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Partial Sheffer Operations

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A partial operation f on a finite set A is called Sheffer if the partial algebra $\langle A, f\rangle$ is complete, i.e. if all partial operations on A are compositions of f (or definable in terms of f). In this paper we describe all partial Sheffer operations for $|A| = 2$ and all binary Sheffer operations for $|A| = 3$.

1. INTRODUCTION

Let A be a finite non-empty universe. For every positive integer n, an n-ary partial operation on A is a map $f: D_f \to A$ where $D_f \subseteq A^n$. Denote by $\mathbf{P}_A^{(n)}$ the set of all n-ary partial operations on A and put $\mathbf{P}_A := \bigcup_{n\geq 1} \mathbf{P}_A^{(n)}$. Next define on $\mathbf{P}_A$ a binary operation $*$, called superposition, as follows.

For $f \in \mathbf{P}_A^{(n)}$, $g \in \mathbf{P}_A^{(m)}$ put $t := m + n - 1$ and define $h = f * g \in \mathbf{P}_A^{(t)}$ by setting

$$D_h := \{(a_1, \ldots, a_t) \in A^t: (a_1, \ldots, a_m) \in D_g \quad \text{and} \quad (g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_t) \in D_f\}$$

and $h(x_1, \ldots, x_t) := f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_t)$ for all $(x_1, \ldots, x_t) \in D_h$, i.e. the left-hand side is defined whenever the right-hand side is defined.

We define three unary operations ζ, τ and Δ on $\mathbf{P}_A$ as follows. Let $n > 1$ and $f \in \mathbf{P}_A^{(n)}$. We define $\zeta(f) \in \mathbf{P}_A^{(n)}$, $\tau(f) \in \mathbf{P}_A^{(n)}$ and $\Delta(f) \in \mathbf{P}_A^{(n-1)}$ by setting

$$D_{\zeta(f)} = \{(a_1, a_2, \ldots, a_n): (a_2, \ldots, a_n, a_1) \in D_f\},$$
$$D_{\tau(f)} = \{(a_1, a_2, \ldots, a_n): (a_2, a_1, \ldots, a_n) \in D_f\},$$
$$D_{\Delta(f)} = \{(a_1, \ldots, a_{n-1}): (a_1, a_1, a_2, \ldots, a_{n-1}) \in D_f\}$$

and

$$\zeta(f)(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),$$
$$\tau(f)(x_1, \ldots, x_n) = f(x_2, x_1, \ldots, x_n),$$
$$\Delta(f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1}).$$

For $n = 1$ we put $\zeta(f) = \tau(f) = \Delta(f) = f$. For a positive integer n and $0 < i \leq n$, the ith *projection* (or trivial operation) e_i^n is the n-ary operation defined by setting $e_i^n(x_1, \ldots, x_n) = x_i$ for all $(x_1, \ldots, x_n) \in A^n$. Let $E = \{e_i^n: 0 < i \leq n < \aleph_0\}$ denote the set of all projections.

DEFINITION 1. A *partial clone* on A is a set of partial operations closed under $*$, ζ, τ, Δ and containing the set E (for an equivalent definition see [5]).

The set of partial clones on A, ordered by inclusion, forms an algebraic lattice L_A in which arbitrary infinimum is the (set-theoretical) intersection. For $F \subseteq \mathbf{P}_A$ the partial clone $\bar{F}$ *generated* by F is the least partial clone containing F.

DEFINITION 2. A set H of partial operations is *complete* (or $\langle A, H\rangle$ is primal) if H generates $\mathbf{P}_A$ (i.e. $\bar{H} = \mathbf{P}_A$).

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Partial completeness was first studied by Freivalds [2, 3] in the case $|A| = 2$. A general partial completeness criterion is based on the following concept:

DEFINITION 3. Let h and n be positive integers, ρ be an h-ary relation on A (i.e. $\rho \subseteq A^h$) and f be an n-ary partial operation. We say that *f preserves ρ* if for every $h \times n$ matrix $X = (X_{ij})$ the columns X_{*j} of which all belong to ρ ($j = 1, \ldots, n$) and the rows X_{i*} of which are all in D_f ($i = 1, \ldots, h$), we have $(f(X_{1*}), \ldots, f(X_{h*})) \in \rho$.

Let $\mathrm{Pol}\,\rho$ denote the set of all $f \in \mathbf{P}_A$ preserving ρ. It is well known and easy to verify that $\mathrm{Pol}\,\rho$ is a partial clone on A.

EXAMPLE 1. Let $h = 1$. A unary relation is a subset of A. We have

$$\mathrm{Pol}\{0\} = \{f \in \mathbf{P}_A\colon (0, \ldots, 0) \in D_f \Rightarrow f(0, \ldots, 0) = 0\}.$$

Note that trivially $f \in \mathrm{Pol}\{0\}$ whenever $(0, \ldots, 0) \notin D_f$.

EXAMPLE 2. $\mathrm{Pol}\{(0, 1)\}$ consists of all partial operations f such that $f(i, \ldots, i) = i$ ($i = 0, 1$) whenever $(0, \ldots, 0), (1, \ldots, 1) \in D_f$.

EXAMPLE 3. $\mathrm{Pol}\{(0, 1), (1, 0)\}$ is the set of partial self dual operations, i.e. of partial operations f satisfying

$$f(x_1', \ldots, x_n') = f(x_1, \ldots, x_n)'$$

whenever $(x_1, \ldots, x_n), (x_1', \ldots, x_n') \in D_f$.

DEFINITION 4. A *maximal partial clone* is a coatom (or dual atom) of the lattice L_A of partial clones on A, i.e. a proper partial clone C of $\mathbf{P}_A$ such that $C \subset D \subset \mathbf{P}_A$ for no partial clone D.

A lattice L with 1 is *coatomic* if to every element $x \in L\backslash\{1\}$ there is a coatom c such that $x \leq c$.

