Evaluate Fuzzy Riemann Integrals Using the Monte Carlo Method

Hsien-Chung Wu

Department of Information Management, National Chi Nan University, Puli, Nantou 545, Taiwan

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Techniques for using the Monte Carlo method to evaluate fuzzy Riemann integrals and improper fuzzy Riemann integrals are proposed in this paper. Owing to the \( \alpha \)-level set of the (improper) fuzzy Riemann integral being the closed interval whose end points are the classical (improper) Riemann integrals, it is possible to invoke the Monte Carlo method to approximate the end points of the \( \alpha \)-level closed intervals. We develop the strong law of large numbers for fuzzy random variables in order to give the techniques proposed for evaluating the (improper) fuzzy Riemann integrals using the Monte Carlo approach more theoretical support. The membership function of the (improper) fuzzy Riemann integral can be transformed into mathematical programming problems. Therefore, we can obtain the membership value by solving the mathematical programming problems using the commercial optimizer.

Key Words: fuzzy numbers; (improper) fuzzy Riemann integrals; Monte Carlo method; strong law of large numbers; mathematical programming problems.

1. INTRODUCTION

The concept of a fuzzy integral was first introduced by Sugeno [8]. Many formulations of fuzzy integrals have since been developed. Sims and Wang [7] gave a good review of this subject. However, the developments were in the measure-theoretic sense. Therefore it is difficult to provide numerical methods in applications. The concepts of fuzzy Riemann integrals and improper fuzzy Riemann integrals based on closed intervals were introduced by Wu [9, 11], and their numerical integrations were also proposed by Wu [9, 11]. Owing to the \( \alpha \)-level set of the (improper) fuzzy Riemann integral being the closed interval whose end points are the
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classical (improper) Riemann integrals, it is possible to invoke the Monte Carlo method to approximate the end points of the \( \alpha \)-level closed intervals.

In order to give the techniques proposed for evaluating the (improper) fuzzy Riemann integrals using the Monte Carlo approach more theoretical support, we develop the strong law of large numbers for fuzzy random variables. The concepts of the fuzzy random variable and its expectation were introduced by Kwakernaak [4] and Puri and Ralescu [6]. However, the viewpoint of fuzzy random variables discussed in this paper is different from the point of view discussed in Puri and Ralescu [6] for the sake of being more applicable in a discussion of the evaluations. The membership function of the (improper) fuzzy Riemann integral can be transformed into mathematical programming problems. Therefore, we can obtain the membership value by solving the mathematical programming problems using a commercial optimizer, e.g., GAMS.

In Section 2, we introduce the basic concepts of fuzzy Riemann integrals and improper fuzzy Riemann integrals. In Section 3, we introduce the notions of the fuzzy random variable and its expectation for the purpose of deriving the strong law of large numbers for fuzzy random variables in the next section. In Section 4, we introduce the concept of convergence with probability one for fuzzy random variables and derive the strong law of large numbers for fuzzy random variables. In Section 5, the Monte Carlo method is invoked to evaluate the (improper) fuzzy Riemann integrals. In the final section, Section 6, we provide the computational procedure and give examples to clarify the discussions in this paper.

2. (IMPROPER) FUZZY RIEMANN INTEGRALS

Let \( X \) be a universal set. Then a fuzzy subset \( \tilde{A} \) of \( X \) is defined by its membership function \( \xi_{\tilde{A}}: X \to [0, 1] \). We denote by \( A_\alpha = \{ x: \xi_{\tilde{A}}(x) \geq \alpha \} \) the \( \alpha \)-level set of \( \tilde{A} \), where \( A_0 \) is the closure of the set \( \{ x: \xi_{\tilde{A}}(x) \neq 0 \} \).

