Effective metric spaces and representations of the reals

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Abstract

Based on standard notions of classical recursion theory, a natural model of approximate computability for partial functions between effective metric spaces is presented. It generalizes the Ko-Friedman approach to computability of real functions by means of oracle Turing machines, follows the main ideas of Weihrauch’s type 2 theory of effectivity, but it avoids the explicit use of representations. The topological arithmetical hierarchy is introduced and shown to be strict if the underlying space contains an effectively discrete sequence. The domains of computable functions are exactly the $\Pi^0_2$-sets of this hierarchy if the space admits a finitary stratification. Finally, this framework is used to investigate and characterize the standard representations of the real numbers. They are just those functions from the name space onto the reals which have both computable extensions and inversions that are computable as relations.

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1. Introduction

The aim of this paper is to present a natural, handy framework of the theory of approximate computability for partial functions between effective metric spaces and apply it to investigate representations of the real numbers. Our approach is machine-oriented, it is based on the function-oracle Turing machines in the sense of Ko and Friedman. The points of a metric space have to be approximated by fast converging Cauchy sequences of elements from an effectively given dense subset, the so-called skeleton. Nevertheless, we avoid the explicit use of representations. On the one hand,
this simplifies the model, which could be called a pocket model of approximate computability.

On the other hand, the representation-free treatment of computability enables us to investigate the representations of the reals, i.e., surjective partial functions from the name space \( \mathbb{N}^\mathbb{N} \) or \( \{0,1\}^\mathbb{N} \) onto \( \mathbb{R} \), from the point of view of computability. So we can show that the standard representations are just those functions which have both computable extensions and computable inversions. The latter are right-inverse relations computable in a suitably generalized sense.

As a useful tool in dealing with computability over metric spaces, the topological arithmetical hierarchy is introduced and shown to be strict under rather weak suppositions. The domains of approximately computable functions are exactly the \( \Pi^0_2 \)-sets within this hierarchy, at least if the related space admits a finitary stratification.

More precisely, Section 2 of this paper presents the basic concepts of approximate computability between metric spaces. Section 3 deals with the topological arithmetical hierarchy which is used in Section 4 to characterize the domains of computable functions. Finally, Section 5 is devoted to the treatment of standard representations of the real numbers. To improve the readability, notions and basic techniques of classical recursion theory are applied in a rather informal manner. Thus, computability and recursive enumerability over discrete object domains, like \( \mathbb{N} \), \( \mathbb{Z} \), or \( \mathbb{N}^* \), are understood in their standard meaning, and the use of the smn-theorem involved in some proofs is not explicitly demonstrated.

In recent years, various successful attempts have been made to a theory of computability over effective metric spaces or, more generally, to computable analysis. For more details, the reader is referred to [12–14, 6, 1, 7, 10, 2]. The present paper tries to give a fairly simple, natural approach. It is essentially based on main ideas of type 2 theory of effectivity developed by Weihrauch et al., cf. [12–14]. The usual explicit treatment of several representations in order to reduce computability over abstract spaces to computability on the name spaces of representations, however, is avoided by fixing implicitly the special normed Cauchy representation on the underlying space. Nevertheless, the approach presented here includes genuinely partial functions to a considerable extend. Only on this basis and because of the representation-free definition of computability, the new results on standard representations of the reals can be obtained.

2. Effective metric spaces and approximate computability

By an effective metric space (briefly: EMS), we understand a triple \( X = (X, d, S) \), where

- \( (X, d) \) is a complete metric space and
- \( S = (s_n)_{n \in \mathbb{N}} \) is a sequence of elements from \( X \) such that
  - the range, \( \text{ran}(S) = \{s_n : n \in \mathbb{N}\} \), is dense in \( (X, d) \) and
  - the set \( D_\epsilon = \{\langle m, n, k \rangle : d(s_m, s_n) < \epsilon_k\} \) is r.e. (recursively enumerable).
If, moreover,
  o the set $D_\alpha = \{ (m,n,k) : d(s_m,s_n) > q_k \}$ is r.e.,
the triple $\mathcal{X}$ is said to be a strongly effective metric space (briefly: SEMS).

Here $\langle n_1, \ldots, n_k \rangle$ is the Cantor number of a $k$-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$. By $\pi^k_x$, $1 \leq \alpha \leq k$, we shall denote the corresponding projection mappings, i.e., $\langle \pi^1_x(n), \ldots, \pi^k_x(n) \rangle = n$, for all $n \in \mathbb{N}$. Let $v_\mathbb{Q} = (q_k)_{k \in \mathbb{N}}$ be a fixed (total) standard numbering of the set $\mathbb{Q}$ of all rational numbers.

More precisely, $v_\mathbb{Q} : \mathbb{N} \rightarrow \mathbb{Q}$ is a mapping of $\mathbb{N}$ onto $\mathbb{Q}$, and $S : \mathbb{N} \rightarrow X$ is a mapping of $\mathbb{N}$ into the universe $X$ of the space $\mathcal{X}$. To facilitate the understanding of the underlying ideas, we here often prefer the notations as sequences, like $(q_k)_{k \in \mathbb{N}} = (q_k : k \in \mathbb{N}) = (q_0, q_1, q_2, \ldots)$ and $(s_k)_{k \in \mathbb{N}} = (s_0, s_1, s_2, \ldots)$, respectively.

The sequence $S = (s_n)_{n \in \mathbb{N}}$, which provides a dense set, $\text{ran}(S)$, together with a total numbering of it, will also be called the skeleton of the EMS $\mathcal{X}$. It gives the basis for the application of recursion theory to the space $\mathcal{X}$.

Complete and separable metric spaces like $(X,d)$ are also called Polish spaces, cf. [9]. The completeness requirement is useful to simplify some formulations. In order to apply our notions to an arbitrary separable space, one can consider its standard completion by means of a suitable at most countable, dense set (corresponding to the skeleton).

We specify some examples of SEMSs that will be dealt with in the sequel:

1. discrete spaces $(X,d_X,S_X)$, where $d_X(x,x') = 1$ for $x \neq x'$ and $S_X = (x_n)_{n \in \mathbb{N}}$ gives a numbering of the finite or countably infinite set $X = \text{ran}(S_X)$;
2. $(\mathbb{R},d_\mathbb{R},S_\mathbb{R})$, the space of the real numbers, with the natural distance function $d_\mathbb{R}(x,y) = |x - y|$, and the standard numbering of the rational numbers, $S_\mathbb{R} = v_\mathbb{Q}$;
3. $(B,d_B,S_B)$, the Baire space of binary sequences, with $B = \{0,1\}^\mathbb{N}$, the Baire metric $d_B$, here defined by $d_B((\beta_n)_{n \in \mathbb{N}},(\beta'_n)_{n \in \mathbb{N}}) = 2^{-\min\{n : \beta_n \neq \beta'_n\}}$ for $(\beta_n)_{n \in \mathbb{N}} \neq (\beta'_n)_{n \in \mathbb{N}}$, and a skeleton $S_B$ corresponding to a bijective standard numbering of the binary sequences becoming stationary with 0, i.e., $\text{ran}(S_B) = \{(\beta_0, \beta_1, \ldots, \beta_l, 0, 0, \ldots) : \beta_0, \beta_1, \ldots, \beta_l \in \{0,1\}, l \in \mathbb{N}\}$;
4. $(F,d_\mathbb{F},S_\mathbb{F})$, the Baire space of sequences of natural numbers, with $F = \mathbb{N}^\mathbb{N}$, the Baire metric $d_\mathbb{F}$ defined like $d_B$, and $S_\mathbb{F}$ being a bijective standard numbering of the ultimately 0-stationary sequences, i.e., $\text{ran}(S_\mathbb{F}) = \{(v_0,v_1,\ldots,v_l,0,0,\ldots) : v_0,v_1,\ldots,v_l \in \mathbb{N}, l \in \mathbb{N}\}$.

The SEMSs described at 2–4, respectively, will also simply be denoted by $\mathbb{R}$, $B$ and $F$. The underlying metric spaces are even perfect Polish spaces, i.e., all their elements are accumulation points. For more details and further examples, like the product spaces $(\mathbb{R}^n,d_{\max},S_{\mathbb{R}^n})$, with the maximum distance $d_{\max}$ and a standard numbering $S_{\mathbb{R}^n}$ of $\mathbb{Q}^n$, or the function spaces $C[0,1]$ and $L^p$, for computable reals $p \geq 1$, the reader is referred to [13].

One easily shows that SEMSs are characterized by the approximate computability of the distance functions on the skeletons, whereas in the case of an EMS, the distance function is just right-cut computable with respect to the skeleton. To be more precise,
we fix some standard numbering \((\Phi_i: i \in \mathbb{N})\) of the partial recursive unary functions \(\Phi: \mathbb{N} \to \mathbb{N}\).

**Lemma 1.** Let \((X, d)\) be a separable complete metric space and \(S = (s_n)_{n \in \mathbb{N}}\) a sequence of points such that \(\text{ran}(S)\) is dense in \((X, d)\). Then it holds:

(i) \((X, d, S)\) is a SEMS iff there is a recursive total function \(\varphi: \mathbb{N}^2 \to \mathbb{N}\) such that

\[
|q_{\varphi(m, n)}(k) - d(s_n, s_m)| < 2^{-k}, \text{ for all } k, m, n \in \mathbb{N};
\]

(ii) \((X, d, S)\) is an EMS iff there is a recursive total function \(\varphi: \mathbb{N}^2 \to \mathbb{N}\) such that

\[
\{q_{\varphi(m, n)}(k): k \in \mathbb{N}\} = \mathbb{Q} \cap (d(s_m, s_n), \infty), \text{ for all } k, m, n \in \mathbb{N}.
\]

For example, the equation in (ii) means that the unary recursive function \(\Phi_{\varphi(m, n)}\) enumerates the set of all rational numbers belonging to the open interval \((d(s_m, s_n), \infty)\), this is the right cut of the real number \(d(s_m, s_n)\). The proof of Lemma 1 needs only standard techniques of recursion theory. So it can be omitted here.

To apply recursion theory to an EMS \(\mathcal{X} = (X, d, S)\), the elements \(x \in X\) are represented by index sequences \(\sigma = (i_0, i_1, i_2, \ldots) \in \mathbb{F}\) for which the corresponding \(S\)-sequences, \(S \circ \sigma = (s_{i_0}, s_{i_1}, s_{i_2}, \ldots) \in \text{ran}(S)^\mathbb{N}\), converge effectively to \(x\).

We recall that the sequence \(S \circ \sigma = (s_n)_{n \in \mathbb{N}}\) is said to be effectively converging to \(x\) if \(d(x, s_n) < 2^{-n}\), for all \(m \in \mathbb{N}\). Then it is also called a standard Cauchy function of \(x\) (with respect to the skeleton \(S\)). By \(\text{CF}_x^{(S)}\) or simply \(\text{CF}_x\), we shall denote the set of all index sequences \(\sigma\) whose \(S\)-sequences, \(S \circ \sigma\), converge effectively to \(x\), whereas \(\overline{\text{CF}}_x^{(S)}\) is the set of all standard Cauchy functions converging to \(x\).

Let us emphasize again that the elements \(x\) are approximated by Cauchy sequences from \(\overline{\text{CF}}_x^{(S)}\), but they are represented by the related index sequences from \(\text{CF}_x^{(S)} \subseteq \mathbb{F}\). This just corresponds to the normed Cauchy representation of the universe \(X\) with respect to the skeleton \(S\), and it yields the natural approach to approximate computability over \(\mathcal{X}\), cf. [12, 13].