2. The Binary Case

For $A = \underset{\sim}{2} = \{0, 1\}$, we denote L_A and $\mathbf{P}_A$ by L_2 and $\mathbf{P}_2$. Put $0' := 1$ and $1' := 0$. We recall Freivald's Theorem:

THEOREM 5 [2]. *The lattice L_2 is coatomic and has exactly eight coatoms:*
(1) *The partial clone M_2 of all operations which are either everywhere defined or nowhere defined*;
(2) $\mathrm{Pol}\{0\}$;
(3) $\mathrm{Pol}\{1\}$;
(4) $\mathrm{Pol}\{(0, 1)\}$;
(5) $\mathrm{Pol}\{(0, 0), (0, 1), (1, 1)\}$;
(6) $\mathrm{Pol}\{(0, 1), (1, 0)\}$;
(7) $\mathrm{Pol}\,R_1$, *where*

$$R_1 = \{(x, x, y, y)\colon x, y \in \underset{\sim}{2}\} \cup \{(x, y, y, x)\colon x, y \in \underset{\sim}{2}\};$$

(8) $\mathrm{Pol}\,R_2$, *where*

$$R_2 = R_1 \cup \{(x, y, x, y)\colon x, y \in \underset{\sim}{2}\}.$$

Hence a subset F of $\mathbf{P}_2$ is complete iff F is a subset of none of the partial clones listed in (1)–(8).

DEFINITION 6. A partial operation $f \in \mathbf{P}_A$ is *Sheffer* for $\mathbf{P}_A$ if the singleton $\{f\}$ is complete, i.e. every partial operation on A is a composition of f's (may be obtained from f's and projections by finitely many applications of ζ, τ, Δ and $*$ e.g. as $\zeta(\zeta(\tau(f*(f*\zeta(f)))))$.

COROLLARY 7. *An n-ary partial operation f on* $\underset{\sim}{2}$ *with domain* D_f *is Sheffer for* $\mathbf{P}_2$ *iff* $\varnothing \neq D_f \neq \underset{\sim}{2}^n$ *and f does not preserve any of the relations defined in* (2), . . . , (8).

This may be improved:

COROLLARY 8. *Let* $n \geqslant 2$. *An n-ary partial operation f on* $\underset{\sim}{2}$ *is Sheffer for* $\mathbf{P}_2$ *iff*
(i) $\varnothing \neq D_f \neq \underset{\sim}{2}^n$ *and*
(ii) $(0, \ldots, 0), (1, \ldots, 1) \in D_f$ *and* $f(x, \ldots, x) = x'$ $(x = 0, 1)$ *and*
(iii) *there exists* $(a_1, \ldots, a_n) \in D_f$ *such that*

$$(a'_1, \ldots, a'_n) \in D_f \quad \text{and} \quad f(a'_1, \ldots, a'_n) = f(a_1, \ldots, a_n).$$

PROOF. ($\Rightarrow$) By Theorem 5 f does not belong to any of the maximal partial clones described in (1), (2), (3) and (6).

($\Leftarrow$) Let f be an n-ary partial operation on $\underset{\sim}{2}$ satisfying (i)–(iii). Then clearly f does not belong to any of the partial clones listed in (1)–(6) in Theorem 5.

Put $a := f(a_1, \ldots, a_n)$ and consider the $4 \times n$ matrix

$$X = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_n \\ 1 & 1 & \ldots & 1 \\ a'_1 & a'_2 & \ldots & a'_n \end{pmatrix}.$$

Note that $(0, x, 1, x') \in R_1$ for $x = 0, 1$; hence all the columns of X belong to $R_1 \subseteq R_2$. Applying f to the rows of X we obtain

$$(f(X_{1\cdot}), f(X_{2\cdot}), f(X_{3\cdot}), f(X_{4\cdot})) = (1, a, 0, a) \notin R_2.$$

Since all columns of X are in R_i while the row values are not in R_i $(i = 1, 2)$ we have $f \notin \operatorname{Pol} R_1 \cup \operatorname{Pol} R_2$ and therefore f is Sheffer for $\mathbf{P}_2$. □

REMARK 9 [2]. A Sheffer operation for $\mathbf{P}_2$ is of arity greater than 2.

PROOF. Let $n = 2$. Conditions (ii) and (iii) of Corollary 8 imply $D_f = \underset{\sim}{2}^2$; hence f being everywhere defined does not satisfy (i).

APPLICATION. Using Corollary 8 we may calculate the number of n-ary partial Sheffer operations for $\mathbf{P}_2$.

There are 3^{2^n-2} partial functions satisfying $f(x, \ldots, x) = x'$ for $x = 0, 1$ (condition (ii)). One should subtract from this number all the functions which do not satisfy condition (iii) of Corollary 8. These functions clearly satisfy the opposite property: $f(a'_1, \ldots, a'_n) = f(a_1, \ldots, a_n)$ (= either 0 or 1) for no $a_1, \ldots, a_n \in \underset{\sim}{2}$. There are $2^{n-1} - 1$ pairs $(f(a'_1, \ldots, a'_n), f(a_1, \ldots, a_n))$ (all except the pair $(0, \ldots, 0), (1, \ldots, 1)$). Now the 'acceptable' values for each such pair are $(0, 1)$, $(1, 0)$, $(0, *)$, $(*, 0)$, $(1, *)$, $(*, 1)$, $(*, *)$ (here $*$ means undefined) and 'non-acceptable' values are $(0, 0)$ and $(1, 1)$. Thus there are 7 possible values for each pair. Hence the number of n-ary partial Sheffer operations for $\mathbf{P}_2$ is

$$|F| = 3^{2^n-2} - 7^{2^{n-1}-1}.$$

Let $\mathbf{O}_2$ be the subset of $\mathbf{P}_2$ consisting of the everywhere defined operations on $\underline{2}$. We say that $f \in \mathbf{O}_2$ is Sheffer for $\mathbf{O}_2$ if $\{\bar{f}\} = \mathbf{O}_2$. It is known that (see [4] and [16, p. 159]) $f \in \mathbf{O}_2^{(n)}$ is Sheffer for $\mathbf{O}_2$ iff f satisfies (ii) and (iii), i.e.

$$f(0, \ldots, 0) = 1, \qquad f(1, \ldots, 1) = 0$$

and

$$f(a_1, \ldots, a_n) = f(a_1', \ldots, a_n') \qquad \text{for some } (a_1, \ldots, a_n) \in \underline{2}^n.$$

A similar but easier counting yields that the number of n-ary Sheffer operations for $\mathbf{O}_2$ is $2^{2^n-2} - 2^{2^{n-1}}$ (cf. [4]).

