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HELICOPTER ROTOR DYNAMICS BY FINITE ELEMENT TIME APPROXIMATION

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Abstract—Starting from a variational formulation, based on Hamilton's principle, the paper exploits the finite element technique in the time domain in order to generate, by an automated procedure, a numerical approximation of the nonlinear periodic equations peculiar to the dynamics of helicopter rotors. With this method, a unified approach is developed, which can be used to solve both the response and stability problem. Two simple examples are presented that show how the method can be used to study the trimmed response of a fully articulated rigid blade and the mechanical instability analysis of a four-bladed rotor.

INTRODUCTION

The analysis of the dynamic behaviour of an helicopter rotor is generally performed in two consecutive steps. At first, the periodic blade motion in steady flight is computed; then the stability of the trimmed solution is evaluated.

From an analytical point of view, in the first phase the investigation is addressed only to the periodic solution of the nonlinear differential equations describing the blade motion, while in the second phase it can be extended to study both the linearized stability of an equilibrium solution and the time history response after a control excitation has been applied.

Different methods[1–10] are available in the literature to solve each of the two previously cited steps, but in general, different schemes are used to attack trim and stability analyses.

The helicopter rotor is generally modelled with a finite number of generalized coordinates by a separation of variables in the space and time domain[7].

Typically the space motion is approximated by the mode shapes of a blade rotating in vacuum, while the time dependency is provided by the amplitude of such modes.

In this way an approximation of the rotor dynamics is obtained, which is based on a set of second-order nonlinear ordinary differential equations in the time domain.

These equations can be solved in different ways. The first way can be easily used when a complete linearization of the equations is acceptable. In this case a periodic solution is assumed in the form of a Fourier series, and the unknown coefficients are determined by solving a linear system of algebraic equations. The method can be easily extended to nonlinear problems by iterating upon successive linearization until convergence to a periodic solution, but the analytical burden posed by the coefficient matrix computation becomes substantial in realistic analyses.

The second way makes use of a step-by-step explicit numerical integration method, which seems able to compute the solution without any limitation in the form of the differential equations, and thus it can handle more general systems, provided they can be reduced to first order.

Here a step-by-step method of integration seems to be the most efficient approach in solving the set of nonlinear differential equations describing the motion of the rotor, but even such a method does not seem to be free from drawbacks. In fact, it must be noted that, while the search of a periodic solution corresponds to the imposition of boundary conditions, the step-by-step procedure requires the knowledge of initial conditions, and in general, for nonlinear equations, it may be difficult to find which initial conditions are related to a particular periodic solution. Then the procedure is arbitrarily started and the integration performed until a steady state periodic solution is reached. It is obvious that the computer effort, and the related cost, is strictly dependent on the stability of the system and that lack of stability makes the method fail to converge. Such an event may be related to a true instability of the system, but no systematic information is provided for a stability analysis, which has to be dealt with by another approach.

This work shows how an appropriate use of Hamilton's principle combined with a finite element approximation in time domain can provide an unified approach to the determination of the periodic response of nonlinear dynamic systems and to the analysis of its stability[11].

Another relevant feature of the finite element application to Hamilton's principle is its capability to provide an automatic way to derive the equations of the dynamic system, with a minimal analytical effort and with the consequent reduction in manual time spent in developing and checking the needed formulas.

FEM APPROXIMATION OF HAMILTON'S PRINCIPLE

We assume that the configuration of an arbitrary holonomic mechanical system is described by a set of generalized coordinates $\{q\}$.

Denoting with $\{Q\}$ the generalized external forces and with $\mathcal{L}$ the Lagrangian of the system, Hamilton's principle can be written in the following synthetic way:

$$\int_{t_I}^{t_F} \delta\{Y\}^T \{l\}\, dt = \delta\{Y\}^T \{b\} \Big|_{t_I}^{t_F}, \tag{1}$$

where

$$\delta\{y\} = \begin{Bmatrix} \delta\{\dot{q}\} \\ \delta\{q\} \end{Bmatrix}, \quad \{l\} = \begin{Bmatrix} \{\mathcal{L}_{\dot{q}}\} \\ \{\mathcal{L}_q\} + \{Q\} \end{Bmatrix}, \quad \{b\} = \begin{Bmatrix} 0 \\ \{p\} \end{Bmatrix}.$$

$\{\mathcal{L}\dot{q}\}$ and $\{\mathcal{L}q\}$ are the partial derivatives of $\mathcal{L}$ with respect to $\{\dot{q}\}$ and $\{q\}$, while $\{p\} = \mathcal{L}\dot{q}$ is the column vector of the generalized moment.