In this sense, an element \(x \in X\) is said to be computable on \(\mathcal{X}\) if \(\text{CF}_x^{(S)}\) contains a recursive sequence (of natural numbers). Then a standard numbering \(\nu\) of the class of all recursive total unary functions \(\varphi: \mathbb{N} \to \mathbb{N}\) induces a standard numbering \(\nu_{\text{Compel}(\mathcal{X})}\) of the class of all computable elements,

\[
\text{Compel}(\mathcal{X}) = \{x \in X: x \text{ is computable on } \mathcal{X}\},
\]

\[
\nu_{\text{Compel}(\mathcal{X})}(n) = x \text{ iff } \Phi_n \in \text{CF}_x^{(S)}.
\]

Of course, \(\nu_{\text{Compel}(\mathcal{X})}\) is only a partial numbering, i.e., a partial function from \(\mathbb{N}\) onto \(\text{Compel}(\mathcal{X})\).

In the related way, approximate computability of partial functions \(f: X \to X'\), for EMSs \(\mathcal{X} = (X, d, S)\) and \(\mathcal{X}' = (X', d', S')\), can naturally be defined based on the skeletons \(S\) and \(S'\). To do this, we consider function-oracle Turing machines (briefly: OTMs), as they have been used by Ko and Friedman [7, 8] for real functions.

To compute the function \(f\), an OTM \(\mathcal{M}\) has to produce an output \(j_n = \mathcal{M}^{\varphi}(n) \in \mathbb{N}\), for any input \(n \in \mathbb{N}\), in such a way that \((j_n)_{n \in \mathbb{N}} \in \text{CF}_{f(x)}^{(S')}\) if the argument \(x \in X\) is given.
Fig. 1. An OTM $\mathcal{M}$ computing a partial function $f : X \rightarrow X'$.

Given EMSs $\mathcal{X} = (X, d, S)$ and $\mathcal{X}' = (X', d', S')$, a partial function $f : X \rightarrow X'$ is said to be (approximately) computable if there is an OTM $\mathcal{M}$ such that:

1. for all $x \in \text{dom}(f)$ and all index sequences $\sigma \in \text{CF}_x^{(S)}$, $\mathcal{M}^\sigma(n)$ always exists, and it holds $S' \circ (\mathcal{M}^\sigma(n))_n \in \text{CF}_{f(x)}^{(S')}$;
2. for all $x \notin \text{dom}(f)$ and all $\sigma \in \text{CF}_x^{(S)}$, there is an input $n \in \mathbb{N}$ for which $\mathcal{M}^\sigma(n)$ is undefined.

This notion of approximate computability of functions between EMSs is a straightforward generalization of a concept which was introduced for real functions by Weihrauch and Kreitz and has been rediscovered and substantially used in [3, 4]. The original notion of computability used by Ko and Friedman [8] is more restrictive. For a discussion of these relationships and for more details, the reader is referred to [3].

Let $\text{Compfu}(\mathcal{X}, \mathcal{X}')$ denote the class of all partial functions $f : X \rightarrow X'$ which are computable in the sense just defined. A standard numbering $\nu_{\text{OTM}}$ of the set of all OTMs yields canonically a standard numbering $\nu_{\text{Compfu}(\mathcal{X}, \mathcal{X}')} = (\mathcal{X}, \mathcal{X}')$ of the set $\text{Compfu}(\mathcal{X}, \mathcal{X}')$:

$$
\nu_{\text{Compfu}(\mathcal{X}, \mathcal{X}'))(n) = f \iff \nu_{\text{OTM}}(n) \text{ computes } f.
$$

Already under rather weak suppositions (p.e., if $\text{card}(X') > 1$ or the space $(X, d)$ contains an accumulation point) one can show that $\nu_{\text{Compfu}(\mathcal{X}, \mathcal{X}')} = (\mathcal{X}, \mathcal{X}')$ is a partial numbering, since there are OTMs which don’t compute functions. Nevertheless, $\nu_{\text{Compfu}(\mathcal{X}, \mathcal{X}')} = (\mathcal{X}, \mathcal{X}')$ is often effectively equivalent to a total numbering.
For EMSs

**Lemma 3.** By mutual simulation, one easily shows into the details, we remark that even computability of functions of type computable in $\mathcal{F}_{[\text{finite operations over the real numbers, cf. Section 5}}$. Without going Richards [10] for $eFV—\text{ective Banach spaces. In the related way, Hertling [5] deals with as pairs,}

$n$ could be dealt with on the base of OTMs. To this purpose, the inputs are interpreted from $X$ is computable in the sense sketched above icVT it is computable as a unary function which is canonically induced by $X$ is continuous on its domain.

The approximate computability of $k$-ary functions $f : X^k \Rightarrow X'$, $k \in \mathbb{N}_+$, can straightforwardly be defined by OTMs using oracle sequences $\sigma = (\sigma_1, \ldots, \sigma_k) \in \text{CF}_{x_1}^{(S)} \times \cdots \times \text{CF}_{x_k}^{(S)}$, for arguments $x = (x_1, \ldots, x_k) \in X^k$. Then the oracle queries “$m?$” are interpreted as Cantor numbers of pairs, i.e., $m = \langle x, \mu \rangle$, and they have to be answered with the $\mu$th index $i_{x\mu}$ of the sequence $\sigma_x = (i_{x\mu})_{\mu \in \mathbb{N}}$, for $1 \leq x \leq k$. Now the definition of computability for a $k$-ary function $f$ is quite analogous to that for the special case $k = 1$ described above.

On the other hand, there is an EMS $\mathcal{X}^{(k)} = (X^k, d_{\text{max}}^{(k)}, S^{(k)})$ over the universe $X^k$ which is canonically induced by $\mathcal{X} = (X, d, S)$:

$$d_{\text{max}}^{(k)}((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \max_{x \in 1} d(x_\mu, y_\mu),$$

$$S^{(k)} = ((s^{(k)}_{i_1^{(1)}}, \ldots, s^{(k)}_{i_k^{(1)})}; i \in \mathbb{N}).$$

By mutual simulation, one easily shows

**Lemma 2.** If $f \in \text{Compfu}(\mathcal{X}, \mathcal{X}')$ and $g \in \text{Compfu}(\mathcal{X}, \mathcal{X}'')$, for EMS $\mathcal{X}, \mathcal{X}', \mathcal{X}''$, then $g \circ f \in \text{Compfu}(\mathcal{X}, \mathcal{X}'')$ and $f$ is continuous on its domain.

The model of OTM is even well suited to deal with infinitary functions of type $f : X^\mathbb{N} \Rightarrow X'$. Also in this case, there is a canonical way to define a distance $\bar{d}$ and a skeleton $\bar{S}$ such that $\mathcal{X}^\mathbb{N} = (X^\mathbb{N}, \bar{d}, \bar{S})$ is an EMS, we refer to [1] for the details. But even if the related notion of computability for functions $f : X^\mathbb{N} \Rightarrow X'$ with respect to $\mathcal{X}^\mathbb{N}$ and $\mathcal{X}'$ is equivalent to ours, we prefer the following explicit version which directly refers to the component spaces, cf. Fig. 2.

More precisely, a function $f : X^\mathbb{N} \Rightarrow X'$ is said to be computable (with respect to the EMSs $\mathcal{X}$ and $\mathcal{X}'$) if there is an OTM $\mathcal{M}$ such that

1. for all $x = (x_0, x_1, x_2, \ldots) \in \text{dom}(f)$ and all $\sigma \in \text{CF}_{x_0}^{(S)} \times \text{CF}_{x_1}^{(S)} \times \text{CF}_{x_2}^{(S)} \times \cdots$, the outputs $\mathcal{M}^\sigma(n)$ always exist and fulfill $(\mathcal{M}^\sigma(n))_{n \in \mathbb{N}} \in \text{CF}_{f(x)}^{(S')}$;
2. for all $x \notin \text{dom}(f)$ and all $\sigma \in \text{CF}_{x_0}^{(S)} \times \text{CF}_{x_1}^{(S)} \times \text{CF}_{x_2}^{(S)} \times \cdots$, there is an input $n \in \mathbb{N}$ for which $\mathcal{M}^\sigma(n)$ is undefined.

Here the oracles $\sigma$ are double sequences as they have been considered by Pour-El and Richards [10] for effective Banach spaces. In the related way, Hertling [5] deals with computable infinitary operations over the real numbers, cf. Section 5. Without going into the details, we remark that even computability of functions of type $f : X^\mathbb{N} \Rightarrow X'^\mathbb{N}$ could be dealt with on the base of OTMs. To this purpose, the inputs are interpreted as pairs, $n = (x', y)$, and the outputs $\mathcal{M}^\sigma(n) = j_{x'y}$ are taken as the $v$th element of
For a universe $X$ of objects one wants to deal with, very often a metric $d$ and a skeleton $S$, on which computability has to be based, are naturally given. Sometimes, however, it may be a non-trivial problem to choose $d$ and $S$ in such a way that an appropriate concept of computability over the EMS $\mathcal{X} = (X, d, S)$ is induced. The solution always depends on the aims and the intended applications of the theory. As an instructive illustration of this problem, we refer to the contrary results by Pour-El and Richards [10] that the wave propagator is not computable and by Weihrauch and Zhong [15] that it is computable. This paradoxical situation can be solved by considering different distance measures and skeletons, for the related function spaces.

Here we only deal with the easier theoretical question to characterize the computational equivalence of different skeletons over the same Polish space $(X, d)$. For EMSs $\mathcal{X}_i = (X, d, S_i)$, $i = 1, 2$, we shall write $S_1 \equiv S_2$ if both $\text{Compfu}(\mathcal{X}_1, \mathcal{X}_1) \subseteq \text{Compfu}(\mathcal{X}_2, \mathcal{X}_2)$ and $\text{Compfu}(\mathcal{X}_1, \mathcal{X}_2) \subseteq \text{Compfu}(\mathcal{X}_2, \mathcal{X}_1)$, for all EMSs $\mathcal{X}''$.

Since the identical function $\text{id}_X$ belongs to $\text{Compfu}(\mathcal{X}_2, \mathcal{X}_2) \cap \text{Compfu}(\mathcal{X}_1, \mathcal{X}_1)$, $S_1 \equiv S_2$ implies that $\text{id}_X \in \text{Compfu}(\mathcal{X}_1, \mathcal{X}_2) \cap \text{Compfu}(\mathcal{X}_2, \mathcal{X}_1)$. Conversely, from this property, by means of Lemma 2, one obtains that $S_2 \equiv S_1$. Thus, $S_1 \equiv S_2$ if $S_2 \equiv S_1$, and we shall better write $S_1 \equiv S_2$ and call the skeletons $S_1$ and $S_2$ (computationally) equivalent in this case.

The following proposition establishes a relationship to the effective equivalence (i.e., the mutual effective reducibility) of the induced standard numberings of computable elements.

**Proposition 1.** For EMSs $(X, d, S_1)$ and $(X, d, S_2)$, it holds

$$S_1 \equiv S_2 \iff \forall \text{Compel}(\mathcal{X}_1) \equiv \forall \text{Compel}(\mathcal{X}_2).$$

For the proof of the first direction, let $S_1 \equiv S_2$. Then there is an OTM $\mathcal{M}$ computing $\text{id}_X \in \text{Compfu}(\mathcal{X}_1, \mathcal{X}_2)$. Given a recursive sequence $\sigma = \Phi_l : \mathbb{N} \to \mathbb{N}$, $l \in \mathbb{N}$, with
\[ \sigma \in \text{CF}^{(S_1)}_x \] for some \( x \in \text{Compel}(X_1) \). \( \mathcal{M} \) transforms it into a recursive sequence \( \sigma' = \Phi_I \in \text{CF}^{(S_2)}_x \). Using the smn-theorem, one shows that the transformation of \( I \) into \( I' \) is (partial) recursive. Hence, \( \text{vCompel}(X_1) \) is effectively reducible to \( \text{vCompel}(X_2) \).