\( \tilde{A} \) is called a normal fuzzy set if there exists an \( x \) such that \( \xi_{\tilde{A}}(x) = 1 \). \( \tilde{A} \) is called a convex fuzzy set if \( \xi_{\tilde{A}}(\lambda x + (1 - \lambda)y) \geq \min\{\xi_{\tilde{A}}(x), \xi_{\tilde{A}}(y)\} \) for \( \lambda \in [0, 1] \) (that is, \( \xi_{\tilde{A}} \) is a quasi-concave function).

Proposition 2.1. (Zadeh [12]). \( \tilde{A} \) is a convex fuzzy set if and only if \( \{ x: \xi_{\tilde{A}}(x) \geq \alpha \} \) is a convex set for all \( \alpha \).

\( \tilde{a} \) is a fuzzy number if \( \tilde{a} \) is a normal and convex fuzzy set. \( \tilde{a} \) is called a closed fuzzy number if \( \tilde{a} \) is a fuzzy number and its membership function \( \xi_{\tilde{a}} \) is upper semicontinuous. \( \tilde{a} \) is called a bounded fuzzy number if \( \tilde{a} \) is a fuzzy number and its membership function \( \xi_{\tilde{a}} \) has compact support.
From Proposition 2.1, we have the following result.

**Proposition 2.2.** If \( \tilde{a} \) is a closed fuzzy number then the \( \alpha \)-level set of \( \tilde{a} \) is a closed interval, which is denoted by \( \tilde{a}_\alpha = [a_\alpha^L, a_\alpha^R] \). (That is why we call \( \tilde{a} \) a closed fuzzy number.)

Now we are going to discuss the fuzzy Riemann integrals. We say that \( \tilde{f}(x) \) is a fuzzy-valued function if \( \tilde{f}(x) : \mathbb{R} \to \mathcal{F} \), where \( \mathcal{F} \) is the set of all fuzzy numbers. We say that \( \tilde{f}(x) \) is a closed fuzzy-valued function if \( \tilde{f}(x) : \mathbb{R} \to \mathcal{F}_c \), where \( \mathcal{F}_c \) is the set of all closed fuzzy numbers. We say that \( \tilde{f}(x) \) is a bounded fuzzy-valued function if \( \tilde{f}(x) : \mathbb{R} \to \mathcal{F}_b \), where \( \mathcal{F}_b \) is the set of all bounded fuzzy numbers.

**Proposition 2.3.** (i) (Zadeh [13]: Resolution Identity). Let \( \tilde{A} \) be a fuzzy set with membership function \( \xi_{\tilde{A}} \) and \( \tilde{A}_\alpha = \{x : \xi_{\tilde{A}}(x) \geq \alpha\} \). Then

\[
\xi_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\tilde{A}_\alpha}(x).
\]

(ii) (Negoita and Ralescu [5]) Let \( A \) be a set and \( \{A_\alpha : \alpha \in [0,1]\} \) be a family of subsets of \( A \) such that

(a) \( A_0 = A \),
(b) \( A_\beta \subseteq A_\alpha \) for \( \alpha < \beta \),
(c) \( A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n} \) for \( \alpha_n \uparrow \alpha \).

Then the function \( \xi : A \to [0,1] \) defined by

\[
\xi(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(x)
\]

has the property that

\[
A_\alpha = \{x : \xi(x) \geq \alpha\} \quad \text{for all } \alpha \in [0,1].
\]

Inspired by the “resolution identity” in Proposition 2.3, we propose the following definition.

**Definition 2.1.** Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \( [a,b] \). Suppose that \( \tilde{f}_L(x) \) and \( \tilde{f}_R(x) \) are Riemann-integrable on \( [a,b] \) for all \( \alpha \in [0,1] \). Let

\[
A_\alpha = \left[ \int_a^b \tilde{f}_L(x) \, dx, \int_a^b \tilde{f}_R(x) \, dx \right].
\]