In many text books of analytical dynamics the right-hand side term of eqn (1) is dropped, and Hamilton's principle is often regarded as a variational principle with constrained extrema. On the contrary, in view of the numerical application, these boundary terms play such an important role that they must not be dropped. To emphasize this aspect, hereafter Hamilton's principle in the form (1) will be reffered as Hamilton's weak principle (HWP).

In view of the method used to obtain a numerical solution of eqn (1), i.e. Newtonian or quasi-Newtonian methods[12,13], and since $\mathcal{L}$ is generally a nonlinear function of $\{\dot{q}\}$ and $\{q\}$, it is useful to provide a linear approximation of eqn (1).

Thus, after a linearization of the left-hand side around a given state vector $\{\bar{Y}\}$, the following expression is derived:

$$\int_{t_I}^{t_F} \delta\{Y\}^T\{\bar{l}\}\, dt + \int_{t_I}^{t_F} \delta\{Y\}^T[\bar{K}]\, \Delta\{Y\}\, dt = \delta\{Y\}^T\{p\}\Big|_{t_I}^{t_F}, \tag{2}$$

where the local tangent matrix $[\bar{K}]$ is defined as

$$[\bar{K}] = \begin{bmatrix} [\mathcal{L}\, \dot{q}\dot{q}] & [\mathcal{L}\, \dot{q}q] \\ ([\mathcal{L}\, q\dot{q}] + [Q\dot{q}]) & ([\mathcal{L}\, qq] + [Qq]) \end{bmatrix},$$

in which $[\mathcal{L}\dot{q}\dot{q}]$, $[\mathcal{L}\dot{q}q]$, $[\mathcal{L}qq]$, $[Q\dot{q}]$ and $[Qq]$ indicate second and first derivatives with respect to the suffixes. In eqn (2) both $\delta\{Y\}$ and $\Delta\{Y\}$ are synchronous infinitesimal changes of the state vector $\{Y\}$. It is now easy to apply a finite element approximation to the linearized HWP as expressed by eqn (2). For this purpose we define the nodal vector

$$\{X\}^T = [\{q_1\}^T, \{q_2\}^T, \ldots, \{q_k{}^T\}, \ldots, \{q_{n-1}\}^T]$$

in which each vector $\{q_K\}$ represents the value of the generalized coordinates at the time nodes $t_I = t_1, t_2, \ldots, t_K, \ldots, t_{N-1} = t_F$, that specify the finite element. In each element, the state vector $\{Y\}$ is interpolated among the nodal values $\{X\}_e$ of the element itself, i.e. an appropriate partition of $\{X\}$, through suitable shape functions $[f]$ as it follows:

$$\{Y\} = [N]\{X\}_e, \quad [N] = \begin{bmatrix} [\dot{f}] \\ [f] \end{bmatrix}.$$

Thus eqn (2) assumes the following variational form:

$$\delta\{X\}^T(\{\bar{L}\} + [\bar{K}]\Delta\{X\} - \{B\}) = 0, \tag{3}$$

where

$$\{B\}^T = [-\{p_2\}^T, 0, \ldots, 0, \ldots, 0, \{p_{n+1}\}^T],$$

and $\{\bar{L}\}$ and $[\bar{K}]$ are obtained through the FEM assembling process performed on the corresponding element matrices

$$\{\bar{L}\}_e = \int_e [N]^T\{\bar{l}\}\,\mathrm{d}t,$$
$$[\bar{K}] = \int_e [N]^T[K][N]\,\mathrm{d}t.$$

The variational statement eqn (3) can be used for solving both initial and periodic boundary problems. In the first case $\delta\{X\}$ is completely free, and this implies

$$[\bar{K}]\Delta\{X\} = \{B\} - \{\bar{L}\}. \tag{4}$$

This set of linearized equations can be solved for the $N+1$ unknown vectors $\{p_{n-1}\}$ and $\Delta\{q_K\}$ ($k=2,3,\ldots,N+1$), by making $\Delta\{q_1\} = 0$, in order to reflect the imposition of the initial conditions on $\{q_1\}$ and $\{p_1\}$. In the second case, i.e. when we enforce the periodicity constraints,

$$\{q_{N+1}\} = \{q_1\}, \qquad \{p_{N+1}\} = \{p_1\},$$

assuming a period $\tau = t_F - t_I$, we simply have $\delta\{X\}^T\{B\} = 0$. Then from eqn (3) we obtain

$$\{\bar{\bar{L}}\} + [\bar{\bar{K}}]\Delta\{X\} = 0, \tag{5}$$

where the matrix $[\bar{\bar{K}}]$ and the vector $\{\bar{\bar{L}}\}$ are obtained by folding the $(N+1)$th row and column on the first ones of $[\bar{K}]$ and $[\bar{L}]$. Then eqn (5) can be directly solved for the N unknown vectors $\{q_K\}$, ($K = 1,2,\ldots,N$).