The reducibility \( \text{vCompel}(X_2) \leq \text{vCompel}(X_1) \) follows analogously from \( \text{id}_X \in \text{Compfu}(X_2, X_1) \).

Now let \( \text{vCompel}(X_1) \leq \text{vCompel}(X_2) \) via a partial recursive function \( \varphi \). To show that \( \text{id}_X \in \text{Compfu}(X_1, X_2) \), let an OTM \( \mathcal{M} \) on an input \( n \) first query for the \((n + 1)\)st index \( i_{n+1} \) of the oracle sequence \( \sigma = (i_m)_{m \in \mathbb{N}} \in \text{CF}^{(S_1)}_x \), \( x \in X \). Then let \( \mathcal{M} \) compute the number \( l \) such that \( \Phi_l = (i_{n+1}, i_{n+1}, i_{n+1}, \ldots) \) and produce the output \( j_n = \Phi_{\varphi(l)}(n+1) \).

For \( S_1 = (s^{(1)}_m)_{m \in \mathbb{N}} \) and \( S_2 = (s^{(2)}_m)_{m \in \mathbb{N}} \), it holds that
\[
\begin{align*}
d(s^{(2)}_{j_n}, x) &\leq d(s^{(2)}_{j_n}, s^{(1)}_{i_{n+1}}) + d(s^{(1)}_{i_{n+1}}, x) \\
&< 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.
\end{align*}
\]

Thus, \( \mathcal{M} \) works correctly.

Analogously, one shows that \( \text{vCompel}(X_2) \leq \text{vCompel}(X_1) \) implies \( \text{id}_X \in \text{Compfu}(X_2, X_1) \).

By Proposition 1, from \( S_1 \equiv S_2 \) it follows that \( \text{Compel}(X_1) = \text{Compel}(X_2) \). The converse is not true, as one can show by considering discrete EMSs with appropriate different skeletons over the universe \( \{0, 1\} \).

There are several equivalent modifications of the condition \( \text{vCompel}(X_1) = \text{vCompel}(X_2) \). For example, it is equivalent that \( S_1 \) can effectively be approximated by \( S_2 \), and conversely. More precisely, this means that there are recursive functions \( \varphi_i : \mathbb{N}^2 \to \mathbb{N} \), \( i = 1, 2 \), such that
\[
\begin{align*}
d(s^{(2)}_{\varphi_i(m,n)}, s^{(1)}_m) \leq 2^{-n} \quad \text{and} \quad d(s^{(1)}_{\varphi_i(m,n)}, s^{(2)}_m) \leq 2^{-n} \quad \text{for all} \ m, n \in \mathbb{N}.
\end{align*}
\]

One easily shows that if \( S_1 \equiv S_2 \) and \((X, d, S_1)\) is a SEMS, then \((X, d, S_2)\) is a SEMS, too. Equivalent skeletons can also be replaced for each other without changing the notions of computability for \( k \)-ary functions and even for infinitary functions, as they have been explained above.

Examples of skeletons over the real numbers, which are computationally equivalent to \( S_\mathbb{R} \), are obtained by standard numberings of \( \mathbb{D} \), the set of dyadic rational numbers, as well as by standard numberings of many other dense subsets of \( \mathbb{R} \).

A skeleton of an infinite SEMS can be assumed to be injective, without loss of generality:

**Lemma 4.** To every infinite SEMS \( X = (X, d, S) \), there is an injective skeleton \( S' \) which is equivalent to \( S \).

To show this, let \( S = (s_n)_{n \in \mathbb{N}} \). By means of a recursive enumeration of the set \( D_> = \{ \langle m, n, k \rangle : d(s_m, s_n) > q_k \} \), one first gets an enumeration of the set \( \{ \langle m, n, k \rangle : d(s_m, s_n) > 0 \} = \{ \langle m, n, k \rangle : s_m \neq s_n \} \) and finally a total recursive function \( \varphi : \mathbb{N} \to \mathbb{N} \).
such that
\[
\text{ran}(S \circ \varphi) = \{s_{\varphi(n)} : n \in \mathbb{N}\} = \text{ran}(S) \quad \text{and} \quad s_{\varphi(m)} \neq s_{\varphi(n)} \text{ for } m \neq n.
\]

Let \( S' = S \circ \varphi = (s_{\varphi(n)} : n \in \mathbb{N}) \). Then \( \mathcal{X}' = (X, d, S') \) is a SEMS too. By the smn-theorem and using an enumeration of \( D_\mathcal{E} \), one shows that \( \text{vCompel}(\mathcal{X}) \equiv \text{vCompel}(\mathcal{X}') \).

3. The topological arithmetical hierarchy

The **topological arithmetical hierarchy** (briefly: **TAH**) over an EMS is a useful tool in dealing with approximate computability. For the space of real numbers, it has been introduced and applied in [3, 4], but at least the classes of the first and second level of the TAH were already well known from computable analysis. For related concepts within descriptive set theory, see [9].

Let an EMS \( \mathcal{X} = (X, d, S) \) be fixed. The **open balls** with radii \( r \in \mathbb{R} \) around points \( x_0 \in X \) are denoted as usual,

\[
\text{Ball}_r(x_0) = \{ x \in X : d(x_0, x) < r \}.
\]

By \( (\text{ball}_n : n \in \mathbb{N}) \), we mean the standard numbering of the open balls with rational radii around the points of the skeleton, which is given by

\[
\text{ball}_{(m,n)} = \text{Ball}_{q_m}(s_n),
\]

with respect to the numbering \( v_\mathbb{Q} = (q_m)_{m \in \mathbb{N}} \) of \( \mathbb{Q} \). Notice that \( v_\mathbb{Q} \) includes the negative rationals. It holds \( \text{ball}_{(m,n)} = \emptyset \) iff \( q_m \leq 0 \), and this property is recursively decidable with respect to the index \( m \).

For \( k \in \mathbb{N}_+ \), let \( \Sigma^a_k \) denote the class of sets \( A \subseteq X \) for which there is a recursive \( k \)-ary (total) function \( \varphi : \mathbb{N}^k \to \mathbb{N} \) such that

\[
A = \begin{cases} 
\bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcap_{n_k \in \mathbb{N}} \text{ball}_{\varphi(n_1,n_2,\ldots,n_k)} & \text{if } k \text{ is odd}, \\
\bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcap_{n_k \in \mathbb{N}} \text{ball}_{\varphi(n_1,n_2,\ldots,n_k)} & \text{if } k \text{ is even}.
\end{cases}
\]

The overline \( \overline{\ } \) denotes the complement of a set with respect to the universe \( X \).

Let \( \Pi^a_k \) be the class of complements of \( \Sigma^a_k \)-sets: \( A \in \Pi^a_k \iff \overline{A} = X \setminus A \in \Sigma^a_k \). Finally, \( \Delta^a_k = \Sigma^a_k \cap \Pi^a_k \).

From the literature, the members of \( \Sigma^1_k \) are known as recursively open sets, \( \Pi^1_k \) consists just of the recursively closed sets, \( \Sigma^2_k \) is the class of recursively \( F_0 \) sets, and \( \Pi^2_k \) is the class of recursively \( G_\delta \) sets. This indicates already that the TAH is an effective counterpart of the hierarchy of Borelian subsets of finite order over the metric space \( (X, d) \).

Here we restrict ourselves to classes of subsets of \( X \) within the TAH. The generalization to members \( A \subseteq X^k, \ k \in \mathbb{N}_+ \), within the classes is straightforward, simply by considering the EMS \( \mathcal{X}^{(k)} = (X^k, d^{(k)}_{\max}, S^{(k)}) \), cf. Section 2.
Proposition 2. Equivalent skeletons over a complete metric space define the same TAH.

To prove this, let $\mathcal{X} = (X, d, S)$ and $\mathcal{X}' = (X, d, S')$, with computationally equivalent skeletons $S = (s_n)_{n \in \mathbb{N}}$ and $S' = (s'_n)_{n \in \mathbb{N}}$. The corresponding numberings of balls with rational radii around the points of the skeletons are $(\text{ball}_n: n \in \mathbb{N})$ and $(\text{ball}'_n: n \in \mathbb{N})$. First we show that there is a recursive function $\psi: \mathbb{N}^2 \to \mathbb{N}$ satisfying

$$\text{ball}_n \cup \bigcup_{l \in \mathbb{N}} \text{ball}'_{\psi(n, l)} \text{ for all } n \in \mathbb{N}.$$ 

Indeed,

$$\text{ball}_{(\tilde{m}, \tilde{n})} = \{x: d(s_{\tilde{m}}, x) < q_{\tilde{n}}\} = \bigcup \{\{x: d(s'_{\tilde{n}}, x) < q_m\}: d(s_{\tilde{m}}, s'_{\tilde{n}}) < q_l, q_m + q_l < q_{\tilde{n}} \text{ for some } l \in \mathbb{N}\} = \bigcup \{\text{ball}'_{(m, n)}: d(s_{\tilde{m}}, s'_{\tilde{n}}) < q_l, q_m + q_l < q_{\tilde{n}} \text{ for some } l \in \mathbb{N}\}.$$ 

Using an effective reduction from $\nu_{\text{Compel}}(\mathcal{X}')$ to $\nu_{\text{Compel}}(\mathcal{X})$ and the recursive enumerability of $D_\infty$ (with respect to the skeleton $S$), one can show that the set $\{\{\tilde{m}, \tilde{n}, n, m, l\}: d(s_{\tilde{m}}, s'_{\tilde{n}}) < q_l, q_m + q_l < q_{\tilde{n}}\}$ is r.e. Thus, there is a recursive function $\varphi: \mathbb{N} \to \mathbb{N}$ such that for all $\tilde{m}, \tilde{n} \in \mathbb{N},$

$$\{\varphi((\tilde{m}, \tilde{n}, l)) : l \in \mathbb{N}\} = \{(m, n) : d(s_{\tilde{m}}, s'_{\tilde{n}}) < q_l, q_m + q_l < q_{\tilde{n}} \text{ for some } l \in \mathbb{N}\}.$$ 

Notice that the latter set is infinite, since $\nu_\mathbb{Q} = (q_m: m \in \mathbb{N})$ enumerates also the negative rationals. Therefore, $\varphi$ can be assumed to be a total function. It follows that

$$\text{ball}_{(\tilde{m}, \tilde{n})} = \bigcup_{l \in \mathbb{N}} \text{ball}'_{\varphi((\tilde{m}, \tilde{n}, l))}$$

and the function $\psi(n, l) = (\pi_1^2(\varphi(\pi_1^2(n), \pi_2^2(n), l)), \pi_2^2(\varphi(\pi_1^2(n), \pi_2^2(n), l)))$ fulfills the equation stated above.

Now let $k$ be odd and $A \in \Sigma_k^{\text{TA}}$ with respect to $S$, i.e.,

$$A = \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_k \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_k)},$$

for some recursive function $\varphi$. Then we have

$$A = \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_k \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} \text{ball}'_{\varphi(n_1, n_2, \ldots, n_k, l)}$$

$$= \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_k \in \mathbb{N}} \text{ball}'_{\varphi(n_1, n_2, \ldots, n_k-1, n_k, l)}$$

Thus, $A \in \Sigma_k^{\text{TA}}$ with respect to $S'$. 
For an even number $k$, the proof is analogous:

$$A = \bigcup_{m_1 \in \mathbb{N}} \bigcap_{n_1 \in \mathbb{N}} \cdots \bigcup_{m_k \in \mathbb{N}} \bigcap_{n_k \in \mathbb{N}} \operatorname{ball}_\varphi(n_1, n_2, \ldots, n_k)$$

$$= \bigcup_{m_1 \in \mathbb{N}} \bigcap_{n_1 \in \mathbb{N}} \cdots \bigcup_{m_k \in \mathbb{N}} \bigcap_{n_k \in \mathbb{N}} \operatorname{ball}_\varphi(n_1, n_2, \ldots, n_k)$$

$$= \bigcup_{m_1 \in \mathbb{N}} \bigcap_{n_1 \in \mathbb{N}} \cdots \bigcup_{m_k \in \mathbb{N}} \bigcap_{n_k \in \mathbb{N}} \operatorname{ball}_\gamma(\varphi(n_1, n_2, \ldots, n_k), \varphi^2(n_k))$$

$$= \bigcup_{m_1 \in \mathbb{N}} \bigcap_{n_1 \in \mathbb{N}} \cdots \bigcup_{m_k \in \mathbb{N}} \bigcap_{n_k \in \mathbb{N}} \operatorname{ball}_\gamma(\varphi(n_1, n_2, \ldots, n_k), \varphi^2(n_k))$$

So we have shown that the $\Sigma^0_k$ sets with respect to the skeleton $S$ are also $\Sigma^0_k$ sets with respect to $S'$. From this, the analogue for the classes $\Pi^0_k$ and $\Delta^0_k$ follows immediately.