Since the (full) n-ary Sheffer operations for $\mathbf{O}_2$ belong to F and are not partial Sheffer operations for $\mathbf{P}_2$ (as $D_f = \underline{2}^n$), we see that the number S of n-ary partial Sheffer operations for $\mathbf{P}_2$ is

$$S = 3^{2^n-2} - 7^{2^{n-1}-1} - 2^{2^n-2} + 2^{2^{n-1}}.$$

Since $|\mathbf{P}_2^{(n)}| = 3^{2^n}$, we deduce that the probability for an n-ary partial operation on $\underline{2}$ to be Sheffer is

$$p = \frac{3^{2^n-2} - 7^{2^{n-1}} - 1 - 2^{2^n-2} + 2^{2^{n-1}}}{3^{2^n}}$$

which converges fast to 1/9.

Note that for a large n almost 25% of n-ary (full) operations are Sheffer for $\mathbf{O}_2$ ([4] and [16, p. 159]) and so there are relatively fewer n-ary Sheffer for $\mathbf{P}_2$ than Sheffer for $\mathbf{O}_2$.

3. The 3-Element Case

The general completeness criterion for $\mathbf{P}_3$ was found by Lau in 1977 [9] and by Romov in 1980 ([14], three maximal clones were inadvertently omitted, see also [10, p. 12]). This, as well as the above Freivald's criterion, are included in the partial completeness criterion for arbitrary finite A [5, 6]. Let $A = \underline{k} := \{0, \ldots, k-1\}$.

Let $h \geqslant 2$ and E_h be the set of all binary equivalence relations on $\underline{h} = \{0, \ldots, h-1\}$ (note that E_h is ordered by inclusion). We also denote an equivalence relation $\varepsilon \in E_h$ by $\{B_1, \ldots, B_l\}$, where $B_1, \ldots, B_l$ are the blocks (or the equivalence classes) of ε the cardinality of which is greater than 1, e.g. $\varepsilon = \{i, j\}$ means that ε has exactly one block $\{i, j\}$ and all the other blocks are singleton. Put $\omega_h := \{(i, i): i \in \underline{h}\}$. For each $\varepsilon \in E_h$ put

$$\Delta_\varepsilon := \{(x_0, \ldots, x_{h-1}) \in \underline{k}^h: (i, j) \in \varepsilon \Rightarrow x_i = x_j\}$$

(Δ_ε is said to be a *diagonal* relation).

Let S_h denote the symmetric group of all permutations of $\underline{h}$ and let ρ be an h-ary relation on $\underline{k}$ (i.e. $\rho \subseteq \underline{k}^h\}$. For $\pi \in S_h$ put

$$\rho^{(\pi)} := \{(x_{\pi(0)}, \ldots, x_{\pi(h-1)}) \in \underline{k}^h: (x_0, \ldots, x_{h-1}) \in \rho\}.$$

The relation ρ is *areflexive* [6] if $\rho \cap \Delta_\varepsilon = \varnothing$ for every $\varepsilon \in E_h \backslash \{\omega_h\}$. A subset $F \subseteq E_h$ is an *antichain* if the elements of F are pairwise incomparable (i.e. if $\varepsilon \subset \varepsilon'$ for no $\varepsilon, \varepsilon' \in F$). Let F be an antichain of E_h and ρ be an h-ary relation on $\underline{k}$ of the form $\rho = \sigma \cup (\bigcup_{\varepsilon \in F} \Delta_\varepsilon)$, where σ is an h-ary areflexive relation.

Put $G := \{\pi \in S_h: \sigma \cap \sigma^{(\pi)} \neq \varnothing\}$. The *model* of ρ is the h-ary relation

$$\mu_{GF} := \{(\pi(0), \ldots, \pi(h-1)): \pi \in G\} \cup \left\{\left(\bigcup_{\varepsilon \in F} \{(\alpha_0, \ldots, \alpha_{h-1}) \in \underline{h}^h: (i, j) \in \varepsilon \Rightarrow \alpha_i = \alpha_j\right)\right\},$$

on the set $\underline{h}$. For $\pi \in S_h$ and $\varepsilon \in E_h$ put $\varepsilon(\pi) := \{(\pi(i), \pi(j)) : (i, j) \in \varepsilon\}$ and $\pi(F) := \{\pi(\varepsilon) : \varepsilon \in F\}$. We assume that h, F and σ satisfy one of the 5 following conditions:
(1) $2 \leqslant h \leqslant k$, $F = \varnothing$ and $\sigma \neq \varnothing$;
(2) $2 \leqslant h \leqslant k$, $F = \{\varepsilon\}$, where $\varepsilon \neq \omega_h$ and $\sigma \neq \varnothing$;
(3) $h = 4$ and $F = \{\{0, 1\}, \{2, 3\}\} \cup \{\{0, 3\}, \{1, 2\}\}$;
(4) $h = 4$ and $F = \{\{0, 1\}, \{2, 3\}\} \cup \{\{0, 3\}, \{1, 2\}\} \cup \{\{0, 2\}, \{1, 3\}\}$;
(5) $h \neq 2$, $h < k$, $F = \bigcup_{0 \leqslant i < j \leqslant h-1} \{i, j\}$ and $\sigma \neq \underline{k}^h \backslash (\bigcup_{\varepsilon \in F} \Delta_\varepsilon)$.

The relation ρ is said to be *coherent* if:
(i) for all $\pi \in G$, we have $\sigma^{(\pi)} = \sigma$ and $\pi(F) = F$ for the first four cases and $G = S_h$ for the fifth;
(ii) for every non-empty subrelation σ' of σ, there exists a relational homomorphism $\psi : \underline{k} \to \underline{h}$ from σ' to μ_{GF} (i.e. for every $(x_0, \ldots, x_{h-1}) \in \sigma'$, we have

$$(\psi(x_0), \ldots, \psi(x_{h-1})) \in \mu_{GF})$$

such that

$$(\psi(i_0), \ldots, \psi(i_{h-1})) = (0, \ldots, h-1)$$

for at least one h-tuple $(i_0, \ldots, i_{h-1}) \in \sigma'$.