STABILITY ANALYSIS OF PERIODIC PROBLEMS

In the linearized stability analysis of any periodic solution $\{X\}$, eqn (3) can be used in the following form:

$$[K]\Delta\{X\} = \Delta\{B\}, \tag{6}$$

where $[K]$ is the tangential matrix, computed at the solution, and $\Delta\{X\}$, $\Delta\{B\}$, are the perturbations of the solution and of the boundary conditions.

Equation (6) can be partitioned as follows:

$$\begin{bmatrix} [K_{aa}] & [K_{ao}] \\ [K_{oa}] & [K_{oo}] \end{bmatrix} \begin{Bmatrix} \Delta\{X_a\} \\ \Delta\{X_o\} \end{Bmatrix} = \begin{Bmatrix} \Delta\{B_a\} \\ 0 \end{Bmatrix}, \tag{7}$$

where $\Delta\{X_a\}^T = [\Delta\{q_1\}^T, \Delta\{q_{N+1}\}^T]$ is the analysis set of unknowns and $\Delta\{B_a\}^T = [-\Delta\{p_1\}^T, \Delta\{p_{N+1}\}^T]$ is the related boundary term, while $\Delta\{X_o\}^T = [\Delta\{q_2\}^T, \ldots, \Delta\{q_N\}^T]$ corresponds to the omitted set.

From eqn (7) we have:

$$\Delta\{X_o\} = -[K_{oo}]^{-1}[K_{oa}]\,\Delta\{X_a\}, \tag{8}$$

$$([K_{aa}] - [K_{ao}][K_{oo}]^{-1}[K_{ao}])\Delta\{X_a\} = \Delta\{B_a\}. \tag{9}$$

The coefficient matrix of eqn (9) can be further partitioned as

$$\begin{bmatrix} [K_{II}] & [K_{IF}] \\ [K_{FI}] & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \Delta\{q\}_I \\ \Delta\{q\}_F \end{Bmatrix} = \begin{Bmatrix} -\Delta\{p\}_I \\ \Delta\{p\}_F \end{Bmatrix}, \tag{10}$$

where the subscripts I and F denote the initial and final time nodes, respectively. At last eqn (10) can be rearranged in the following form:

$$\begin{Bmatrix} \Delta\{q\}_F \\ \Delta\{p\}_F \end{Bmatrix} = \begin{bmatrix} -[K_{FF}]\,[K_{IF}]^{-1} & [K_{FI}] - [K_{FF}]\,[K_{IF}]^{-1}[K_{II}] \\ -[K_{IF}]^{-1} & -[K_{IF}]^{-1}[K_{II}] \end{bmatrix} \begin{Bmatrix} \Delta\{q\}_I \\ \Delta\{p\}_I \end{Bmatrix}, \tag{11}$$

in which the coefficient matrix is an approximation of the Floquet transition matrix of the periodic problem. As in Floquet theory, we look for homogeneous solutions of eqn (11) of the type

$$\begin{Bmatrix} \Delta\{q\}_F \\ \Delta\{p\}_F \end{Bmatrix} = \lambda \begin{Bmatrix} \Delta\{q\}_I \\ \Delta\{p\}_I \end{Bmatrix},$$

which leads to an eigenvalue problem that can be directly solved[14] in the range of the highest λ of interest. The eigenvalues λ and the corresponding eigenvectors $[\Delta\{q\}_I^T, \Delta\{p\}_I^T]^T$ are in general pairs of complex conjugate ones. Moreover, an approximation $\Delta\{X\}$ of the complete eigensolution $\{q(t)\}$ at nodal time points can be obtained, remembering that $\Delta\{X\}^T = [\Delta\{X_a\}^T, \Delta\{X_o\}^T]$, with the solution of eqn (8), by making $\Delta\{X_a\} = [\Delta\{q\}_I, \lambda\Delta\{q\}_I]$, with $\Delta\{q\}_I$ and λ taken from the eigensolution under consideration. Equation (10) can be interpreted as finite difference equations of the state variables $\Delta\{q\}_I$, $\Delta\{p\}_I$ and $\Delta\{q\}_F$, $\Delta\{p\}_F$ at the beginning and at the end of each period, since for periodic solutions, the tangent matrix is constant irrespective of the period. Therefore, for the period following the one considered in eqn (10), we have