Since the assertion is symmetric with respect to $S$ and $S'$, Proposition 2 has been shown.

By the usual technique, one proves

**Lemma 5.** The classes of the TAH are closed under finite unions and intersections of sets.

Now we are going to show the hierarchy properties for the TAH under rather weak suppositions on the underlying EMS $\mathcal{X}$. This is done by establishing a relationship to the classical arithmetical hierarchy (briefly: AH), whose classes are here denoted by $\Sigma^0_k$, $\Pi^0_k$, and $\Delta^0_k$ $(k \in \mathbb{N}_+)$. They consist of sets of natural numbers. By a classical representation lemma, for $k \in \mathbb{N}_+$ and $M \subseteq \mathbb{N}$ it holds: $M \in \Sigma^0_k$ iff there is a r.e. set $E \subseteq \mathbb{N}$ such that

$$M = \begin{cases} \{m : \exists (n_1 \in \mathbb{N}) \forall (n_2 \in \mathbb{N}) \cdots \forall (n_{k-1} \in \mathbb{N}) \} & \text{if } k \text{ is odd,} \\ \{m, n_1, n_2, \ldots, n_{k-1} \in E \} & \text{if } k \text{ is even} \end{cases}$$

The relationship we want to use between TAH and AH is based on considering the index sets of the members of classes from the TAH, with respect to a suitable subsequence of the skeleton $S$.

By an effectively discrete sequence (briefly: EDS) of the EMS $\mathcal{X} = (X, d, S)$, we mean a pair $(\eta, \delta)$ of recursive total functions $\eta, \delta : \mathbb{N} \to \mathbb{N}$ such that

$$d(s_{\eta(n)}, s_{\eta(n')}) > 2^{-\delta(n)}$$

for all $n, n' \in \mathbb{N}$ with $n \neq n'$. It follows $s_{\eta(n)} \neq s_{\eta(n')}$, for $n \neq n'$. One easily shows that if with EDS $\mathcal{X}$ possesses an EDS, there is such one $(\eta, \delta)$, for which $\eta$ is strictly monotonic, i.e., it indeed defines an effective subsequence $S \circ \eta = s_{\eta(n)}$ of the skeleton $S$. The function $\delta$ separates effectively the points of ran$(S \circ \eta)$ each from the other. Fig. 3 gives an illustration.
Obviously, the universe $X$ has to be infinite if $X$ possesses an EDS. On the other hand, for all infinite SEMSs we have mentioned so far, the existence of EDSs can be shown.

For example, over the EMS $\mathbb{R}$, let $\eta$ be an injective enumeration of $S_\mathbb{R}$-indices of integers, and $\delta(n) = 1$ for all $n \in \mathbb{N}$. Over $\mathbb{Z}$, let $\eta(n) \in S_\mathbb{Z}^{-1}((1, \ldots, 1, 0, 0, \ldots))$ and $\delta(n) = n + 2$. Over $\mathbb{F}$, $\eta(n) \in S_\mathbb{F}^{-1}((n, 0, 0, \ldots))$ and $\delta(n) = 1$ fulfill the required properties. More general, we have

**Proposition 3.** *If the SEMS $\mathcal{X}$ is unbounded or contains a computable accumulation point, $\mathcal{X}$ possesses an EDS.*

Indeed, if $\mathcal{X}$ is unbounded, one recursively defines a function $\eta$ such that

$$\eta(0) = 0,$$

$$\eta(n + 1) \in \{ m \colon d(s_m, s_{n'}) > 1 \text{ for all } n' \in \{0, \ldots, n\} \}.$$

This can be done by using an enumeration of the set $D_\succ$. Then the pair $(\eta, \delta)$, with $\delta(n) = 0$ (i.e., $2^{-\delta(n)} = 1$) for all $n \in \mathbb{N}$, is an EDS.

Now let $x_0 \in X$ be a computable accumulation point and $\varphi_0 : \mathbb{N} \to \mathbb{N}$ be a recursive function computing $x_0$, i.e.,

$$d(s_{\varphi_0(n)}, x_0) < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

Let $\eta(0)$ be defined in such a way that

$$d(s_{\eta(0)}, s_{\varphi_0(m_0)}) > 2 \cdot 2^{-m_0},$$

for some $m_0 \in \mathbb{N}$. Both $\eta(0)$ and $m_0$ can effectively be found using a recursive enumeration of $D_\succ$. If $\eta(n)$ and $m_n$ are defined, let $\eta(n+1)$ and $m_{n+1}$ be similarly chosen,
by using enumerations of $D_<$ and $D_>$, such that

$$2^{-m_n} > d(s_{q(n+1)}, s_{q_0(m_n+1)}) > 2 \cdot 2^{-m_n+1}.$$ 

The pair $(\eta, \delta)$ with $\delta(n) = m_n$ is an EDS.

Obviously, the premise of Proposition 3 is fulfilled for all SEMSs whose underlying metric space is perfect Polish.

We now suppose that the EMS $\mathcal{F} = (X, d, S)$ possesses an EDS $(\eta, \delta)$ which is fixed in the remaining part of this section.

For a set $A \subseteq X$, let the **index set** of $A$ with respect to $(\eta, \delta)$ be defined by

$$\text{ind}(A) = \{ n \in \mathbb{N} : s_\eta(n) \in A \}.$$ 

The **index class** of a class $\Gamma$ of the TAH is

$$\text{ind}(\Gamma) = \{ \text{ind}(A) : A \in \Gamma \}.$$ 

**Lemma 6.** For any $k \in \mathbb{N}_+$, $\text{ind}(\Sigma_k^\alpha) \subseteq \Sigma_k^0$, $\text{ind}(\Pi_k^\alpha) \subseteq \Pi_k^0$, $\text{ind}(A_k^\alpha) \subseteq A_k^0$.

To prove this, let $A \in \Sigma_k^\alpha$, i.e.,

$$A = \left\{ \begin{array}{ll}
\bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_k \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_k)} & \text{if } k \text{ is odd}, \\
\bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcap_{n_k \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_k)} & \text{if } k \text{ is even},
\end{array} \right.$$

with a recursive function $\varphi : \mathbb{N}^k \rightarrow \mathbb{N}$. Then $m \in \text{ind}(A)$ iff

$$\left\{ \begin{array}{ll}
\exists (n_1 \in \mathbb{N}) \forall (n_2 \in \mathbb{N}) \cdots \forall (n_{k-1} \in \mathbb{N}) \\
\text{if } k \text{ is odd,}
\end{array} \right.$$ 

$$\left\{ \begin{array}{ll}
\exists (n_1 \in \mathbb{N}) \forall (n_2 \in \mathbb{N}) \cdots \forall (n_{k-1} \in \mathbb{N}) \\
\text{if } k \text{ is even,}
\end{array} \right.$$ 

It holds $s_{\eta(m)} \in \bigcap_{n_1 \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_k)}$ iff $s_{\eta(m)} \in \bigcup_{n_1 \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_{k-1}, n_k)}$ iff $s_{\eta(m)} \notin \bigcup_{n_1 \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_{k-1}, n_k)}$. The set $\{ (m, n_1, \ldots, n_{k-1}) : s_{\eta(m)} \in \bigcup_{n_1 \in \mathbb{N}} \text{ball}_{\varphi(n_1, n_2, \ldots, n_{k-1}, n_k)} \}$ is r.e., since $D_<$ is r.e.. Thus, $\text{ind}(A) \in \Sigma_k^0$.

For $A \in \Pi_k^\alpha$, i.e., $A \in \Sigma_k^\alpha$, we have $\mathbb{N} \setminus \text{ind}(A) = \text{ind}(\bar{A}) \in \Sigma_k^0$, and $\text{ind}(A) \in \Pi_k^0$. The assertion for $A_k^\alpha$ follows immediately.

Notice that the previous proof didn’t use the characteristic property of the EDS. This is just needed in order to obtain the converse inclusions.

**Lemma 7.** For any $k \in \mathbb{N}_+$, $\Sigma_k^0 \subseteq \text{ind}(\Sigma_k^\alpha)$, $\Pi_k^0 \subseteq \text{ind}(\Pi_k^\alpha)$. 
If $k$ is odd and $M = \{ m : \exists (n_1 \in \mathbb{N}) \forall (n_2 \in \mathbb{N}) \cdots \forall (n_{k-1} \in \mathbb{N}) (m, n_1, n_2, \ldots, n_{k-1}) \in E \}$, with a r.e. set $E \subseteq \mathbb{N}$, let

$$A_M = \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcap_{n_{k-1} \in \mathbb{N}} \bigcup_{(m, n_1, \ldots, n_{k-1}) \in E} \text{ball}_{(\delta(m), \eta(m))}.$$  

Then $\text{ind}(A_M) = M$ and $A_M \in \Sigma_k^\Delta$.

If $k$ is even and $M = \{ m : \exists (n_1 \in \mathbb{N}) \forall (n_2 \in \mathbb{N}) \cdots \exists (n_{k-1} \in \mathbb{N}) (m, n_1, n_2, \ldots, n_{k-1}) \notin E \}$, with a r.e. $E \subseteq \mathbb{N}$, then let

$$A_M = \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_{k-1} \in \mathbb{N}} \bigcup_{(m, n_1, \ldots, n_{k-1}) \in E} \text{ball}_{(\delta(m), \eta(m))}.$$  

Again we have $\text{ind}(A_M) = M$ and $A_M \in \Sigma_k^\Delta$.

Given a set $M \in \Pi_k^0$, it follows $\mathbb{N} \setminus M \in \Sigma_k^0$. Then we obtain $A_{\mathbb{N}\setminus M} \in \Sigma_k^\Delta$, and $\text{ind}(A_{\mathbb{N}\setminus M}) = \mathbb{N} \setminus M$. Since $\text{ind}(A_{\mathbb{N}\setminus M}) = M$ and $A_{\mathbb{N}\setminus M} \in \Pi_k^0$, Lemma 7 has been proved.

The class $A_1^\Delta$ contains only clopen (i.e., closed and open) subsets of $X$. Thus, it may be rather small. For example, for the EMS of real numbers ($\mathbb{R}, d_{\mathbb{R}}, \mathbb{S}_{\mathbb{R}}$), we have $A_1^\Delta = \{\emptyset, \mathbb{R}\}$, therefore $\text{ind}(A_1^\Delta) = \{\emptyset, \mathbb{N}\} \subset A_1^\Delta$ in this case.

**Lemma 8.** For every EDS $(\eta, \delta)$, $\text{ran}(S \circ \eta) \in A_2^\Delta$. For $k \geq 2$, $A_k^\Delta \subseteq \text{ind}(A_k^\Delta)$.