Note that, in cases 3 and 4, if $\sigma = \varnothing$ then ρ is automatically coherent, furthermore condition (i) in the fifth case implies the coherence. For $n > 0$ let p_n be the partial n-ary operation with $D_{p_n} = \varnothing$ (i.e. nowhere defined). Put $M_k := \mathbf{O}_k \cup \{p_n : 0 < n < \omega\}$. Now we can formulate the general completeness criterion:

THEOREM 10 [6]. *Let $k \geqslant 3$ be an integer. Every proper partial clone on $\underline{k}$ extends to a maximal one. Furthermore, a partial clone C on $\underline{k}$ is maximal iff either $C = M_k$ or $C = \operatorname{Pol} \rho$ for some coherent relation ρ on $\underline{k}$.*

COROLLARY 11 [6]. *A subset $H \subseteq \mathbf{P}_k$ is complete iff H has an n-ary partial operations g such that $\varnothing \neq D_g \neq \underline{k}^n$, and for every coherent relation ρ, the set H contains an operation not preserving ρ.*

COROLLARY 12 [9]. *The maximal partial clones on $\underline{3}$ are exactly M_3 and $\operatorname{Pol} \rho_1, \ldots, \operatorname{Pol} \rho_{57}$, where*

(1) $\rho_1 = \{(0, 1), (1, 0)\}$,
$\rho_2 = \{(1, 2), (2, 1)\}$,
$\rho_3 = \{(0, 2), (2, 0)\}$,
$\rho_4 = \{(0, 1), (1, 0), (1, 2), (2, 1)\}$,
$\rho_5 = \{(0, 1), (1, 0), (0, 2), (2, 0)\}$,
$\rho_6 = \{(0, 2), (2, 0), (1, 2), (2, 1)\}$,
$\rho_7 = \{(0, 1)\}$,
$\rho_8 = \{(0, 2)\}$,
$\rho_9 = \{(1, 2)\}$,
$\rho_{10} = \{(0, 1), (0, 2)\}$,
$\rho_{11} = \{(0, 1), (2, 1)\}$,
$\rho_{12} = \{(0, 2), (1, 2)\}$,
$\rho_{13} = \{(0, 1, 2)\}$,

$\rho_{14}=\{(0,1,2),(0,2,1)\}$,
$\rho_{15}=\{(0,1,2),(1,0,2)\}$,
$\rho_{16}=\{(0,1,2),(2,1,0)\}$,
$\rho_{17}=\{(0,1,2),(2,0,1),(1,2,0)\}$,
$\rho_{18}=\{(0,1,2),(2,0,1),(1,2,0),(0,2,1),(1,0,2),(2,1,0)\}$;

(2) $\rho_{19}=\omega_3\cup\{(0,1)\}$,
$\rho_{20}=\omega_3\cup\{(0,2)\}$,
$\rho_{21}=\omega_3\cup\{(1,2)\}$,
$\rho_{22}=\omega_3\cup\{(0,1),(0,2)\}$,
$\rho_{23}=\omega_3\cup\{(0,1),(2,0)\}$,
$\rho_{24}=\omega_3\cup\{(0,1),(1,2)\}$,
$\rho_{25}=\omega_3\cup\{(0,1),(2,1)\}$,
$\rho_{26}=\omega_3\cup\{(0,2),(1,2)\}$,
$\rho_{27}=\omega_3\cup\{(0,2),(2,1)\}$,
$\rho_{28}=\omega_3\cup\{(0,1),(0,2),(1,2)\}$,
$\rho_{29}=\omega_3\cup\{(0,1),(0,2),(2,1)\}$,
$\rho_{30}=\omega_3\cup\{(0,1),(2,0),(2,1)\}$,
$\rho_{31}=\omega_3\cup\{(0,1),(1,0)\}$,
$\rho_{32}=\omega_3\cup\{(0,2),(2,0)\}$,
$\rho_{33}=\omega_3\cup\{(1,2),(2,1)\}$,
$\rho_{34}=\omega_3\cup\{(0,1),(1,0),(0,2),(2,0)\}$,
$\rho_{35}=\omega_3\cup\{(0,1),(1,0),(1,2),(2,1)\}$,
$\rho_{36}=\omega_3\cup\{(0,2),(2,0),(1,2),(2,1)\}$,
$\rho_{37}=\{(0,1,2)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{38}=\{(0,1,2),(0,2,1)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{39}=\{(0,1,2),(1,0,2)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{40}=\{(0,1,2),(2,1,0)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{41}=\{(0,1,2),(1,2,0),(2,0,1)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{42}=\{(0,1,2),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(2,1,0)\}\cup\{(x,x,x): x\in\underset{\sim}{3}\}$,
$\rho_{43}=\{(0,1,2)\}\cup\{(x,x,y): x,y\in\underset{\sim}{3}\}$,
$\rho_{44}=\{(0,1,2),(1,0,2)\}\cup\{(x,x,y): x,y\in\underset{\sim}{3}\}$,
$\rho_{45}=\{(0,1,2)\}\cup\{(x,y,y): x,y\in\underset{\sim}{3}\}$,
$\rho_{46}=\{(0,1,2),(0,2,1)\}\cup\{(x,y,y): x,y\in\underset{\sim}{3}\}$,
$\rho_{47}=\{(0,1,2)\}\cup\{(x,y,x): x,y\in\underset{\sim}{3}\}$,
$\rho_{48}=\{(0,1,2),(2,1,0)\}\cup\{(x,y,x): x,y\in\underset{\sim}{3}\}$,
$\rho_{49}=\{(x,x,y,y): x,y\in\underset{\sim}{3}\}\cup\{(x,y,y,x): x,y\in\underset{\sim}{3}\}$,
$\rho_{50}=\{(x,x,y,y): x,y\in\underset{\sim}{3}\}\cup\{(x,y,y,x\}: x,y\in\underset{\sim}{3}\}\cup\{(x,y,x,y): x,y\in\underset{\sim}{3}\}$,
$\rho_{51}=\{0\}$,

$\rho_{52} = \{1\}$,
$\rho_{53} = \{2\}$,
$\rho_{54} = \{0, 1\}$,
$\rho_{55} = \{0, 2\}$,
$\rho_{56} = \{1, 2\}$,
$\rho_{57} = \{(x, x, y): x, y \in \underset{\sim}{3}\} \cup \{(x, y, x): x, y \in \underset{\sim}{3}\} \cup \{(x, y, y): x, y \in \underset{\sim}{3}\}$.

Note that 54 of these relations are cited in [14].

COROLLARY 13. *An n-ary partial operation f on $\underset{\sim}{3}$ is Sheffer iff $\varnothing \neq D_f \neq \underset{\sim}{3}^n$ and f does not preserve any relation ρ_i for $i = 1, \ldots, 57$.*

Note that an example of a partial Sheffer operation for $\mathbf{P}_3$ is given in [14]. We need the following general remark.