$$\begin{bmatrix} [K_{II}] & [K_{IF}] \\ [K_{FI}] & [K_{FF}] \end{bmatrix} \begin{Bmatrix} \Delta\{q\}_{I+1} \\ \Delta\{q\}_{F+1} \end{Bmatrix} = \begin{Bmatrix} -\Delta\{p\}_{I+1} \\ \Delta\{p\}_{F+1} \end{Bmatrix}, \tag{12}$$

and since the relations

$$\begin{aligned} \Delta\{p\}_{I+1} &= \Delta\{p\}_F, \\ \Delta\{q\}_{F+1} &= \Delta\{q\}_{I+2} \end{aligned} \tag{13}$$

must hold for the continuity of the solution, we obtain the following finite difference equation:

$$[K_{IF}]\Delta\{q\}_{I+2} + ([K_{II}] + [K_{FF}])\,\Delta\{q\}_{I+1} + [K_{FI}]\Delta\{q\}_I = 0, \tag{14}$$

which entails a solution of the type

$$\Delta\{q\}_{I+1} = \lambda\,\Delta\{q\}_I. \tag{15}$$

By substituting eqn (15) into eqn (14), the following quadratic eigenvalue problem is obtained:

$$(\lambda^2[K_{IF}] + \lambda([K_{II}] + [K_{FF}]) + [K_{FI}])\,\Delta\{q\}_I = 0. \tag{16}$$

It is interesting to note that the eigenproblems of eqns (16) and (11) have the same eigenvalues, as it can be trivially proved by substituting the vector $\Delta\{p\}_I$ in eqn (11) with the linear relationship

$$\Delta\{p\}_I = -(\lambda\, [K_{IF}] + [K_{II}])\, \Delta\{q\}_I.$$

This approach to the analysis of the stability of periodic systems closely resembles the one of Floquet's method. In fact eqn (6) replaces the integration for n-independent sets of initial conditions, i.e. the procedure usually adopted when the transition matrix cannot be obtained analytically[1,15].

As it is well known[15], any fundamental solution of a linear system with periodic coefficients of period τ is expressed as

$$\{q(t)\} = e^{\beta t}\, \{\varphi(t)\}, \tag{17}$$

where $\{\varphi(t)\}$ are periodic functions of the same period τ, and for a solution such as $q(t+\tau) = \lambda q(t)$, β is given by $\beta = (1/\tau)\ \ln\lambda$. When λ is complex, the imaginary part of β is not uniquely determined, since a $\pm 2n\pi$ can be added, n being an arbitrary integer. When this arbitrariness is removed by choosing $n = 0$, from eqn (17) we have

$$\{\varphi(t)\} = \rho^{-t/\tau}\, e^{-i\theta t/\tau}\, \{q(t)\}, \tag{18}$$

where ρ and θ indicate the modulus and the principal argument of $\lambda(-\pi < \theta < \pi)$. The nodal approximation $\Delta\{X\}$ of each eigensolution $\{q(t)\}$ can now be used in eqn (18) to set N values of its periodic part $\{\varphi\}$:

$$\{\phi\}^T = [\{\varphi_1\}^T, \{\varphi_2\}^T, \ldots, \{\varphi_N\}^T].$$

Provided that the period τ has been divided into an even number N of equal intervals, by means of time nodes $t_K = (K-1)\tau/N$ $(K = 1, 2, \ldots, N+1)$, $\{\varphi(t)\}$ can be approximated with a truncated Fourier series by using $\{\phi\}$ as sampling values. Therefore, the eigensolution $\{q(t)\}$, eqn (17), is available in the form

$$\{q(t)\} = \rho^{t/\tau} \sum_{1K}^{N} e^{i(\omega_K + \theta/\tau)t}\, \{C_K\}. \tag{19}$$

where the expansion coefficients are given by

$$\{C_K\} = \frac{1}{N} \sum_{1I}^{N} \rho^{-(I-1)/N}\, e^{-i(\omega_K + \theta/\tau)(I-1)\tau/N}\, \{q_I\}, \tag{20a}$$

and

$$\varphi_K = \begin{cases} (k-1)2\pi/\tau, & \text{for } 1 \le K \le 1 + N/2, \\ (k-1-N)2\pi/\tau, & \text{for } 1 + N/2 < k \le N. \end{cases} \tag{20b}$$

When the interest is focused on limit stability, the sole eigenvalue analysis is required. In fact, stability is provided by the condition $\rho \le 1$, while any $\rho > 1$ denotes instability. Obviously a complete eigensolution analysis, carried out with eqn (19), provides more informations on the instability mechanisms themselves. Moreover, eqn (19) shows that the argument θ acts as a frequency shift, and the indetermination of $\mp 2n\pi$ is meaningless, since for eqn (20) it corresponds to a redifinition of the integer K. Due to the finite sampling of N values, the folding frequency limit $\omega_F = N\pi/\tau$ must be taken into account[17]. Consequently, for a correct choice of the number N of nodal points, a physical insight of the specific problem under investigation is necessary. It is important to note that the function $\varphi(t)$ is in general periodic, and not simply harmonic: therefore, the eigensolutions $\{q(t)\}$ cannot be characterized by means of a single frequency. However, in many practical cases the periodic function $\varphi(t)$ is well approximated by a single harmonic function, so that a single frequency characterization can be accepted in such cases.