Then we say that \( \tilde{f}(x) \) is fuzzy Riemann-integrable on \( [a,b] \), and the membership function of \( \int_a^b \tilde{f}(x) \, dx \) is defined by

\[
\xi_{\int_a^b \tilde{f}(x) \, dx}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_\alpha}(r),
\]

for \( r \in A_0 \).
Theorem 2.1. (Wu [11]). Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \([a, b]\). If \( \tilde{f}(x) \) is fuzzy Riemann-integrable on \([a, b]\), then \( \int_a^b \tilde{f}(x) \, dx \) is a closed fuzzy number. Furthermore, the \( \alpha \)-level set of \( \int_a^b \tilde{f}(x) \, dx \) is
\[
\left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha = \left[ \int_a^b \tilde{f}_L(x) \, dx, \int_a^b \tilde{f}_R(x) \, dx \right].
\]

Theorem 2.2. (Wu [11]). If \( \tilde{f}(x) \) is a closed and bounded fuzzy-valued function on \([a, b]\) and if \( \tilde{f}_L(x) \) and \( \tilde{f}_R(x) \) are continuous on \([a, b]\) for all \( \alpha \in [0, 1] \), then \( \tilde{f}(x) \) is fuzzy Riemann-integrable on \([a, b]\). Furthermore, we have
\[
\left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha = \left[ \int_a^b \tilde{f}_L(x) \, dx, \int_a^b \tilde{f}_R(x) \, dx \right].
\]

Next we consider the improper fuzzy Riemann integrals.

Definition 2.2. (i) Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \([a, +\infty)\). Suppose that \( \tilde{f}_L^a(x) \) and \( \tilde{f}_R^a(x) \) are improper Riemann-integrable on \([a, +\infty)\) for all \( \alpha \in [0, 1] \). Let
\[
A_\alpha = \left[ \int_a^{+\infty} \tilde{f}_L^a(x) \, dx, \int_a^{+\infty} \tilde{f}_R^a(x) \, dx \right].
\]
Then we say that \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \([a, +\infty)\), and the membership function of \( \int_a^{+\infty} \tilde{f}(x) \, dx \) is defined by
\[
\xi_{\int_a^{+\infty} \tilde{f}(x) \, dx}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_\alpha}(r),
\]
for \( r \in A_0 \).

(ii) Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \((-\infty, a]\). Suppose that \( \tilde{f}_L^a(x) \) and \( \tilde{f}_R^a(x) \) are improper Riemann-integrable on \((-\infty, a]\) for all \( \alpha \in [0, 1] \). Let
\[
A_\alpha = \left[ \int_{-\infty}^a \tilde{f}_L^a(x) \, dx, \int_{-\infty}^a \tilde{f}_R^a(x) \, dx \right].
\]
Then we say that \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \((-\infty, a]\), and the membership function of \( \int_{-\infty}^a \tilde{f}(x) \, dx \) is defined by
\[
\xi_{\int_{-\infty}^a \tilde{f}(x) \, dx}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_\alpha}(r),
\]
for \( r \in A_0 \).
(iii) Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \(-\infty, +\infty\). Suppose that \( \tilde{f}^L_a(x) \) and \( \tilde{f}^R_a(x) \) are improper Riemann-integrable on \(-\infty, +\infty\) for all \( \alpha \in [0, 1] \). Let

\[ A_\alpha = \left[ \int_{-\infty}^{+\infty} \tilde{f}^L_a(x) \, dx, \int_{-\infty}^{+\infty} \tilde{f}^R_a(x) \, dx \right]. \]

Then we say that \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \(-\infty, +\infty\), and the membership function of the improper fuzzy Riemann integral \( \int_{-\infty}^{+\infty} \tilde{f}(x) \, dx \) is defined by

\[ \xi_{\int_{-\infty}^{+\infty} \tilde{f}(x) \, dx}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_\alpha}(r), \]

for \( r \in A_0 \).