REMARK. 14. Let s be a permutation on A and $f \in \mathbf{P}_A^{(n)}$ with domain D_f. Define $g := f^{(s)}$ by setting

$$D_g := s(D_f) = \{(s(x_1), \ldots, s(x_n)): (x_1, \ldots, x_n) \in D_f\}$$

and

$$g(x_1, \ldots, x_n) = s(f(s^{-1}(x_1), \ldots, s^{-1}(x_n))) \qquad \text{for all } (x_1, \ldots, x_n) \in D_g.$$

Clearly, $g \in \mathbf{P}_A^{(n)}$ and the map $\phi(f) = f^{(s)}$ is a self-map of $\mathbf{P}_A$. It is easy to verify that ϕ is an automorphism of $\langle P_A, *, \zeta, \tau, \Delta, e_1^2\rangle$ (i.e. $(f*g)^{(s)} = f^{(s)} * g^{(s)}$, $(\zeta(f))^{(s)} = \zeta(f^{(s)})$, $(\Delta(f))^{(s)} = \Delta(f^{(s)})$ and $(e_1^2)^{(s)} = e_1^2$ hold for all $f, g \in \mathbf{P}_A$). It follows that f is Sheffer for $\mathbf{P}_A$ iff $f^{(s)}$ is Sheffer.

Suppose an n-ary f is Sheffer for $\mathbf{P}_3$. Then $f \notin \operatorname{Pol}\{0\} \cup \operatorname{Pol}\{1\} \cup \operatorname{Pol}\{2\}$ and so $\{x, x, \ldots, x\} \in D_f$ and $f^+(x) = f(x, \ldots, x) \neq x$ for all $x \in \underset{\sim}{3}$. We have two cases: (1) f^+ is not injective; and (2) f^+ is injective. First we study case (1). We can write $\underset{\sim}{3} = \{a, b, c\}$, where $f^+(a) = f^+(b) = c$ and $f^+(c) = b$. Put $s(a) := 0$, $s(b) := 1$ and $s(c) := 2$. Then $g := f^{(s)}$ satisfies $g(0, 0, \ldots, 0) = s(f^+(s^{-1}(0))) = s(f^+(a)) = s(c) = 2$. Similarly, $g(1, \ldots, 1) = 2$ and $g(2, \ldots, 2) = 1$. Thus, without loss of generality, we may assume that f already satisfies

$$f(0, 0, \ldots, 0) = f(1, 1, \ldots, 1) = 2 \qquad \text{and} \qquad f(2, 2, \ldots, 2) = 1. \tag{1}$$

Note that from (1) it follows that $f \notin \operatorname{Pol}\{0, 1\} \cup \operatorname{Pol}\{0, 2\}$. Since $f \notin \operatorname{Pol}\{1, 2\}$ we have

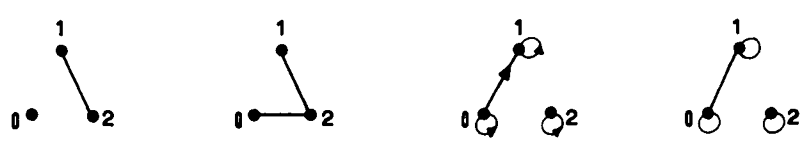
$$f(\underset{\sim}{a}) = 0 \qquad \text{for some } \underset{\sim}{a} = (a_1, \ldots, a_n) \in D_f \cap \{1, 2\}^n. \tag{2}$$

Let us recall that

$$\rho_2 = \{(1, 2), (2, 1)\}, \qquad \rho_6 = \{(0, 2), (2, 0), (1, 2), (2, 1)\},$$

$$\rho_{19} = \omega_3 \cup \{(0, 1)\}, \qquad \rho_{31} = \omega_3 \cup \{(0, 1), (1, 0)\}.$$

Their graphs are respectively:

We have: 

LEMMA 15. *Let $f \in P_A^{(n)}$ be such that $0 \neq D_f \subset \underset{\sim}{3}^n$ and f satisfies* (1) *and* (2). *Then f is Sheffer for* $\mathbf{P}_3$ *iff* $f \notin \operatorname{Pol} \rho_2 \cup \operatorname{Pol} \rho_6 \cup \operatorname{Pol} \rho_{19} \cup \operatorname{Pol} \rho_{31}$.

PROOF. Necessity is obvious. For sufficiency we prove that $f \notin \operatorname{Pol} \rho_i$ for $i \in \{1, \ldots, 57\} \backslash \{2, 6, 19, 31\}$.
(1) $i \in \{1, 4, 5, 7, 10, 11\}$, $(0, 1) \in \rho_i$, $(2, 2) \notin \rho_i$ and $f(0, \ldots, 0) = f(1, \ldots, 1) = 2$.
(2) $i \in \{3, 8, 12, 20, 22, 26, 28, 32, 34\}$. Use $(0, 2) \in \rho_i$, $(2, 1) \notin \rho_i$ and (1).
(3) $i = 23$. Use $(2, 0) \in \rho_i$, $(1, 2) \notin \rho_i$ and (1).
(4) $i \in \{9, 21, 24\}$. Use $(1, 2) \in \rho_i$, $(2, 1) \notin \rho_i$ and (1).
(5) $i \in \{13, \ldots, 18, 45, \ldots, 48\}$. Use $(0, 1, 2) \in \rho_i$ $(2, 2, 1) \notin \rho_i$ and (1).
(6) $i \in \{25, 27, 29, 30\}$. Use $(2, 1) \in \rho_i$, $(1, 2) \notin \rho_i$ and (1).
(7) $i \in \{33, 35\}$. Note that the elements a_j from (2) satisfy $(a_j, 1) \in \rho_i$ $(j = 1, \ldots, n)$, while $(f(\underline{a}), f(1, \ldots, 1)) = (0, 2) \notin \rho_i$.
(8) $i = 36$. Similarly, $(a_j, 2) \in \rho_{36}$ for $j = 1, \ldots, n$, while $(f(\underline{a}), f(2, \ldots, 2)) = (0, 1) \notin \rho_{36}$.
(9) $i \in \{37, \ldots, 42\}$. Use $(0, 1, 2) \in \rho_i$, $(2, 2, 1) \notin \rho_i$ and (1).
(10) $i = 43$. Since $f \notin \operatorname{Pol} \rho_{19}$ there are $\underline{b} = (b_1, \ldots, b_n) \in D_f$, $\underline{c} = (c_1, \ldots, c_n) \in D_f$ such that $(b_j, c_j) \in \rho_{19}$ while $(f(\underline{b}), f(\underline{c})) \notin \rho_{19}$. Note that $(b_j, c_j, 2) \in \rho_{43}$ $(j = 1, \ldots, n)$ while $(f(\underline{b}), f(\underline{c}), f(2, \ldots, 2)) = (f(\underline{b}), f(\underline{c}), 1) \notin \rho_{43}$.
(11) $i = 44$. Since $f \notin \operatorname{Pol} \rho_{31}$, there are $\underline{d} = (d_1, \ldots, d_n) \in D_f$, $\underline{e} = (e_1, \ldots, e_n) \in D_f$ such that $(d_j, e_j) \in \rho_{31}$ for $j = 1, \ldots, n$, while $(f(\underline{d}), f(\underline{e})) \notin \rho_{31}$. Now $(d_j, e_j, 2) \in \rho_{43}$ for $j = 1, \ldots, n$ while $(f(\underline{d}), f(\underline{e}), 1) \notin \rho_{43}$.
(12) $i \in \{49, 50\}$. Since $f \notin \operatorname{Pol} \rho_2$ there are $\mathbf{g} = (g_1, \ldots, g_n) \in D_f$ and $\mathbf{h} = (h_1, \ldots, h_n) \in D_f$ such that $(g_j, h_j) \in \rho_2$ for $j = 1, \ldots, n$ while $(f(\underline{g}), f(\underline{h})) \notin \rho_2$. Note that $(1, g_j, 2, h_j) \in \rho_i$ for $j = 1, \ldots, n$ while $(2, f(\underline{g}), 1, f(\underline{h})) \notin \rho_i$.
(13) $i \in \{51, \ldots, 56\}$. See comments before the lemma.
(14) $i = 57$. As $(a_j, 1, 2) \in \rho_{57}$ for $j = 1, \ldots, n$ while $(0, 2, 1) \notin \rho_{57}$. Therefore $f \notin \operatorname{Pol} \rho_i$ for $i = 1, \ldots, 57$ and thus is Sheffer for $\mathbf{P}_3$. □