The approach illustrated above gets rid of all the troubles related to the solution of the indetermination of $\pm 2n\pi$ by physical insight or by continuous variation of a system parameter[1], and it affords a reliable method to automatically decide when the frequency content of the eigensolution is purely harmonic. In fact, in such cases, a single coefficient C_K is numerically predominant in the series expansion, and its corresponding frequency $\omega_K + \theta/\tau$ is assumed as the characteristic frequency. Finally, some considerations have to be made about the different requirements that the numerical approximation must satisfy, depending on the type of problem, i.e. periodic solution or stability analysis, under consideration.

In fact, the determination of the periodic response leads to the solution of a boundary value problem, in which only precision requirements are relevant; on the other hand, the study of the stability implies the solution of an initial value problem, in which both precision and stability consideration must be taken into account, if spurious numerical instabilities have to be avoided[11].

NUMERICAL EXAMPLES

The method here described can be applied to many classes of periodic problems[11]. Here its use is demonstrated by two simple yet significant examples. The first one is concerned with the nonlinear periodic response of a fully articulated rigid blade, while the second refers to the "ground resonance" stability analysis and the results compared with those of Ref. 18.

In Fig. 1 a fully articulated rigid blade is sketched, and the significant dimensions of the rotor blade are given. The blade and the hub are assumed as rigid; nonlinearities and control couplings are taken into account by simulating the swashplate through the imposition of the inextensibility of the pitch linkrod. A quasi steady aerodynamic model and an uniform distribution of induced velocity are assumed.

The flapping, drag and pitch angles are chosen as degrees of freedom (Fig. 2). The pushrod inextensibility constraint has been imposed by means of a nodal collocation of each time node, and the pitch increment $\Delta\theta$ is eliminated, during Newton–Raphson iteration, by solving the constraint relation at each time node of the finite element time discretization. Such elimination procedure has been preferred to the use of Lagrange's multipliers, which although more general, in this case, would have doubled the order of the system.

The calculations are performed in steady flight conditions, and flight control settings are imposed by fixing swashplate location and rotation.

Some results of this numerical application are here briefly discussed.

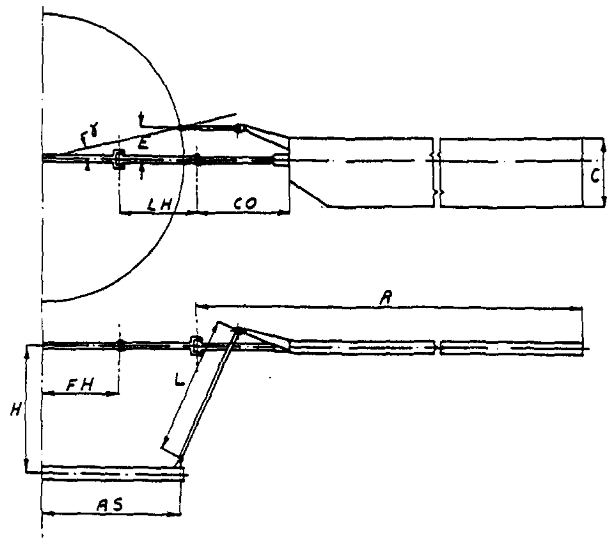
If the array vector $\{q\}$ represents the generalized coordinates of this problem, i.e. the flap, drag and pitch angle of the blade, then the velocity $\{V\}$ of the centroid of any blade section in a reference frame rotating with the hub can be computed with the expression

$$\{V\} = [C]\{\dot{q}\} + \{V_t\},$$

where the matrix $[C]$ is function of $\{q\}$ and its analytical expression can be easily determined from Fig. 2, by means of simple trigonometric relationships and $\{V_t\}$ is the velocity entrained by the hub.