**Theorem 2.3.** (Wu [9]). (i) Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \([a, +\infty)\) (or on \((-\infty, a]\)). For any fixed \( \alpha \), assume \( \tilde{f}^L_a(x) \) and \( \tilde{f}^R_a(x) \) are Riemann-integrable on \([a, b]\) for every \( b \geq a \), and assume there are two positive constants \( M^L_a \) and \( M^R_a \) such that \( \int_a^b |\tilde{f}^L_a(x)| \, dx \leq M^L_a \) and \( \int_a^b |\tilde{f}^R_a(x)| \, dx \leq M^R_a \) for every \( b \geq a \). Then \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \([a, +\infty)\) (or on \((-\infty, a]\)), and the improper fuzzy Riemann integral \( \int_{a}^{+\infty} \tilde{f}(x) \, dx \) (or \( \int_{-\infty}^{a} \tilde{f}(x) \, dx \)) is a closed fuzzy number. Furthermore, we have

\[ \left( \int_{a}^{+\infty} \tilde{f}(x) \, dx \right)_a = \left[ \int_{a}^{+\infty} \tilde{f}^L_a(x) \, dx, \int_{a}^{+\infty} \tilde{f}^R_a(x) \, dx \right] \]

or

\[ \left( \int_{-\infty}^{a} \tilde{f}(x) \, dx \right)_a = \left[ \int_{-\infty}^{a} \tilde{f}^L_a(x) \, dx, \int_{-\infty}^{a} \tilde{f}^R_a(x) \, dx \right]. \]

(ii) Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \((-\infty, +\infty)\). Suppose that \( |\tilde{f}^L_a(x)| \) and \( |\tilde{f}^R_a(x)| \) are improper Riemann-integrable on \((-\infty, +\infty)\) for all \( \alpha \in [0, 1] \). Then \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \((-\infty, +\infty)\), and the improper fuzzy Riemann integral \( \int_{-\infty}^{+\infty} \tilde{f}(x) \, dx \) is a closed fuzzy number. Furthermore, we have

\[ \left( \int_{-\infty}^{+\infty} \tilde{f}(x) \, dx \right)_a = \left[ \int_{-\infty}^{+\infty} \tilde{f}^L_a(x) \, dx, \int_{-\infty}^{+\infty} \tilde{f}^R_a(x) \, dx \right]. \]

**Theorem 2.4.** (Wu [9]). Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \((-\infty, +\infty)\). If \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \((-\infty, a]\) and \([a, +\infty)\) then \( \tilde{f}(x) \) is improper fuzzy Riemann-integrable on \((-\infty, +\infty)\) and

\[ \int_{-\infty}^{+\infty} \tilde{f}(x) \, dx = \int_{-\infty}^{a} \tilde{f}(x) \, dx \oplus \int_{a}^{+\infty} \tilde{f}(x) \, dx. \]
3. THE FUZZY RANDOM VARIABLE AND ITS EXPECTATION

In order to make fuzzy random variables more applicable in this paper, the viewpoint of fuzzy random variables discussed in this paper is different from the viewpoint discussed in Puri and Ralescu [6].

Let \((X, \mathcal{M})\) be a measurable space and \((\mathbb{R}, \mathcal{B})\) be a Borel measurable space. Let \(f: X \to \mathcal{P}(\mathbb{R})\) (power set of \(\mathbb{R}\)) be a set-valued function. According to Aumann [1], \(f\) is called measurable if and only if \(\{(x, y) : y \in f(x)\}\) is \(\mathcal{M} \times \mathcal{B}\)-measurable. If \(\tilde{f}\) is a fuzzy-valued function then \(\tilde{f}_\alpha\) is a set-valued function for all \(\alpha \in [0, 1]\). \(\tilde{f}\) is called (fuzzy-valued) measurable if and only if \(\tilde{f}_\alpha\) is (set-valued) measurable for all \(\alpha \in [0, 1]\).

In order to make fuzzy random variables more tractable mathematically, we need a stronger sense of measurability for fuzzy-valued functions. From Wu [10], the following two statements are equivalent.

(i) \(\tilde{f}_\alpha^L(x)\) and \(\tilde{f}_\alpha^R(x)\) are (real-valued) measurable for all \(\alpha \in [0, 1]\).