We can say more for $n = 2$.

DEFINITION 16. A partial operation $f \in \mathbf{P}_A^{(n)}$ is said to have *injective diagonal* if $(x, \ldots, x) \in D_f$ for all $x \in A$ and $f^+(x) = f(x, \ldots, x) (x \in A)$ is a permutation of A.

COROLLARY 17. *Let h be a binary partial operation on $\underline{3}$ with non-injective diagonal. Then h is Sheffer for $\mathbf{P}_3$ iff $h = f^{(s)}$ for some $s \in S_3$ and some $f \in \mathbf{P}_3^{(s)}$ such that:*
(1) $\varnothing \neq D_f \neq \underline{3}^2$;
(2) $f(0, 0) = f(1, 1) = 2$, $f(2, 2) = 1$;
(3) $(1, 2), (2, 1) \in D_f$ and for $a := f(1, 2)$, $b := f(2, 1)$;

either $g := f$ or $g := \tau f$ satisfies one of the following conditions:

(α) $(a, b) \notin \{(1, 2), (2, 1)\}$ and $(0, 1) \in D_g$, $g(0, 1) \neq 2$;
(β) $a = b \neq 2$ and $(0, 2) \in D_g$, $g(0, 2) = 2$;
(γ) $a = b = 2$ and $(0, 2) \in D_g$, $g(0, 2) \neq 2$;
(δ) $\{a, b\} = \{0, 2\}$, $g(1, 2) = 0$, $g(2, 1) = 2$ and either $(0, 2) \in D_g$, $g(0, 2) \in D_g$, $g(0, 2) = 2$ or $(2, 0) \in D_g$, $g(2, 0) \neq 2$;
(ε) $\{a, b\} = \{0, 1\}$, $(0, 2) \in D_g$, $g(0, 2) = 2$.

PROOF. Necessity: we know that $f \notin \operatorname{Pol} \rho_2 \cup \operatorname{Pol} \rho_6 \cup \operatorname{Pol} \rho_{19} \cup \operatorname{Pol} \rho_{31}$.
From $f \notin \operatorname{Pol} \rho_2$ and $f(1, 1) = 2$, $f(2, 2) = 1$, the only possibility is $(1, 2), (2, 1) \in D_f$ and $(a, b) \notin \{(1, 2), (2, 1)\}$. Now $g \notin \operatorname{Pol} \rho_6$. If $(0, 1) \in D_g$ and $g(0, 1) \neq 2$, then condition (α) holds. Otherwise from $f \notin \operatorname{Pol} \rho_6 \cup \operatorname{Pol} \rho_{19} \cup \operatorname{Pol} \rho_{31}$ and depending on the values of a and b, one of the conditions in (β), (γ), (δ) and (ε) occurs.

Sufficiency: conditions (1)–(3) imply that $f \notin \mathrm{Pol}\,\rho_2 \cup \mathrm{Pol}\,\rho_6 \cup \mathrm{Pol}\,\rho_{19} \cup \mathrm{Pol}\,\rho_{31}$ and by Remark 14 and Lemma 15 f is Sheffer for $\mathbf{P}_3$. □

Let us look at the case in which f has an injective diagonal.

Define $c \in S_3$ by $c(0)=1$, $c(1)=2$ and $c(2)=0$.

PROPOSITION 18. *Let $f \in \mathbf{P}_3^{(n)}$ have injective diagonal and let $\varnothing \neq D_f \subset \underline{3}^n$. Then f is Sheffer for $\mathbf{P}_3$ iff there is $i \in \{1, 2\}$ such that $f(x, \ldots, x) = c^i(x)$ for all $x \in \underline{3}$ and f preserves none of $\{\rho_{17}, \rho_{18}, \rho_{41}, \rho_{42}, \rho_{49}, \rho_{50}, \rho_{57}\}$.*

PROOF. Without loss of generality we may assume that $f(x, \ldots, x) = c(x)$ for all $x \in \underline{3}$.