Taking into account only the translational kinetic energy of the blade, we have

$$T = \frac{1}{2}\int_0^R m\{V\}^T\{V\}\,\mathrm{d}r = \frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\} + \{Z\}^T\{\dot{q}\} + \{T_0\}.$$



γ	FH	LH	CO	R	C	AS	L	E
36°	150	85	1630	5265	335	180	240	125

Fig. 1. Blade geometry.

where

$$[M] = \int_0^R m\,[C]^T[C]\,dr,$$

$$\{Z\} = \int_0^R m\,\{V_t\}^T[C]\,dr,$$

$$T_0 = \frac{1}{2}\int_0^R m\,\{V_t\}^T\{V_t\}\,dr.$$

Thus the first and second derivatives of the kinetic energy T with respect to $\{q\}$ and $\{\dot{q}\}$ can be easily computed from first and second derivatives of $[C]$ and $\{V_t\}$, once the rotor configuration and motion are known. The integrals with respect to the radial coordinate r are numerically computed by means of Gauss quadrature formulae.

Making use of the well known blade element aerodynamic theory, the aerodynamic forces acting on each section have the form

$$\{F\} = [A]\,\{w\},$$

where $\{w\}$ is the airspeed of the section, and the matrix $[A]$ is a function of the profile aerodynamic coefficients and of $\{w\}$.

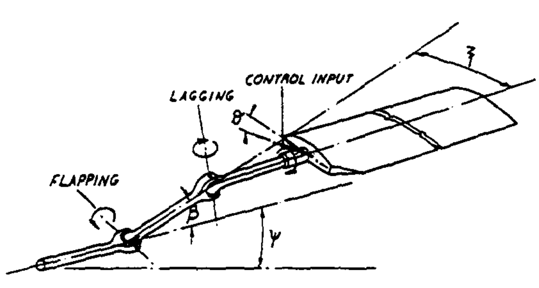


Fig. 2. Rigid blade degrees of freedom.

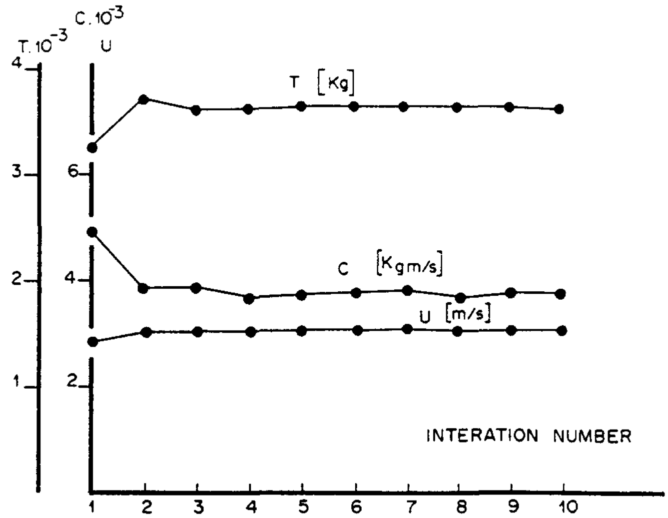


Fig. 3. Convergence behaviour.

Therefore, the generalized forces are given by

$$\{Q\} = \int_0^R [C]^T\{F\}\,\mathrm{d}r,$$

and the computation of the derivatives $\{Q_{\dot{q}}\}$ and $\{Q_q\}$ can be easily performed, once the rotor configuration and motion are assigned.

The aerodynamic field induced by the rotor is approximated in a rather simple way by the Glauert relationship.

The drag hinge is fitted with a linear viscous damper and the blade is assumed to possess a constant twist angle of one degree per meter.

Parabolic shape functions and Gauss numerical quadrature formulae are used to obtain the element matrices for the finite element time discretization.

The harmonic content of the solution suggests that six parabolic elements spanning the period of rotation can afford results of acceptable precision. Some numerical tests to check the convergence for finer and finer time discretization have confirmed the previous assumption.

We will now show the results obtained in some calculation performed with no cyclic pitch and null rotor disc angle of attack.

A typical way in which the rotor thrust T, torque C and the mean induced velocity converge to trimmed solutions, is shown in Fig. 3 versus the number of iterations. Figures 4–6 are related

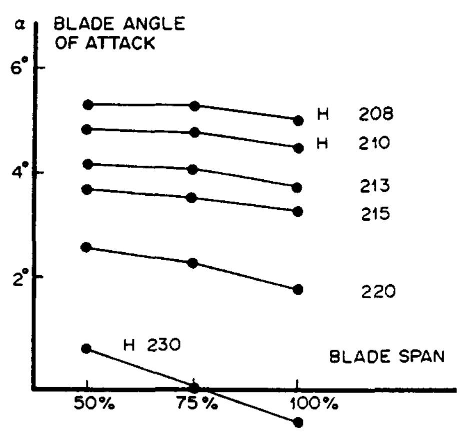


Fig. 4. Blade angle of attack distribution (hovering).