Indeed, one easily sees that

$$\text{ran}(S \circ \eta) = \{s_{\eta(m)} : m \in \mathbb{N}\} = \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}, q_n > 0} \text{ball}_{(n, \eta(m))} = \bigcap_{n \in \mathbb{N}, q_n > 0} \bigcup_{m \in \mathbb{N}} \text{ball}_{(n, \eta(m))}.$$  

Thus, $\text{ran}(S \circ \eta) \in \Sigma_k^\Delta \cap \Pi_k^\Delta$.

Now let $M \in A_k^\Delta$, for $k \geq 2$, i.e., $M \in \Sigma_k^\Delta \cap \Pi_k^0$. By Lemma 7, there are sets $A, A' \subseteq X$ such that $\text{ind}(A) = \text{ind}(A') = M$ and $A \in \Sigma_k^\Delta$, $A' \in \Pi_k^\Delta$. Since the classes of the TAH are closed under finite intersections, we have $A \cap \text{ran}(S \circ \eta) \in \Sigma_k^\Delta, A' \cap \text{ran}(S \circ \eta) \in \Pi_k^\Delta$. Moreover, $\text{ind}(A \cap \text{ran}(S \circ \eta)) = \text{ind}(A' \cap \text{ran}(S \circ \eta)) = M$. Thus, $A \cap \text{ran}(S \circ \eta) = \{s_{\eta(m)} : m \in M\} = A' \cap \text{ran}(S \circ \eta) \in A_k^\Delta, i.e., M \in \text{ind}(A_k^\Delta)$.

By Lemmas 6, 7 and 8, we have

**Proposition 4.** For all $k \in \mathbb{N}_+$, $\text{ind}(\Sigma_k^\Delta) = \Sigma_k^0$ and $\text{ind}(\Pi_k^\Delta) = \Pi_k^0$. For all $k \geq 2$, it holds $\text{ind}(A_k^\Delta) = A_k^0$. 

Applying the well-known hierarchy properties of the AH, we immediately obtain

**Theorem 1.** If the EMS $\mathcal{X}$ possesses an EDS $X$, then for all $k \in \mathbb{N}_+$,

$$A_k^\alpha \subset \{ \Sigma_k^\alpha \} \subset \Sigma_k^\alpha \cup \Pi_k^\alpha \subset A_{k+1}^\alpha,$$

$$\Sigma_k^\alpha \neq \Pi_k^\alpha, \quad \text{and} \quad \Pi_k^\alpha \neq \Sigma_k^\alpha.$$

Finally, in contrast to Theorem 1, we notice that the TAH collapses over all finite EMSs.

**Lemma 9.** If the EMS $\mathcal{X}$ has a finite universe $X$, then for every subset $A \subseteq X$, $A \in A_1^\alpha$.

### 4. Domains of computable functions

In this section, we shall characterize the domains of approximately computable functions as being the $\Pi_2^\alpha$-sets of the TAH, at least for those EMSs we are mainly interested in. For the space of real numbers, this result goes back to Weihrauch and Kreitz. It has also been proved and essentially used in [3, 4]. Remember that the domains of real functions computable in the Ko–Friedman sense are just the $\Sigma_1^\alpha$-sets over $\mathbb{R}$, i.e., the recursively open sets.

First we show that any $\Pi_2^\alpha$-set occurs as the domain of a computable function.

**Proposition 5.** Let $A \in \Pi_2^\alpha$, for an EMS $\mathcal{X} = (X, d, S)$. Then $\text{id}_A \in \text{Compfu}(\mathcal{X}, \mathcal{X})$.

For the proof, we suppose that

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \text{ball}_{\varphi(n,m)},$$

where $\varphi : \mathbb{N}^2 \rightarrow \mathbb{N}$ is a recursive function. To compute the identical function $\text{id}_A$, an OTM $\mathcal{M}$ can work as follows, on any input $n \in \mathbb{N}$ and oracle sequences $\sigma \in \text{CF}_x$:

Let $\mathcal{M}$ enumerate simultaneously the sequences $(\text{ball}_{\varphi(n,k)} : k \in \mathbb{N})$ and $\sigma = (i_l)_{l \in \mathbb{N}}$ as well as the set $D_\prec$ up to reaching a pair $(k, l)$ such that $d(s_{\varphi,\varphi(n,k)}, s_l) + 2^{-l} < q_{\varphi,\varphi(n,k)}$. When such a pair is reached, let $\mathcal{M}$ output the $n$th index $i_n$ of $\sigma$; if no such pair exists, $\mathcal{M}^\sigma(n)$ remains undefined.

If $x \in A$, then $x \in \bigcup_{m \in \mathbb{N}} \text{ball}_{\varphi(n,m)}$, for all $n \in \mathbb{N}$. Thus, for all $n$ and all oracles $\sigma \in \text{CF}_x$, a pair $(k, l)$ of the above characterized kind exists and is found finally by the simultaneous enumeration. It follows $(\mathcal{M}^\sigma(n))_{n \in \mathbb{N}} = \sigma \in \text{CF}_x$.

If $x \notin A$, then for all $\sigma \in \text{CF}_x$ there is an input $n \in \mathbb{N}$ such that $x \notin \bigcup_{m \in \mathbb{N}} \text{ball}_{\varphi(n,m)}$. For such an $n$, no pair $(k, l)$ of the requested kind can be found, and $\mathcal{M}^\sigma(n)$ is undefined.

So we have shown that $\mathcal{M}$ approximately computes the function $\text{id}_A$.

Applying Lemma 2, the proposition yields a lot of further computable function with the domain $A$, namely as compositions $g \circ \text{id}_A$, for $g \in \text{Compfu}(\mathcal{X}, \mathcal{X'})$ with $A \subseteq \text{dom}(g)$. 


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Some more effort is needed in order to prove the conversion of Proposition 5, i.e.,
that the domains of computable functions are always $\Pi^0_2$-sets. More precisely, we are
not able to show this in general. The proof technique we are now going to deal with
applies to our standard examples of EMSs, however.

By a finitary stratification of (a bound of) ambiguity $b \in \mathbb{N}_+$ over an EMS $\mathcal{X} =
(X, d, S)$, we mean a recursive partial function $\varphi : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

(i) $X = \bigcup_{m \in \mathbb{N}, \varphi(n,m) \downarrow} \text{Ball}_2^{-n}(s_{\varphi(n,m)})$;

(ii) $\bigcap_{m \in M} \text{Ball}_2^{-n}(s_{\varphi(n,m)}) = \emptyset$, for all sets $M \subseteq \{m : \varphi(n,m) \downarrow\}$ with $\text{card}(M) > b$.

As usual, by $\varphi(n,m) \downarrow$ we indicate that $(n, m) \in \text{dom}(\varphi)$.

Let $\varphi_n$ denote the partial function defined by $\varphi_n(m) \simeq \varphi(n,m)$. The condition (i)
of the definition requires that $\varphi_n$ enumerates a $2^{-n}$-net of the space $(X, d)$, namely the set

$$\text{ran}(S \circ \varphi_n) = \{s_{\varphi(n,m)} : m \in \mathbb{N}, \varphi(n,m) \downarrow\}.$$ 

This $n$th stratum of the stratification $\varphi$ covers the universe $X$ in such a way that
any point $x \in X$ belongs to at most $b$ balls of form $\text{Ball}_2^{-n}(s_{\varphi(n,m)})$ with $\varphi_n(m) \downarrow$, by condition (ii). The function $\varphi$ is allowed to be partial in order to include also the case of
finite strata.

All the EMSs specified as examples at the beginning of Section 2 have finitary stratifications:

1. For a discrete space with a finite universe, $X = \{x_0, x_1, \ldots, x_k\}$, let all $\varphi_n$ injectively
enumurate the index set $\{0, 1, \ldots, k\}$. If $X = \{x_n : n \in \mathbb{N}\}$ is the universe of a discrete
space, $x_i \neq x_j$ for $i \neq j$, and the skeleton $S_X$ is injective, the function $\varphi(n, m) = m$ is
a finitary stratification. In both cases, we have the ambiguity $b = 1$.

2. For $(\mathbb{R}, d_{\mathbb{R}}, S_\mathbb{Q})$, the $n$th stratum can be taken as $\mathbb{D}_n = \{k \cdot 2^{-n} : k \in \mathbb{Z}\}$, the set of
dyadic rationals of precision $n$. Here the ambiguity $b = 2$ is possible.

3. For the Baire space $(\mathbb{B}, d_{\mathbb{B}}, S_\mathbb{B})$, let the $n$th strata be defined in such a way that it
holds $\text{ran}(S_\mathbb{Q} \circ \varphi_n) = \{(\beta_0, \beta_1, \ldots, \beta_n, 0, 0, \ldots) : \beta_0, \beta_1, \ldots, \beta_n \in \{0, 1\}\}$. This is always
a finite set. The related stratification has the ambiguity 1.

4. Over $(\mathbb{F}, d_{\mathbb{F}}, S_{\mathbb{F}})$, let $\text{ran}(S_{\mathbb{F}} \circ \varphi_n) = \{(v_0, v_1, \ldots, v_n, 0, 0, \ldots) : v_0, v_1, \ldots, v_n \in \mathbb{N}\}$. Again,
the ambiguity 1 is reachable. This example shows that the existence of a finitary stratification
does not imply the local compactness of the underlying metric space.

For the examples 3 and 4, it is essential that the Baire metric is defined in the
special way we specified in Section 2. Remark also that, given a finitary stratification
of an ambiguity $b$ over an EMS $\mathcal{X}$, one easily gets such one of ambiguity $b^k$ for the
product space $\mathcal{X}^{(k)}$ as it has been specified in Section 2.

Every finitary stratification $\varphi$ defines canonically a skeleton $S_\varphi$ which is computationally equivalent to $S$, the skeleton of the underlying EMS $\mathcal{X}$. If $\varphi$ is a total function,
then let $S_\varphi = S \circ \varphi(\pi_1^2(n), \pi_2^2(n))$. If the stratification is a partial function, one has first
to define a total function with the same range (which is not necessarily a stratification).
Then $S_\varphi$ can be defined like above.

Unfortunately, the property of having a finitary stratification is not invariant under
the equivalence of EMSs. We shall say that an EMS $\mathcal{X}$ admits a finitary stratification
if there is an EMS $X'$ which is equivalent to $X$ and has a finitary stratification of some finite ambiguity $b \in \mathbb{N}_+$. 

**Proposition 6.** Let the EMS $X$ admit a finitary stratification. Then for every function $f \in \text{Compfu}(X, X')$, where $X'$ is an arbitrary EMS, it holds $\text{dom}(f) \in \Pi^0_2$.

It is enough to prove the proposition for an EMS $X = (X, d, S)$, with $S = (s_n)_{n \in \mathbb{N}}$ and a finitary stratification $\phi$, say of ambiguity $b \in \mathbb{N}_+$. We consider an OTM $M$ computing a function $f \in \text{Compfu}(X, X')$, for some EMS $X'$. Without loss of generality, we suppose that $f$ is constant, even $f(x) = s'_0$, the first element of the skeleton of $X'$, for all $x \in \text{dom}(f)$. By $M_{\sigma\downarrow}$ is indicated that the machine halts on input $n$ with the oracle sequence $\sigma$. Moreover, we can suppose that $M_{\sigma\downarrow} \downarrow$ implies that $M_{\sigma\downarrow}(n') \downarrow$ and $M_{\sigma\downarrow}(n') = 0$, for all inputs $n' \leq n$.

So we have

$$\text{dom}(f) = \{ x \in X : \text{for all } n \in \mathbb{N} \text{ and all } \sigma \in \text{CF}_x, M_{\sigma\downarrow}(n) = 0 \}.$$ 

To obtain a $\Pi^0_2$-representation of $\text{dom}(f)$, we deal with the finite initial parts of index sequences corresponding to standard Cauchy functions of a special kind, for the elements $x \in X$.

