F E & \to & H_1 \otimes F E \\
\downarrow h_m & & \leftarrow \downarrow h_m^{-1} \\
R \otimes F E & \to & H_2 \otimes F E \\
\downarrow f \otimes 1 & & \leftarrow \downarrow f \otimes 1 \\
R \otimes F E & \to & H_2 \otimes F E + R \otimes F E
\end{array}
$$

By construction we have $(g_2f \otimes 1)(a_i) = b_i$. Next we define the map $h : A \otimes F E \to A \otimes F E$ by $h = (g_2f \otimes 1) h_m$. In the following we will write $g_2f$ instead of $g_2 \otimes 1, f \otimes 1$. Then the restriction and corestriction of $h$ to $H_2 \otimes F E$ is an automorphism of $H_2 \otimes F E$. Denote by $\bar{A}_i, i = 1, \ldots, m$ the simple factors of $(A \otimes F E)/(R \otimes F E)$ and by $\bar{e}_i$ the identities of these factors. In order to show that $h | H_2 \otimes F E$ leaves the center of $H_2 \otimes F E$ invariant it suffices by our hypothesis to show that $h(g_2(\bar{A}_i)) = g_2(\bar{A}_i)$ for $i = 1, \ldots, m$, and hence $h(g_2(\bar{e}_i)) = g_2(\bar{e}_i)$. But this is clear since

$$h(g_2(\bar{A}_i)) = g_2fh_m g_2(\bar{A}_i) = g_2f g_2(\bar{A}_i) = g_2(\bar{A}_i),$$

since the inner automorphism $h_m$ is generated by an element in the kernel of $f$. 


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and \( f g_2 = 1_{(A \otimes E)/(R \otimes E)} \). Now we may apply an immediate corollary of the

theorem of Skolem-Noether ([2], Th. 1, p. 110) and conclude that there exists

an inner automorphism \( h' \) generated by an element \( a' \) of \( H_2 \otimes E \) such that

\[ h: H_2 \otimes E \to h' \cdot H_2 \otimes E. \]

By construction \( h' h(a) = b \cdot a = a + \sum \nu_i c_i \),

i.e., modulo \( R \otimes E \) the inner automorphism \( h' \) induces the identity map.

Hence \( f(a') \) is contained in the center of \( A \). Thus \( g_2(f(a')) \) is in the center of

\( H_2 \otimes E \) and the inner automorphism \( h_a \) generated by \( a \) \( g_2(f(a'))^{-1} \)

agrees with \( h_a' \) on \( H_2 \otimes E \). Hence we still have \( h_a h_a(a) = b \), but also

\( f(a) = 1 \) which means \( a = 1 - r \) with \( r = \sum \{ r^k c_k : k \in f(k) \} \), \( f(k) \) finite and

\( r^k \in E \). If we define \( (1 - x) \cdot (1 - y) = x \cdot R \otimes E \) and also

satisfies the equations

\[(1 - x) a_i = b_i (1 - x).\]

Replacing \( r \) by \( \sum r^k c_k \) and \( b_i \) by \( a_i + \sum \nu_i c_i \) yields the equations

\[ \sum r^k (c_i a_i + a_i c_k + \sum \nu_i c_i c_k) = \sum \nu_i c_i. \]

These finitely many vector equations correspond to a finite system of linear

equations with coefficients in \( K \), which has a solution in \( E \), namely the \( r^k \)'s.

Thus by Lemma 13 there exists a solution \( n = \sum n^i c_i \in R \), satisfying the

equations \((*)\). This finishes the proof.

16 Theorem. Let \( A \) be a compact ring and suppose the radical in \( A \) splits.

Then any two complements of the radical in \( A \) are conjugate by an inner auto-
morphism generated by an element in the radical.

Proof. Let \( H_1, H_2 \) be two complements of \( R(A) \) and \( e_1, e_2 \) the identities

of \( H_1, H_2 \) respectively. By Corollary 4 applied to the sets \( e_1 \) and \( e_2 \) there

exists an inner automorphism generated by an element in the radical mapping

\( e_1 A e_2 \) onto \( e_2 A e_2 \). Taking this into account we may assume \( A \) has an identity.

But then \( A \) is totally disconnected ([10], Th. 1, p. 447) and hence has ideal

neighborhoods of zero. For compact rings with ideal neighborhoods of zero

the proof will be given together with the proof of the next theorem.

17 Theorem. Let \( A \) be a linearly compact ring with ideal neighborhoods of

zero which is an algebra over a field \( K \). Suppose \( A \) splits over its radical semi-
directly and the simple factors of \( A / R \) are finite dimensional, separable algebras

over \( K \). Then any two complements of the radical are conjugate under an inner

automorphism generated by an element of the radical.

Remark. In the light of ([9], 4.3 and Th. 16) one can formulate a similar

theorem for strictly linearly compact algebras (for definitions see [I], p. 246).

As this would be somewhat lengthy we shall not explicitly spell it out.
Proof. By ([11], Th. 12, 13, p. 258) $A/R(A)$ is isomorphic to a product of simple rings with identities $\{e_i : i \in I\}$. By ([12], 1.15, p. 297) the radical in $A$ is closed. Furthermore any complement of the radical actually is a complement of the radical in the category of topological rings ([8], [24]). By these remarks the hypotheses of Theorem 5 are satisfied and we may assume $A$ is a primary ring with identity. By ([7], [20], [21]) $A$ is a matrix ring over a completely primary ring $B$. Now we are ready to apply Theorem 1, which assures us that the Malcev theorem is true for $A$, whenever it is true for the ring $B$. In case $B$ is compact $B$ is an algebra over a field $GF(p)$ ([8], Th. 6) and any ring complement of the radical is a finite dimensional algebra over this field. With this and the hypothesis of Theorem 17 in mind, the Malcev theorem will be true for the ring $B$ by Theorem 14.

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