For $\rho \subseteq \underline{3}^h$ put $c(\rho) := \{(c(a_0), \ldots, c(a_{h-1})) : (a_0, \ldots, a_{h-1}) \in \rho\}$. It is easy to see that $f \notin \mathrm{Pol}\,\rho$ if $c(\rho) \nsubseteq \rho$. A direct check shows that, exactly, the relations listed satisfy $c(\rho) \subseteq \rho$. □

REMARK. Suppose $f \notin \mathrm{Pol}\,\rho_{17}$. Then there exist $i_1, \ldots, i_n \in \underline{3}$ such that $(c^{i_1}(x), \ldots, c^{i_n}(x)) \in D_f$ for $x = 0, 1, 2$, while $h(x) := f(c^{i_1}(x), \ldots, c^{i_n}(x))$ $(x \in \underline{3})$ satisfies $(h(0), h(1), h(2)) \notin \rho_{17}$. If $h(0) = h(1) = h(2)$ then clearly also $f \notin \mathrm{Pol}\,\rho_{18}$. If $|\mathrm{im}\,h| = 2$ then $f \notin \mathrm{Pol}\,\rho_{18} \cup \mathrm{Pol}\,\rho_{41} \cup \mathrm{Pol}\,\rho_{42}$. If h is a permutation then in view of $h \notin \{c^0, c^1, c^2\}$ we have $f \notin \mathrm{Pol}\,\rho_{41}$.

We can say more if f is a binary operation.

COROLLARY 19. *Let $f \in \mathbf{P}_3^{(2)}$ have injective diagonal and let $\varnothing \neq D_f \subset \underline{3}^2$. Then f is Sheffer for $\mathbf{P}_3$ iff there is $i \in \{1, 2\}$ such that $f(x, x) = c^i(x)$ for all $x \in \underline{3}$ and f preserves neither ρ_{17} nor ρ_{49}.*

PROOF. Sufficiency: we have to show that f preserves none of $\{\rho_{18}, \rho_{41}, \rho_{42}, \rho_{50}, \rho_{57}\}$. Without loss of generality assume that $g := f$ or $g := \tau f$ (where $\tau f(x, y) = f(y, x)$) satisfies $g(x, x) = c^2(x)$ $(= x + 2 \bmod (3))$ for all $x \in \underline{3}$ and $X := \{(0, 1), (1, 2), (2, 0)\} \subseteq D_g$ is such that $(x_1, x_2, x_3) := (g(0, 1), g(1, 2), g(2, 0)) \notin \rho_{17}$. Since $g \notin \mathrm{Pol}\,\rho_{49}$ there are $a, b, c, d \in \underline{3}$ such that

$$Y := \{(a, c), (a, d), (b, c), (b, d)\} \subseteq D_g \quad \text{and} \quad (g(a, c), g(a, d), g(b, c), g(b, d)) \notin \rho_{49}. \tag{3}$$

In view of $|D_g| \leq 8$ we have $X \cap Y \neq \varnothing$. Note that the cases $(0, 1) \in Y$, $(1, 2) \in Y$, $(2, 0) \in Y$ are similar. Thus we assume $(0, 1) \in Y$. We can arrange the elements of Y so that $(0, 1) = (a, c)$. In view of (3) we have $b \neq 0$ and $d \neq 1$. Therefore the matrix

$$A := \begin{pmatrix} a & c \\ a & d \\ b & c \\ b & d \end{pmatrix} \in \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 2 & 1 \\ 2 & 2 \end{pmatrix} \right\}.$$

Case I. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$(g(0, 1), g(0, 0), g(1, 1), g(1, 0)) \notin \rho_{49}. \tag{4}$$

(a) If $(x_1, \alpha, 1) := (g(0, 1), g(1, 0), g(2, 2)) \in \rho_{18}$, then $(x_1, \alpha) \in \{(0, 2), (2, 0)\}$ which contradicts (4); therefore $g \notin \mathrm{Pol}\, \rho_{18}$.

(b) Assume that $(x_1, x_2, x_3) = (g(0, 1), g(1, 2), g(2, 0)) \in \rho_{41} \backslash \rho_{17}$ (hence $x_1 = x_2 = x_3$) and take the two matrices

$$B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the columns of which belong to ρ_{41}. Since $g(1, 2) = g(2, 0)$ and $g(1, 1) \neq g(0, 0)$, at least one of $\mathbf{b} := (g(1, 2), g(1, 0), g(1, 1))$ or $\mathbf{c} := (g(2, 0), g(0, 0), g(1, 0))$, say $\mathbf{b}$, does not belong to the diagonal (i.e. is not equal to (x, x, x) for some $\mathbf{x} \in \underset{\sim}{3}$). Let us assume that $\mathbf{b} \in \rho_{41}$; then $\mathbf{b} = (g(1, 2), g(1, 0), 0) \in \rho_{17}$. Thus $x_2 = g(1, 2) = 2$ and $g(1, 0) = 0$. Now $g(2, 0) = x_3 = x_2$; hence $g(2, 0) = 2$ and $\mathbf{c} = (g(2, 0), g(0, 0), g(1, 0)) = (2, 2, 1) \notin \rho_{41}$. The proof is the same if we assume that $\mathbf{c} \in \rho_{17}$. Therefore $g \notin \mathrm{Pol}\, \rho_{41}$.

(c) We consider the matrix

$$C_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix},$$

the columns of which belong to $\rho_{42} = \rho_{18} \cup \{(x, x, x) : x \in \underset{\sim}{3}\}$ and assume that $(x_1, \alpha, 1) = (g(0, 1), g(1, 0), g(2, 2)) \in \rho_{42}$. In view of what was shown in (a) we have $g(0, 1) = g(1, 0) = g(2, 2) = 1$.

Note that the columns of the matrices B and C belong to ρ_{42}. If $x_3 = g(2, 0) \neq 0$, then $(g(2, 0), g(0, 0), g(1, 0)) = (g(2, 0), 2, 1) \notin \rho_{42}$ and the matrix C shows that $f \notin \mathrm{Pol}\, \rho_{42}$. Otherwise, let $g(2, 0) = 0$. In view of $(g(0, 1), g(1, 2), g(2, 0)) \notin \rho_{17}$ we have $g(1, 2) \neq 2$. Therefore $(g(1, 2), g(1, 0), g(1, 1)) = (g(1, 2), 1, 0) \notin \rho_{42}$ and the matrix B shows that $g \notin \mathrm{Pol}\, \rho_{42}$.

(d) We have shown in (a) that $g(0, 1) = 2$ implies $g(1, 0) \neq 0$. Thus the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

shows that $g \notin \mathrm{Pol}\, \rho_{50}$.

(e) If $1 \in \{g(0, 1), g(1, 0)\}$ then the first (or last) three rows of the matrix A show that $g \notin \mathrm{Pol}\, \rho_{57}$. Thus let us assume that $g(0, 1) \neq 1 \neq g(1, 0)$. Here, the only possibilities are $g(0, 1) = g(1, 0) = 0$ or 2.