By $\mathbb{N}^*$, the set of all finite sequences of natural numbers is denoted as usual. Let an injective total standard numbering $\nu_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}^*$ be fixed. Any finite sequence $\tau = (i_0, i_1, \ldots, i_l) \in \mathbb{N}^*$ defines a set of points within the space $X$,

$$\text{set}(\tau) = \bigcap_{j=0}^l \text{Ball}_{2^{-j}}(s_{i_j}).$$

By a regular finite index sequence, we understand a sequence

$$\tau = (i_0, i_1, \ldots, i_l) \in \mathbb{N}^*$$

such that $i_j \in \text{ran}(\phi_\lambda)$, for all $\lambda \in \{0, 1, \ldots, l\}$, and $\text{set}(\tau) \neq \emptyset$. This means that any index $i_j$ within $\tau$ defines an element of the $\lambda$th stratum with respect to the stratification $\phi$, and the intersection of the related open balls is not empty.

Let $\text{Rfis}$ denote the set of all regular finite index sequences.

**Lemma 10.** The set $\nu_{\mathbb{N}}^{-1}(\text{Rfis})$ is r.e., and there is a recursive total function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\text{set}(\nu_{\mathbb{N}}^{-1}(m)) = \bigcup_{k \in \mathbb{N}} \text{ball}_{\psi(m, k)} \text{ for all } m \in \mathbb{N}.$$ 

Indeed, for $\nu_{\mathbb{N}}^{-1}(m) = \tau = (i_0, i_1, \ldots, i_l)$, it can effectively be recognized if $i_\lambda \in \text{ran}(\phi_\lambda)$ for all $\lambda \in \{0, 1, \ldots, l\}$. Moreover, $\text{set}(\tau) \neq \emptyset$ iff there are a number $k$ and a point $s_n$ of the skeleton such that $q_k > 0$ and

$$d(s_n, s_{i_\lambda}) + q_k < 2^{-\lambda} \text{ for all } \lambda \in \{0, 1, \ldots, l\}.$$
This can also effectively be recognized by means of a recursive enumeration of the set $D_s$ of skeleton $S$. Thus, we have shown that $v_{∧}^4(\text{RFis})$ is r.e.

However, it always holds

$$\text{set}(\tau) = \bigcup \{ \text{ball}_{(k,n)}: k,n \in \mathbb{N}, d(s_n,s_{i_k}) + q_k < 2^{-\lambda} \text{ for all } \lambda \in \{0,1,\ldots,l\} \}.$$ 

So there is a recursive total function $\psi: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $\psi(m, \cdot)$ enumerates all Cantor numbers $\langle k, n \rangle$ belonging to the set on the right hand side of this equation. $\psi$ can be assumed to be total, since ball$_{(k,n)} = \emptyset$ if $q_k \leq 0$.

For a finite sequence $\tau = (i_0, i_1, \ldots, i_l) \in \text{RFis}$ and $n \in \mathbb{N}$, we write $\mathcal{M}^\tau(n) \downarrow$ if, for the oracle sequence $\bar{\tau} = (i_0, i_1, \ldots, i_l, i_1, i_1, \ldots)$ (and then for all oracle sequences $\bar{\tau}$ which begin with the initial part $\tau$), it holds:

$\mathcal{M}^\tau(n) \downarrow$, and the machine puts only oracle queries of form “$m$?” with $m \leq l$, in the course of computing $\mathcal{M}^\tau(n)$.

Proposition 6 is proved by showing that

$$\text{dom}(f) = \bigcap_{n \in \mathbb{N}} A_n, \text{ where } A_n = \bigcup \{ \text{set}(\tau): \tau \in \text{RFis}, \mathcal{M}^\tau(n) \downarrow \}.$$ 

Since $v_{∧}^4(\text{RFis})$ is r.e. and the condition $\mathcal{M}^\tau(n) \downarrow$ can effectively be recognized too (by simulating the work of the machine $\mathcal{M}$ on the input $n$ and the recursive oracle sequence $\bar{\tau}$), by Lemma 10 there is a recursive function $\varphi^\tau: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that

$$\bigcap_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \text{ball}_{\varphi^\tau(n,m)}.$$ 

So, dom$(f) \in \Pi^0_2$ follows from the representation of dom$(f)$ stated above.

The inclusion dom$(f) \subseteq \bigcap_{n \in \mathbb{N}} A_n$ is easily obtained. For any $x \in X$, there is an infinite sequence $\bar{\tau} = (i_j)_{j \in \mathbb{N}}$ such that all $i_j$ belong to the $j$th stratum of the underlying finitary stratification $\varphi$, i.e., $i_j \in \text{ran}(\varphi_j)$, and also $x \in \bigcap_{j \in \mathbb{N}} \text{Ball}_{2^{-j}}(s_{i_j})$. Then $\bar{\tau} \in \text{CF}_x$.

If $x \in \text{dom}(f)$, then for any $n \in \mathbb{N}$ there is a finite initial part $\tau$ of $\bar{\tau}$ such that $\mathcal{M}^\tau(n) \downarrow$. Moreover, $\tau \in \text{RFis}$. Thus, $x \in \bigcap_{n \in \mathbb{N}} A_n$.

To show the converse inclusion, let $x \in \bigcap_{n \in \mathbb{N}} A_n$. Thus, for any $n \in \mathbb{N}$ there is a finite sequence $\tau_n = (i_0, i_1, \ldots, i_n) \in \text{RFis}$ such that $\mathcal{M}^{\tau_n}(n) \downarrow$ and $x \in \text{set}(\tau_n)$. Moreover, we can suppose that $l_n \geq n$. So we have an infinite set $T_x = \{ \tau_n: n \in \mathbb{N} \} \subseteq \text{RFis}$.

There is a sequence $\bar{\tau} \in \text{CF}_x$ such that infinitely many sequences $\tau_n \in T_x$ are initial parts of $\bar{\tau}$. Indeed, otherwise we would have infinitely many $\tau_{nj} \in T_x$, $j \in \mathbb{N}$, which were mutually incomparable with respect to the initial-part relation. This would yield a level $n$ of the stratification with more than $b$ balls $\text{Ball}_{2^{-b}}(s_{\varphi(n,m_k)}), k = 0,1,\ldots,b,\ldots$, each of which containing the element $x$. This is a contradiction to the property (ii) of the stratification of ambiguity $b$.

Thus, $\mathcal{M}^\tau(n) \downarrow$ for infinitely many $n$ and, due to our suppositions on the work of machine $\mathcal{M}$, $\mathcal{M}^\tau(n) = 0$, for all $n \in \mathbb{N}$. So we have shown that $x \in \text{dom}(f)$.

The notion of finitary stratification has been defined in such a way that the proof of Proposition 6 remains fairly clear and easily understandable. It should be noticed, however, that the proof also works for a generalized notion.
By a generalized finitary stratification of ambiguity $b \in \mathbb{N}_+$ over the EMS $\mathcal{X} = (X,d,S)$, we understand a recursive function $\varphi : \mathbb{N}^2 \rightharpoonup \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

(i) $X = \bigcup_{m \in \mathbb{N}, \varphi(n,m)\downarrow} \text{ball}_{\varphi(n,m)}$;

(ii) $\bigcap_{m \in M} \text{ball}_{\varphi(n,m)} = \emptyset$, for all sets $M \subseteq \{m : \varphi(n,m)\downarrow\}$ with $\text{card}(M) > b$;

(iii) if $\tau = (i_0,i_1,\ldots) \in \Gamma$ such that $\varphi(n,i_n)\downarrow$ for all $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} \text{ball}_{\varphi(n,i_n)} \neq \emptyset$, then

$$\lim_{n \to \infty} q_{\pi^2_1 \circ \varphi(n,i_n)} = 0.$$ 

Here the meshes of the $n$th stratum have the radii $q_{\pi^2_1 \circ \varphi(n,i_n)}$, $m \in \mathbb{N}$. Condition (iii) assures that the mesh radii corresponding to a sequence $\tau = (i_n)_{n \in \mathbb{N}}$ converge to 0 if $\tau$ defines a non-empty set $\bigcap_{n \in \mathbb{N}} \text{ball}_{\varphi(n,i_n)}$. Then this intersection is a singleton $\{x\}$, with $x \in X$.

We shall say that an EMS $\mathcal{X}$ admits a generalized finitary stratification if there is an equivalent EMS having such one of some finite ambiguity $b \in \mathbb{N}_+$.

Similarly to the proof of Proposition 6 but with some more technical effort, one shows

**Corollary 1.** If the EMS $\mathcal{X}$ admits a generalized finitary stratification, then for any function $f \in \text{Compfu}(\mathcal{X}, \mathcal{X}')$, with an arbitrary EMS $\mathcal{X}'$, it holds $\text{dom}(f) \in \Pi^a_2$.

Proposition 5 and Corollary 1 yield

**Theorem 2.** Let the EMS $\mathcal{X}$ admit a generalized finitary stratification. Then the domains of computable functions from $\mathcal{X}$ to any EMS $\mathcal{X}'$ are just the $\Pi^a_2$-sets over $\mathcal{X}$.

Unfortunately, we do not know an example of an EMS which admits a generalized finitary stratification but no finitary stratification. Also it is not yet known if the admissance of a (generalized) finitary stratification is necessary for an EMS on which all domains of computable functions belong to $\Pi^a_2$.

We close this section with remarks on the location of some special sets within the TAH. From topology the following result is well-known.

**Fact.** In a perfect Polish space $(X,d)$, there is no countable dense $G_\delta$-set.

So we have

**Lemma 11.** Let $\mathcal{X} = (X,d,S)$ be an EMS over a perfect Polish space $(X,d)$. Then neither $\text{ran}(S)$ nor any countable set including $\text{ran}(S)$ belongs to $\Pi^a_2$.

For example, the lemma applies to the set of all computable elements, $\text{Compel}(\mathcal{X})$. Thus, if $\mathcal{X}$ admits a generalized finitary stratification, neither $\text{ran}(S)$ nor $\text{Compel}(\mathcal{X})$, nor any other countable superset of $\text{ran}(S)$, can occur as domain of a computable function. On the other hand,

$$\text{ran}(S) = \bigcup_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}, x \neq x_k} \overline{\text{Ball}}_{2^{-k}}(s_m).$$

This is a $\Sigma^a_2$-set if $\mathcal{X}$ is a SEMS.
Finally, we consider the set generated by an EDS \((\eta, \delta)\), i.e., the set \(\text{ran}(S \circ \eta)\). Lemma 8 says that \(\text{ran}(S \circ \eta) \in \Delta^0_\alpha\). If the underlying space is perfect Polish, then \(\text{ran}(S \circ \eta) \notin \Sigma^0_\alpha\). Indeed, if \(\text{ran}(S \circ \eta) \in \Sigma^0_\alpha\), this would be a non-empty open set, hence it could not be countable — contradiction. It is possible that \(\text{ran}(S \circ \eta) \in \Pi^0_\alpha\), p.e., \(\mathbb{N}\) or \(\mathbb{Z}\), or the set \(\{(n,0,0,\ldots): n \in \mathbb{N}\}\) in \(\mathbb{F}\). On the other hand, in compact perfect Polish spaces, like \(\mathbb{B}\) or the real interval \([0,1]\) with the natural distance function, it always holds \(\text{ran}(S \circ \eta) \notin \Pi^0_\alpha\).

5. Standard representations of the real numbers

Computability over non-countable object domains is usually introduced and treated by means of representations. In particular in Weihrauch’s type 2 theory of computability, representations are the essential tool for transferring notions of computability and constructivity from domains of sequences of discrete objects, like \(\mathbb{B}\) or \(\mathbb{F}\), to other type 2 domains, like \(\mathbb{R}\).