(ii) \(\tilde{f}(x)\) is (fuzzy-valued) measurable and one of \(\tilde{f}_\alpha^L(x)\) and \(\tilde{f}_\alpha^R(x)\) is (real-valued) measurable for all \(\alpha \in [0, 1]\).

Then \(\tilde{f}(x)\) is called strongly measurable if one of the above two conditions is satisfied. It is easy to see that strong measurability implies measurability.

Let \((X, \mathcal{M}, \mu)\) be a measure space and \((\mathbb{R}, \mathcal{B})\) be a Borel measurable space. Let \(f: X \to \mathcal{P}(\mathbb{R})\) be a set-valued function. For \(K \subseteq \mathbb{R}\), the inverse image of \(f\) is defined by

\[
\tilde{f}^{-1}(K) = \{x \in X : f(x) \cap K \neq \emptyset\}.
\]

Let \((X, \mathcal{M}, \mu)\) be a complete \(\sigma\)-finite measure space. From Hiai and Umehaki [3], the following two statements are equivalent:

(a) for each Borel set \(K \subseteq \mathbb{R}\), \(\tilde{f}^{-1}(K)\) is measurable (i.e., \(\tilde{f}^{-1}(K) \in \mathcal{M}\)),

(b) \(\{(x, y) : y \in f(x)\}\) is \(\mathcal{M} \times \mathcal{B}\)-measurable.

**Proposition 3.1.** Let \((X, \mathcal{M}, \mu)\) be a complete \(\sigma\)-finite measure space. \(\tilde{f}\) is a closed fuzzy-valued function defined on \(X\). Then measurability and strong measurability of this function are equivalent.

**Proof.** It suffices to prove that one of \(\tilde{f}_\alpha^L\) and \(\tilde{f}_\alpha^R\) is measurable for each \(\alpha\). Now we have

\[
\{x : \tilde{f}_\alpha^R(x) \geq y\} = \{x : \left[\tilde{f}_\alpha^L(x), \tilde{f}_\alpha^R(x)\right] = \tilde{f}_\alpha(x)\} \cap [y, +\infty) \neq \emptyset \in \mathcal{M}
\]

(by Condition (a)). This says that \(\tilde{f}_\alpha^R\) is measurable. This completes the proof. \(\blacksquare\)
Let $\Omega, \mathcal{A}, P$ be a probability space (a complete $\sigma$-finite measure space). We know that $X: \Omega \to \mathbb{R}$ is a random variable if $X$ is an $(\mathcal{A}, \mathcal{B})$-measurable function. Thus, we propose the following definition.

**Definition 3.1.** Let $\tilde{X}$ be a closed fuzzy-valued function. $\tilde{X}$ is called a fuzzy random variable if $\tilde{X}$ is measurable (or equivalently, strongly measurable).

Then the following proposition obviously holds true.

**Proposition 3.2.** Let $\tilde{X}$ be a closed fuzzy-valued function. $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X}_L^\alpha$ and $\tilde{X}_R^\alpha$ are random variables for all $\alpha \in [0, 1]$.

**Definition 3.2.** Let $\tilde{X}$ and $\tilde{Y}$ be two fuzzy random variables. We say that $\tilde{X}$ and $\tilde{Y}$ are independent if each random variable in the set $\{\tilde{X}_L^\alpha, \tilde{X}_R^\alpha : 0 \leq \alpha \leq 1\}$ is independent of each random variable in the set $\{\tilde{Y}_L^\alpha, \tilde{Y}_R^\alpha : 0 \leq \alpha \leq 1\}$. We say that $\tilde{X}$ and $\tilde{Y}$ are identically distributed if and only if $\tilde{X}_L^\alpha$ and $\tilde{X}_R^\alpha$ are identically distributed and $\tilde{Y}_L^\alpha$ and $\tilde{Y}_R^\alpha$ are identically distributed for all $\alpha \in [0, 1]$.

Let $\tilde{X}$ be a fuzzy random variable. We want to discuss the expectation of $\tilde{X}$. First of all, we develop the basic theory of fuzzy-valued integrals.