Let $g(0, 1) = g(1, 0) = 0$ and consider the three matrices

$$X_0 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 2 & 2 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \\ 1 & 0 \end{pmatrix},$$

the columns of which belong to ρ_{57}. Let $t := g(2, 0) \in \underset{\sim}{3}$. Then the matrix A_t shows that $g \notin \mathrm{Pol}\, \rho_{57}$. Similarly, if $g(0, 1) = g(1, 0) = 2$ then, depending on the value of $g(1, 2)$, we consider one of the three matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$$

to show that $g \notin \operatorname{Pol} \rho_{57}$. This completes the proof for the case I.

Case II. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$(g(0, 1), g(0, 2), g(1, 1), g(1, 2)) \notin \rho_{49}. \tag{5}$$

(f) Assume that $(x_1, x_2, x_3) \in \rho_{18}$ (otherwise $g \notin \operatorname{Pol} \rho_{18}$) and moreover that $(\beta, x_3, 0) := (g(0, 2), g(2, 0), g(1, 1)) \in \rho_{18}$. Therefore one of the following holds:

$$-\beta = g(0, 2) = 1 \quad \text{and} \quad x_3 = g(2, 0) = 2$$

or

$$-\beta = 2 \quad \text{and} \quad x_3 = 1.$$

(1) Let $g(0, 2) = 1$ and $x_3 = g(2, 0) = 2$. Since $(x_1, x_2, x_3) \notin \rho_{17}$ we have $x_1 = g(0, 1) = 1$ and $x_2 = g(1, 2) = 0$. Thus $(g(0, 1), g(0, 2), g(1, 1), g(1, 2)) = (1, 1, 0, 0) \in \rho_{49}$, which is a contradiction.

(2) Let $g(0, 2) = 2$ and $g(2, 0) = 0$. As in (1) we have $g(0, 1) = 0$ and $g(1, 2) = 2$, which leads to the contradiction $(g(0, 1), g(0, 2), g(1, 1), g(1, 2)) = (0, 2, 0, 2) \in \rho_{49}$. Hence $(\beta, x_3, 0) \notin \rho_{18}$, proving that $g \notin \operatorname{Pol} \rho_{18}$.

(*g*) The proof of $g \notin \operatorname{Pol} \rho_{41}$ is similar to the one given in (b). Here we use the two matrices

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

(h) Let us assume that $(x_1, x_2, x_3) \in \rho_{42} \setminus \rho_{17}$. If $x_1 = x_2 = x_3$, (hence $g(0, 1) = g(1, 2)$) then, applying g on the rows of the two matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 2 \end{pmatrix},$$

we deduce that $g \notin \operatorname{Pol} \rho_{42}$. Therefore assume $(x_1, x_2, x_3) \in \rho_{18}$. Moreover, suppose that $(g(0, 2), g(2, 0), g(1, 1)) \in \rho_{42}$. In view of what was shown in (f) we have $g(0, 2) = g(2, 0) = g(1, 1) = 0$. We claim that $(g(0, 0), g(0, 1), g(0, 2)) = (2, g(0, 1), 0) \notin \rho_{42}$; for otherwise $g(0, 1) = 1$ and since $x_1 = g(0, 1)$, $x_2 = g(1, 2)$ and $x_3 = g(2, 0)$ are pairwise distinct we have $g(1, 2) = 2$, which implies the contradiction $(x_1, x_2, x_3) = (1, 2, 0) \in \rho_{17}$. This proves that $g \notin \operatorname{Pol} \rho_{42}$.

(i) Assume that $(g(0, 1), g(0, 2), g(1, 1), g(1, 2)) \in \rho_{50}$. In view of (5) we have

$g(0,2)=g(1,1)=0$ (and $g(0,1)=g(1,2)$). Now the matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 2 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

shows that $g \notin \operatorname{Pol} \rho_{50}$, since $g(0,2)$, $g(2,2)$ and $g(0,0)$ are pairwise distinct.

(j) We show that $g \notin \operatorname{Pol} \rho_{57}$. Note that if $(g(0,1), g(0,2), g(1,1), g(1,2)) \in \rho_{50}$, then in view of what was shown in (i) the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$$

shows the required result. Thus assume $(g(0,1), g(0,2), g(1,1), g(1,2)) \notin \rho_{50}$. If $|\{g(0,1), g(0,2), g(1,1), g(1,2)\}|=3$, we consider the 3×2 submatrix A' of A such that $|\{g(A'_{1\cdot}), g(A'_{2\cdot}), g(A'_{3\cdot})\}|=3$, which proves that $g \notin \operatorname{Pol} \rho_{57}$. Therefore assume that exactly 3 row values of g on A are equal.

(1) Let $g(0,2)=0$. In this case we consider the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 2 \end{pmatrix}$$

where $(g(0,0), g(0,2) g(2,2)) = (2,0,1) \notin \rho_{57}$.

(2) Let $g(0,2)\neq 0$. Then either $g(0,1)=g(0,2)=g(1,2)$ or $g(0,1)=g(1,1)=g(1,2)=0$. If the first case occurs then, depending on the common value of $g(0,1)$ and $g(1,2)$, we take one of the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

We do the same in the second case with the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}.$$

Finally, since the cases

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 2 \\ 2 & 1 \\ 2 & 2 \end{pmatrix}$$

are similar to case II we omit the proof for such cases. We have shown that g does not preserve any of $\{\rho_{18}, \rho_{41}, \rho_{42}, \rho_{50}, \rho_{57}\}$. Combining this with our assumption and Proposition 18, we deduce that g is Sheffer for $\mathbf{P}_3$. $\square$

Remark 20. The (full) Sheffer operations for $\mathbf{O}_k$, where $k \geq 3$, are completely described in [17]. This description is based on the well known Rosenberg's classification of the (full) maximal clones on $\underset{\sim}{k}$ [15, 16]. The present paper shows that except for $|A|=2$ the partial case is much more complex than the full one. This is not surprising since the number of maximal partial clones on a finite universe $\underline{k}$ exceeds greatly the number of maximal (full) clones [5, 6]. On the other hand, let us mention that if ρ is a given areflexive totally symmetric relation on $\underline{k}$ which is at least ternary, then the problem of deciding whether Pol ρ is a maximal partial clone is known to be NP complete; similar problems for other cases are open (see [1], [5], [6]).

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