In our pocket model of approximate computability introduced in the preceding sections, we avoided the explicit use of representations. Our notions are only implicitly based on the rather natural representation of elements \(x\) of an EMS \(\mathcal{X}\) by index sequences from \(\mathbb{C}\subseteq \mathbb{F}\). So, and since we also studied computability of partial functions between different EMSs \(\mathcal{X}\) and \(\mathcal{X}'\), we are able to evaluate representations from the computability point of view. This idea will now be applied to characterize the standard representations of the real numbers.

In the sequel, by \(\mathcal{X}\) we always denote an arbitrary of the sets \(\mathbb{B}\) or \(\mathbb{F}\), as well as the EMSs over them, which have been specified in Section 2.

A representation (of the real numbers) is a partial function \(g: \mathcal{X} \rightharpoonup \mathbb{R}\), where \(\mathcal{X} \in \{\mathbb{B}, \mathbb{F}\}\) and \(\text{ran}(g) = \mathbb{R}\).

Then a real function \(f: \mathbb{R} \rightharpoonup \mathbb{R}\) is said to be computable with respect to \(g\) if there is a partial function \(g \in \text{Compfu}(\mathcal{X}, \mathcal{X})\) such that

\[ f \circ g(x) = g \circ g(x) \quad \text{for all } x \in \text{dom}(f \circ g).\]

Notice that this implies that \(\text{dom}(f \circ g) \subseteq \text{dom}(g \circ g)\). So any representation determines a related class of computable real functions, and the problem arises to characterize the class of those representations which yield a natural concept of approximate computability over the reals.

In order to compare representations with each other, the relation of (effective) reducibility is introduced. For representations \(g: \mathcal{X} \rightharpoonup \mathbb{R}\) and \(g': \mathcal{X}' \rightharpoonup \mathbb{R}\), we say that \(g\) is reducible to \(g'\) (briefly: \(g \leq g'\)) if there is a function \(g \in \text{Compfu}(\mathcal{X}, \mathcal{X}')\) such that

\[ g(x) = g' \circ g(x) \quad \text{for all } x \in \text{dom}(g).\]

\(g\) and \(g'\) are called equivalent (briefly: \(g \equiv g'\)) if both \(g \leq g'\) and \(g' \leq g\). One easily shows that equivalent representations determine the same class of computable real functions. By several reasons, cf. [12, 14, 5], the representation by index sequences

...corresponding to standard Cauchy functions seems to be a very natural one. So our problem consists in characterizing just those representations which are equivalent to the normed Cauchy representation.

More precisely, the normed Cauchy representation is defined to be the function $q_{ac} : \mathbb{F} \to \mathbb{R}$, where

$$\text{dom}(q_{ac}) = \{ \sigma \in \mathbb{F} : \sigma \in \mathbb{CF}^r_q \text{ for some } r \in \mathbb{R} \}$$

and

$$q_{ac}(\sigma) = r \text{ if } \sigma \in \mathbb{CF}^r_q.$$  

By a standard representation, we understand a representation which is equivalent to $q_{ac}$.

A first attempt to characterize the class of standard representations in a more direct, natural way was made by Hertling [5]. He showed that a representation $q : \mathbb{F} \to \mathbb{R}$ is a standard representation if it makes the structure $(\mathbb{R}; 0, 1; +, -, *, /, \text{NormLim}; <)$ effective, i.e., the basic elements, operations and relation become computable with respect to $q$. So this structure is effectively categorical: up to equivalence there is only one representation which makes it effective. Unfortunately, $\text{NormLim} : \mathbb{R}^N \to \mathbb{R}$ is an infinitary operation over $\mathbb{R}$; its computability has to be understood in the way described in Section 2. So it is approximate in a double sense, or with respect to double sequences. Hertling could also show that the ordered field of real numbers, $(\mathbb{R}; 0, 1; +, -, *, /; <)$, is not effectively categorical. It is still unknown if there is a finitary structure at all over $\mathbb{R}$ which is effectively categorical. For all details of these results, the reader is referred to [5]. Brattka [1] has applied and generalized these notions to his many-sorted approach to computability over topological structures.

Now we are going to characterize the standard representations within our framework. First the computability of the normed Cauchy representation is realized.

**Lemma 12.** $q_{ac} \in \text{Compfu}(\mathbb{F}, \mathbb{R})$.

By definition, an OTM $\mathcal{M}$ computing $q_{ac}$ has to generate approximately an output sequence $\sigma'' \in \mathbb{CF}^r_q$, for any oracle sequence $\sigma \in \mathbb{CF}^r_q$ with $\sigma' \in \mathbb{CF}^r_q$, $r \in \mathbb{R}$. This is simply done by putting out $\sigma'' = \sigma'$. Since the skeleton $S^r_q$ is assumed to be a standard numbering of the ultimately 0-stationary sequences, it is effective and $\sigma'$ can successively be generated from the oracle $\sigma$. The only difficulty is to guarantee that for all oracle sequences $\sigma \in \mathbb{CF}^r_q$ with $\sigma' \notin \bigcup_{r \in \mathbb{R}} \mathbb{CF}^r_q$ there is an input $n \in \mathbb{N}$ such that $\mathcal{M}^\sigma(n)$ is undefined. To this purpose, if $\sigma' = (i_0, i_1, \ldots)$, for any $n \in \mathbb{N}$, $\mathcal{M}$ first checks if $\bigcap_{m=0}^n \text{Ball}_{2^{-m}}(q_{i_m}) \neq \emptyset$. Only when the non-emptyness is recognized, let the machine halt with the output $\sigma'(n)$. The condition holds if and only if there is a pair $(k, l) \in \mathbb{N}^2$ satisfying $q_l < 2^{-n}$ and $d_q(q_k, q_{i_m}) < q_l$ for all $m \in \{0, 1, \ldots, n\}$. This can be recognized by means of an effective enumeration of the set $D_<$ for the skeleton $v_Q$ of $\mathbb{R}$.

Intuitively, the normed Cauchy representation is also effective in the converse direction, in going from a real number $r$ to an index sequence $\sigma' \in q_{ac}^{-1}(r)$. At first...
glance, such a computation seems to require a nondeterministic computation device, since \( q_{sc} \) is not injective. On the other hand, according to the idea of approximate computability, the numbers \( r \) cannot be given to a machine as entities, but only by index sequences \( \sigma \in \text{CF}_r^{(v)} \). And quite generally, the variety of the classes \( \text{CF}_r^{(S)} \), for a given EMS \( \mathcal{X} = (X,d,S) \) and \( x \in X \), involves already an element of nondeterminism which is sufficient to compute relations (i.e., many-valued functions) by means of ordinary, deterministic OTMs.

Let \( \mathcal{X} = (X,d,S) \) and \( \mathcal{X}' = (X',d',S') \) be EMSs and \( q \) a relation from \( X \) into \( X' \). More precisely, \( q \subseteq X \times X' \); let \( \text{dom}(q) = \{ x : \text{there is an } x' \text{ with } (x,x') \in q \} \); \( \text{ran}(q) \) and other notations are analogously understood.

The relation \( q \) is said to be (approximately) computable if there is an OTM \( \mathcal{M} \) such that:

1. for all \( x \in \text{dom}(q) \) and all index sequences \( \sigma \in \text{CF}_x^{(S)} \), \( M^\sigma(n) \) always exists and it holds \( S' \circ (M^\sigma(n))_{n \in \mathbb{N}} \in \overline{\text{CF}}_{x'}^{(S')} \), for some \( x' \) with \( (x,x') \in q \);
2. for all \( (x,x') \in q \) there is a sequence \( \sigma \in \text{CF}_x^{(S)} \) such that \( S' \circ (M^\sigma(n))_{n \in \mathbb{N}} \in \overline{\text{CF}}_{x'}^{(S')} \);
3. for all \( x \notin \text{dom}(q) \) and all \( \sigma \in \text{CF}_x^{(S)} \), there is an input \( n \in \mathbb{N} \) for which \( M^\sigma(n) \) is undefined.

Let \( \text{Comprel}(\mathcal{X},\mathcal{X}') \) denote the set of all computable relations from \( \mathcal{X} \) into \( \mathcal{X}' \). If \( q \) is even a function, i.e., to any \( x \in X \) there is at most one \( x' \) with \( (x,x') \in q \), then our requirements coincide with those of the computability for functions, given in Section 2.

Thus,

\[
\text{Comprel}(\mathcal{X},\mathcal{X}') \cap \{ q : q \text{ is a function} \} = \text{Compfu}(\mathcal{X},\mathcal{X}').
\]

Our notion of computability for relations is not closed under composition, but if \( q_1 \in \text{Comprel}(\mathcal{X},\mathcal{X}') \) and \( q_2 \in \text{Comprel}(\mathcal{X}',\mathcal{X}'') \), then there is a relation \( q_0 \in \text{Comprel}(\mathcal{X},\mathcal{X}'') \) with \( q_0 \subseteq q_2 \circ q_1 \) and \( \text{dom}(q_0) = \text{dom}(q_2 \circ q_1) \).

We do not want to develop a theory of computability for relations here. In this paper, we shall only deal with computations of relations which are right-inverse to representations.

More precisely, by an inversion of a representation \( q : \mathbb{X} \rightarrow \mathbb{R} \), we understand a relation \( q^{-} \) from \( \mathbb{R} \) into \( \mathbb{X} \) such that

\[
q \circ q^{-} = \text{id}_{\mathbb{R}}.
\]

**Lemma 13.** There is an inversion \( q_{sc}^{-} \) of \( q_{sc} \) which is computable, i.e., \( q_{sc}^{-} \in \text{Comprel} (\mathbb{R}, \mathbb{X}) \).

The proof is still easier than that of Lemma 12, since \( q_{sc} \) is surjective. We take \( q_{sc}^{-} \) as the complete inverse of \( q_{sc} : q_{sc}^{-} = \{(r, \sigma) \in \mathbb{R} \times \mathbb{F} : \sigma \in \text{CF}_r^{(v)} \} \). To compute \( q_{sc}^{-} \), let an OTM \( M \) work in such a way that \( (M^\sigma(n))_{n \in \mathbb{N}} \in \overline{\text{CF}}_\sigma^{(S)} \); for example,

\[
M^\sigma(n) = S_{\sigma}^{-1}(\sigma(0), \sigma(1), \ldots, \sigma(n), 0, 0, \ldots).
\]
This can effectively be done, since the standard numbering $S_{\mathcal{F}}$ is supposed to be effective.

To prepare the proof of the next proposition, we need a technical lemma concerning the existence of a special OTM for the name spaces $\mathbb{B}$ and $\mathbb{F}$.

For $X \in \{\mathbb{B}, \mathbb{F}\}$, an index sequence $\sigma = (i_0, i_1, i_2, \ldots)$ is said to be regular if, for all $n \in \mathbb{N}$, $S_X(i_n) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots) \in X$, where

$$x_0, x_1, \ldots, x_n \in \begin{cases} \{0, 1\} & \text{if } X = \mathbb{B}, \\ \mathbb{N} & \text{if } X = \mathbb{F}. \end{cases}$$

Remember that both the skeletons $S_{\mathbb{B}}$ and $S_{\mathcal{F}}$ are defined to be injective. Thus, for any sequence $x \in X$ there is exactly one regular index sequence $\sigma_x \in \text{CF}_x^{(S_X)}$, namely $\sigma_x = (i_0, i_1, i_2, \ldots)$, where $S_X(i_n)$ begins with just the first $n + 1$ elements $x_0, x_1, \ldots, x_n$ within the sequence $x$ and leaves the following elements equal to 0.

**Lemma 14.** For $X \in \{\mathbb{B}, \mathbb{F}\}$, there is an OTM $\mathcal{M}_{\text{reg}}$ computing the identical function $\text{id}_X$ in such a way that, for all $x \in X$ and for all oracles $\sigma \in \text{CF}_x^{(S_X)}$, $(\mathcal{M}_{\text{reg}}^\sigma(n))_{n \in \mathbb{N}}$ is just the regular index sequence in $\text{CF}_x^{(S_X)}$.