**Proposition 3.3.** (i) Let $A_\alpha = \{x : \xi(x) \geq \alpha\}$. Then $\bigcap_{n=1}^\infty A_{\alpha_n} = A_\alpha$ for $\alpha_n \uparrow \alpha$.

(ii) If $\tilde{a}$ is a closed fuzzy number then $\tilde{a}_L^{\alpha_n} \uparrow \tilde{a}_L^\alpha$ and $\tilde{a}_U^{\alpha_n} \downarrow \tilde{a}_U^\alpha$ for $\alpha_n \uparrow \alpha$ (i.e., left-continuous with respect to $\alpha$).

**Proof.** (i) It is easy to see that $A_\alpha \supseteq A_\beta$ for $\alpha > \beta$. Therefore $A_\alpha \subseteq A_{\alpha_n}$ for all $n$. It also says that $A_\alpha \subseteq \bigcap_{i=1}^n A_{\alpha_n}$. Now we assume that $x \in \bigcap_{i=1}^n A_{\alpha_n}$. Then $\xi(x) \geq \alpha_n$ for all $n$; that is, $\xi(x) \geq \alpha$ since $\alpha_n \uparrow \alpha$. Thus $x \in A_\alpha$.

(ii) The result follows immediately from (i).
via the form of “resolution identity” in Proposition 2.3. It is easy to see that $A_\beta \subseteq A_\alpha$ for $\alpha < \beta$. Since $\tilde{f}$ is a closed fuzzy-valued function, we have, for $\alpha_n \uparrow \alpha$, $\tilde{f}_{a_n}^L \uparrow \tilde{f}_a^L$ and $\tilde{f}_{a_n}^R \downarrow \tilde{f}_a^R$ from Proposition 3.3. By Lebesgue’s Dominated Convergence Theorem we have, for $\alpha_n \uparrow \alpha$,

$$\int_E \tilde{f}_{a_n}^L d\mu \rightarrow \int_E \tilde{f}_a^L d\mu \quad \text{and} \quad \int_E \tilde{f}_{a_n}^R d\mu \rightarrow \int_E \tilde{f}_a^R d\mu$$

since $\tilde{f}_{a_n}^L \leq \tilde{f}_a^L$ and $\tilde{f}_{a_n}^R \geq \tilde{f}_a^R$. This says that the interval $A_\alpha$ is continuously shrinking with respect to $\alpha$; that is, $A_\alpha = \bigcap_{\alpha_n = 1}^{\infty} A_{\alpha_n}$ for $\alpha_n \uparrow \alpha$. From Proposition 2.3(ii), we have

$$\left( \int_E \tilde{f} d\mu \right)_\alpha = A_\alpha = \left[ \int_E \tilde{f}_a^L d\mu, \int_E \tilde{f}_a^R d\mu \right].$$

This also says that the fuzzy-valued integral $\int_E \tilde{f} d\mu$ is a closed fuzzy number.

Let $\tilde{X}$ be a fuzzy random variable. The (fuzzy) expectation of $\tilde{X}$ is defined by

$$\tilde{E}[\tilde{X}] = \int_\Omega \tilde{X} dP.$$

Then we have

$$(\tilde{E}(\tilde{X}))_\alpha = \left( \int_\Omega \tilde{X} dP \right)_\alpha = \left[ \int_\Omega \tilde{X}_a^L dP, \int_\Omega \tilde{X}_a^R dP \right] = \left[ E[\tilde{X}_a^L], E[\tilde{X}_a^R] \right]. \quad (1)$$

We also see that the (fuzzy) expectation $\tilde{E}[\tilde{X}]$ is a closed fuzzy number.

### 4. STRONG LAW OF LARGE NUMBERS

Now is the right time to discuss the strong law of large numbers for fuzzy random variables. From Proposition 3.2, it is natural to propose the following definition.