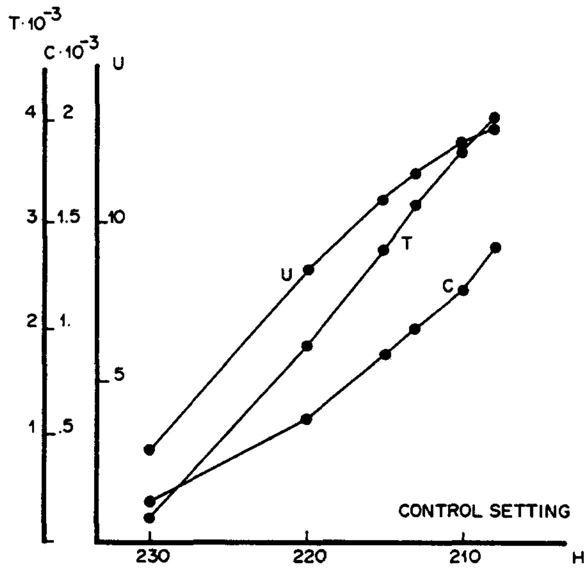


Fig. 5. Variation of the thrust, torque and induced velocity versus swash plate position.

to trimmed hovering conditions, with a rotor speed of 385 rpm, and they show the behaviour of the same quantities, and of blade angle of attack, flap, leadlag and pitch motions versus the swashplate position H, labeled control setting.

Finally Fig. 7 shows the motion of rotor blade and the angle of attack at 75% of the blade span axis at various azimuth angles.

It is of some interest to remark that, even in presence of a severe stall of the retreating blade, the method has shown no difficulty in converging to the solution. The second numerical example concerns the analysis of the mechanical instability of the four bladed rotor taken from Ref. 18 and shown in Fig. 8.

In order to compare the results, the mathematical representation is exactly the same as that previously studied in Ref. 18, and the same numerical values of the parameters are used in the present calculations.

The referred test case has been solved with a finite element mesh which involved four azimuthal time elements, each having four nodes for a total of twelve nodes per revolution period. Floquet's transition matrix is evaluated by assembling the tangent matrix of each element which is numerically computed by means of four Gaussian integration points. This crude

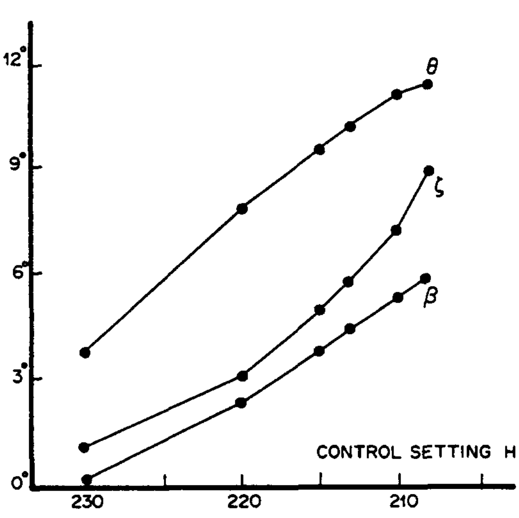


Fig. 6. Flap, lead-lag and pitch angles vs swash plate position.

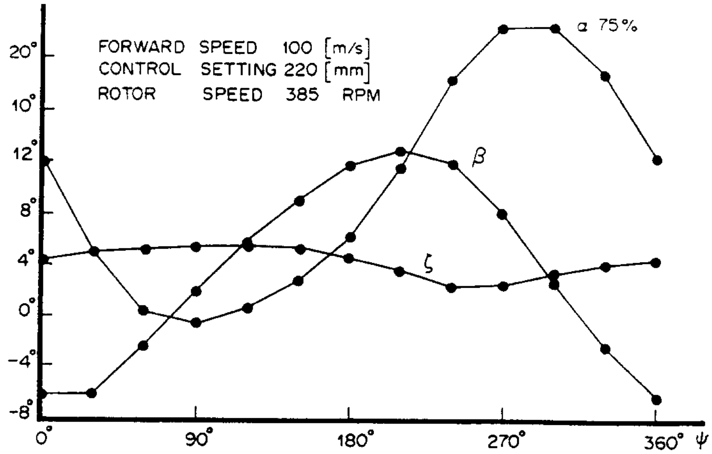


Fig. 7. Flap, lead-lag and blade angle of attack azimuth.

approximation of only four time elements seems to be quite adequate for the present purposes and Fig. 9 shows the results for the non isotropic hub and one blade damper inoperative, compared with those of Ref. 18. The numbers attached to the different modes are used to provide a label for the various modes. These figures illustrate the variation of the real and imaginary part of the characteristic exponents as a function of the rotor speed; continuous curves refer the results of Ref. 18, while the present calculation are marked with black points.