Indeed, using the first $n + 1$ elements of the oracle sequence $\sigma \in \text{CF}_x^{(S_X)}$, the uniquely determined index $i_n$ of the regular sequence in $\text{CF}_x^{(S_X)}$ can effectively be put here.

$\mathcal{M}_{\text{reg}}$ could be called a funnel machine for the space $X$. It unifies the several oracles $\sigma \in \text{CF}_x^{(S_X)}$ to the only regular index sequence related to the same element $x \in X$.

We are now able to state a first, provisional characterization of the standard representations.

**Proposition 7.** For every representation $g : X \rightarrow \mathbb{R}$,

(i) $g \leq g_{\text{ac}}$ iff there is an extension $\bar{g} \supseteq g$ such that $\bar{g} \in \text{Compfu}(X, \mathbb{R})$;

(ii) $g_{\text{ac}} \leq g$ iff there is an inversion $g^{-1}$ of $g$ such that $g^{-1} \in \text{Comprel}(\mathbb{R}, X)$ holds.

We first prove the assertion (i). If $g \leq g_{\text{ac}}$, there is a function $\hat{g} \in \text{Compfu}(X, \mathbb{F})$ such that $g(x) = g_{\text{ac}} \circ \hat{g}(x)$, for all $x \in \text{dom}(g)$. Thus, for $\hat{g} = g_{\text{ac}} \circ g$ the requirement is fulfilled.

Now let $\hat{g} \in \text{Compfu}(X, \mathbb{R})$ be an extension of $g$. Then there is a relation $\varrho_0 \in \text{Comprel}(X, \mathbb{F})$ such that $\varrho_0 \subseteq \varrho_{\text{ac}}^{-1} \circ \hat{g})$. dom($\varrho_0$) = dom($\varrho_{\text{ac}}^{-1} \circ \hat{g}$) = dom($\hat{g}$) and $\varrho_0 \subseteq \varrho_{\text{ac}} \circ \varrho_0$. For $x \in \text{dom}(g)$ and $(x, r) \in \varrho_{\text{ac}} \circ \varrho_0$ we have $r = g(x)$. Unfortunately, $\varrho_0$ is not necessarily a function.

We get a computable function $g$, with $g \subseteq \varrho_0$ and dom($g$) = dom($\varrho_0$) = dom($\varrho_{\text{ac}}^{-1} \circ \hat{g}$), by composing the funnel machine $\mathcal{M}_{\text{reg}}$ from Lemma 14, for the space $X$, with an OTM $\mathcal{M}_0$ computing the relation $\varrho_0$. Then, for any oracle sequence $\sigma \in \text{CF}_x^{(S_X)}$, $x \in \text{dom}(g)$, the machine $\mathcal{M}_{\text{reg}}$ puts here the corresponding regular sequence $\sigma_x \in \text{CF}_x^{(S_X)}$. This is transformed into just one sequence $\sigma' \in \text{CF}_y^{(S_y)}$, $y \in \varrho_{\text{ac}}^{-1}(\hat{g}(x))$.

To show (ii), let first $g_{\text{ac}} \leq g$. Thus, there is a function $g \in \text{Compfu}(\mathbb{F}, X)$ such that $g_{\text{ac}}(x) = g \circ g(x)$, for all $x \in \text{dom}(g_{\text{ac}})$. Let $g^{-1} = g \circ g_{\text{ac}}^{-1}$. Then $g^{-1} \in \text{Comprel}(\mathbb{R}, X)$ and
\[ q \circ g^- = q \circ (g \circ g_{ac}^-) = (q \circ g) \circ g_{ac}^- \supseteq g_{ac} \circ g_{ac}^- = \text{id}_R. \] Moreover, if \( r \in \mathbb{R} \) and \((r, x) \in g_{ac}^-\), then \( g(x) \) exists and \( q \circ g(x) = g_{ac}(x) = r \). This means that \( q \circ g^- = \text{id}_R \).

On the other hand, let \( q^- \) be an inversion of \( q \) and \( g^- \in \text{Comprel}(\mathbb{R}, \mathbb{X}) \). Then \( \text{id}_R = q \circ g^- \), i.e., \( g_{ac}^- = \text{id}_R \circ g_{ac}^- = q \circ (q^- \circ g_{ac}) \supseteq q \circ g_0 \), with a computable relation \( g_0 \subseteq g^- \circ g_{ac} \). \( \text{dom}(q_0) = \text{dom}(g^- \circ g_{ac}) = \text{dom}(g_{ac}) \). Again, applying first a funnel machine \( \mathcal{M}_{\text{eg}} \) over the space \( F \), we get a function \( g \in \text{Compfu}(F, \mathbb{X}) \) which is included in \( g^- \circ g_{ac} \) and satisfies \( g_{ac}(x) = q \circ g(x) \) for all \( x \in \text{dom}(g_{ac}) \).

From Proposition 7, we immediately obtain

**Corollary 2.** A representation \( q: \mathbb{X} \rightarrow \mathbb{R} \) is a standard representation iff there are both a computable extension \( \tilde{q} \supseteq q \), \( \tilde{q} \in \text{Compfu}(\mathbb{X}, \mathbb{R}) \), and a computable inversion \( q^- \) of \( q \), \( q^- \in \text{Comprel}(\mathbb{R}, \mathbb{X}) \).

Moreover, one can show that the range of a computable inversion always includes a \( \Pi_2^a \)-set being also the range of a computable inversion.

**Proposition 8.** Let \( q^- \) be an inversion of a representation \( q: \mathbb{X} \rightarrow \mathbb{R} \). If \( q^- \in \text{Comprel}(\mathbb{R}, \mathbb{X}) \), then there is a \( \Pi_2^a \)-set \( A \subseteq \text{ran}(q^-) \) which is the range of a computable inversion \( q'^- \subseteq q^- \) of \( q \).

For the proof, let a finitary stratification \( \varphi \) of the EMS \( \mathbb{R} \) be fixed. We apply the notations introduced in the proof of Proposition 6. Let \( \mathcal{M} \) be an OTM computing the relation \( q^- \). Without loss of generality, we suppose that, for all finite sequences \( \tau \in \mathbb{N}^* \), \( \mathcal{M}^\tau(n) \downarrow \) iff \( \mathcal{M}^\tau(n') \downarrow \) for all \( n' \leq n \).

By Ball\({ }^\mathbb{X}(S_\mathbb{X}(n))\), we denote the open ball within the space \( \mathbb{X} \), around the skeleton point \( S_\mathbb{X}(n) \) with radius \( q \). Let

\[ A = \cap_{n \in \mathbb{N}} A_n \quad \text{with} \quad A_n = \bigcup \left\{ \cap_{\tau = 0}^n \text{Ball}^\mathbb{X}_i(\mathcal{M}^\tau(n)); \ \tau \in \text{Rfis}, \ \mathcal{M}^\tau(n) \downarrow \right\}. \]

By a similar technique as used in the proof of Proposition 6, one shows that for any \( r \in \mathbb{R} \) there is an \( x \in A \) with \((r, x) \in q^- \). Conversely, if \( x \in A \), there is an \( r \in \mathbb{R} \) with \((r, x) \in q^- \). Thus, \( \varphi_{\mathcal{M}} \) is also a surjection onto \( \mathbb{R} \). \( A \in \Pi_2^a \) follows as usual by means of an effective enumeration of the set \( D_< \), for the space \( \mathbb{X} \in \{ \mathbb{B}, \mathbb{F} \} \).

Moreover, using an enumeration of \( D_< \), one defines an OTM \( \mathcal{M}' \) for which any oracle sequence \( \sigma \in \text{CF}_r^{(S_\mathbb{X})} \), \( r \in \mathbb{R} \), successively generates a sequence \( \vec{\tau} = (i_\tau)_{\tau \in \mathbb{N}} \), such that always \( i_\tau \in \text{ran}(\varphi_\tau) \) and \( r \in \bigcap_{i \in \mathbb{N}} \text{Ball}^\mathbb{X}_{i-1}(s_i) \), and then computes \( \mathcal{M}'(n) \). Thus, any initial part \( \tau \) of \( \vec{\tau} \) belongs to Rfis, and \( \mathcal{M}' \) computes an inversion \( q'^- \subseteq q^- \) of \( q \) such that \( \text{ran}(q'^- 0) = A \).

Now let, for a representation \( q \), \( \tilde{q} \) be an extension and \( q^- \) be an inversion according to Corollary 2, and let the \( \Pi_2^a \)-set \( A \) be chosen according to Proposition 8. We consider the function \( g = \tilde{q} \circ \text{id}_A \). It holds \( g \in \text{Compfu}(\mathbb{X}, \mathbb{R}) \) and \( g \supseteq q \). So \( g \) is a computable representation and a restriction of \( q \). Moreover, \( q'^- 0 \) is also an inversion of both \( q \) and \( \tilde{q} \). So we have
Theorem 3. A surjective function $g : X \to R$, $X \in \{B, F\}$, is a standard representation if there are computable surjections $\overline{g}, \overline{\delta} : X \to R$, $\overline{g}, \overline{\delta} \in \text{Compfu}(X, R)$, such that $\overline{g} \subseteq g \subseteq \overline{\delta}$ and there is a computable inversion $\overline{g}^{-} \subseteq \overline{\delta}^{-} \in \text{Comprel}(R, X)$.

Our characterization of the standard representations according to Theorem 3 looks perhaps rather complicated. This is caused, however, by the usual definition of the notion of standard representation we have taken over from [5]. In particular, there is no restrictive requirement on the domains of standard representations. Therefore, this notion includes a huge class of functions, as we are going to show now.

One easily defines a computable extension of the normed Cauchy representation, $\overline{g}_{ac} : F \to R$, in such a way that it is a standard representation and, moreover, the difference set $\text{dom}(\overline{g}_{ac}) \setminus \text{dom}(g_{ac})$, contains $2^{\aleph_0}$ elements. Then, by Theorem 3, any function $g$ with $g_{ac} \subseteq g \subseteq \overline{g}_{ac}$ is a standard representation, too. So we have

Lemma 15. There are at least $2^{\aleph_0}$ standard representations.

By our impression, it should be sufficient for the aims of type 2 theory of computability to accept only computable functions as standard representations. This means that the domains have to be $\Pi^0_2$-sets. The standard representations in this restricted sense are just those surjective functions $g : X \to R$ which are computable and have inversions which are also computable (as relations). The class of these functions looks rather natural, and it is countable.

Finally, we remember that some disadvantages of the representation-based approaches to computability over the real numbers are already caused by the topological differences between $B$ and $F$ as name spaces and $R$ as the object space. For example, one easily sees that there is no injective standard representation. Indeed, an injective function $g : \overline{\mathbb{N}} \to \overline{R}$, where both $\overline{\mathbb{N}}$ and the only possible inversion $g^{-} = g^{-1}$ were computable, would be a homeomorphism between the totally disconnected set $\text{dom}(g)$, as subspace of $\overline{\mathbb{N}}$, and the connected space $\overline{R}$. This would be a contradiction. Whereas the universe $\overline{F}$ occurs as the domain of a standard representation, there is no standard representation $g$ with $\text{dom}(g) = \overline{B}$. This follows since the domain of a continuous surjective function onto $\overline{R}$ cannot be compact.

So, on the one hand, the spaces $B$ and $F$ do not seem to be very well suited to serve as name spaces for the real numbers. On the other hand, they are always involved if approximate computability is defined by means of machines operating on discrete object domains, like $\mathbb{N}$ or $\{0, 1\}$, at each step. Even in our machine-oriented approach, representations are always implicitly present.

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