In particular the so-called modal damping is computed by $(1/\tau)\ \ln \rho$ while the so-called modal frequency is the frequency corresponding to the highest Fourier expansion coefficient. It is convenient to note that in this case the eigensolution is mostly a harmonic function so that the modal frequency can be also computed as $\omega = a \tan^{-1} (\mathrm{I\omega}\lambda/\mathrm{Re}\lambda)$.

It is noted that the results of this paper show the presence of an extra mode with respect to those of Ref. 18. In this mode the hub, the blade with an inoperative damper and its opposite blade do not oscillate so that the modal damping is simply one-half of the ratio of the lag damping to the mass moment of inertia, and obviously, it is independent to the rotor speed. For this mode the condition of one blade damper inoperative is ineffective. This mode is missed in Ref. 18 for this case, while it is present when all of the blade dampers are working.

The conclusion to be drawn here is that even with the crude approximation of four elements, each spanning over a quarter of a period, a good agreement is obtained with the results of Ref. 18.

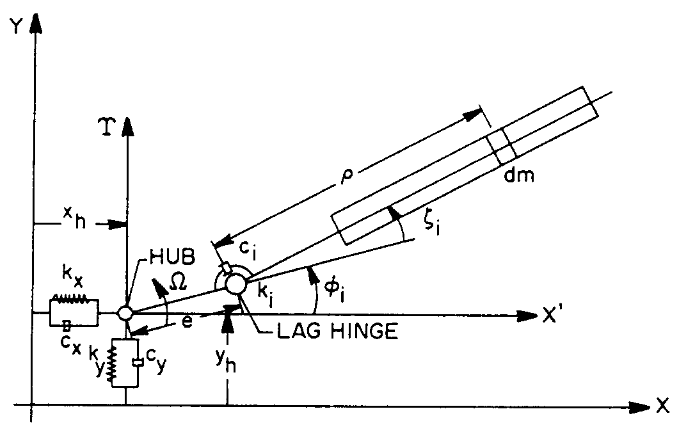


Fig. 8. Mathematical representation of rotor and hub (from Ref. 18).

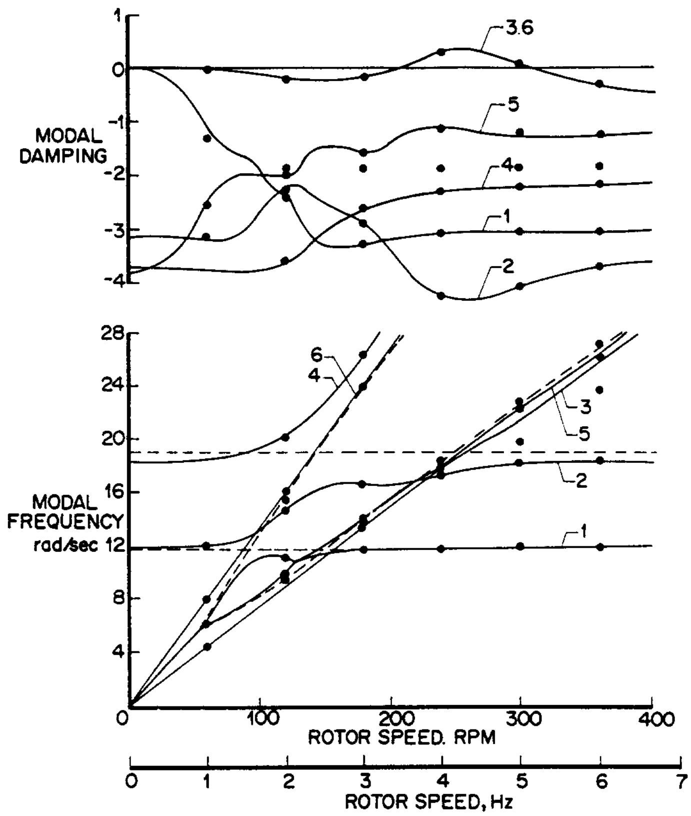


Fig. 9. Modal damping and frequencies for nonisotropic hub. one-blade damper inoperative.

CONCLUSIONS

The method presented proves to be a practical tool for the analyses of nonlinear trimmed responses, and related stability studies, of helicopter rotors. Moreover, this method affords an unified approach to attack both periodic response, stability analysis and transient time history[11] due to a control excitation. Since the explicity development of the differential equations of motion is not required, it leads to an automated and easy-to-check formulation. Thus even systems with many degrees of freedom, as those required in rotor aeroelastic analyses, could be effectively solved. A further advantage of the method is the possibility of exploiting the sound and well-known numerical techniques already developed for different applications of the finite element method.

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