**DEFINITION 4.1.** Let $\tilde{X}_n$ and $\tilde{X}$ be fuzzy random variables defined on the same probability space $(\Omega, \mathcal{A}, P)$. We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ with probability one if $(\tilde{X}_n)_a^L$ converges to $\tilde{X}_a^L$ with probability one and $(\tilde{X}_n)_a^R$ converges to $\tilde{X}_a^R$ with probability one for all $\alpha \in [0, 1]$.

Let $\tilde{a}$ be a fuzzy number. $\tilde{a}$ is called a nonnegative fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x < 0$; $\tilde{a}$ is called a nonpositive fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x > 0$; $\tilde{a}$ is called a positive fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x \leq 0$; and $\tilde{a}$ is called a negative fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x \geq 0$. 
Let “⊙” be any binary operation ⊕, ⊖, or ⊗ between two fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \). The membership function of \( \tilde{a} \circ \tilde{b} \) is defined by

\[
\xi_{\tilde{a} \circ \tilde{b}}(z) = \sup_{x \circ y = z} \min \{ \xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y) \}
\]

for \( \circ = \oplus, \ominus, \text{or } \otimes \) and \( \circ = +, - , \text{or } \times \). The membership function of the inverse of \( \tilde{a} \) is defined by

\[
\xi_{1/\tilde{a}}(z) = \sup_{z = 1/x, x \neq 0} \min \{ \xi_{\tilde{a}}(x) = \xi_{\tilde{a}}(1/z) \}.
\]

The quotient of \( \tilde{a} \) and \( \tilde{b} \) is defined by \( \tilde{a} \circ \tilde{b} = \tilde{a} \otimes (1/\tilde{b}) \).

Let “\( \circ_{int} \)” be any binary operation \( \oplus_{int}, \ominus_{int}, \otimes_{int} \), or \( \otimes_{int} \) between two closed intervals \( \tilde{a} = [\tilde{a}_L, \tilde{a}_R] \) and \( \tilde{b} = [\tilde{b}_L, \tilde{b}_R] \). Then \( \tilde{a} \circ_{int} \tilde{b} \) is defined by

\[
\tilde{a} \circ_{int} \tilde{b} = \{ z \in \mathbb{R} | z = x \circ y, \exists x \in \tilde{a}, \exists y \in \tilde{b} \},
\]

where “\( \circ \)” is a usual binary operation +, −, ×, or ÷.

Note that \( \tilde{a} \circ_{int} \tilde{b} \) is well-defined when \( \tilde{b} \) does not contain zero. Therefore \( \tilde{a} \circ \tilde{b} \) is well-defined when \( \tilde{b} \) is a positive or negative fuzzy number.

Then we have the following result.

**Proposition 4.1.** Let \( \tilde{a} \) and \( \tilde{b} \) be two closed fuzzy numbers. Then \( \tilde{a} \oplus \tilde{b} \) and \( \tilde{a} \otimes \tilde{b} \) are also closed fuzzy numbers. Furthermore, we have

\[
(\tilde{a} \oplus \tilde{b})_a = \tilde{a} \circ_{int} \tilde{b} = [\tilde{a}_L + \tilde{b}_L, \tilde{a}_R + \tilde{b}_R]
\]

and

\[
(\tilde{a} \otimes \tilde{b})_a = \tilde{a} \circ_{int} \tilde{b} = \min \left\{ \tilde{a}_L \tilde{b}_L, \tilde{a}_L \tilde{b}_R, \tilde{a}_R \tilde{b}_L, \tilde{a}_R \tilde{b}_R \right\},
\]

\[
\max \left\{ \tilde{a}_L \tilde{b}_L, \tilde{a}_L \tilde{b}_R, \tilde{a}_R \tilde{b}_L, \tilde{a}_R \tilde{b}_R \right\}.
\]

If \( \tilde{a} \) is a nonnegative closed fuzzy number and \( \tilde{b} \) is a positive closed fuzzy number then \( \tilde{a} \circ \tilde{b} \) is also a closed fuzzy number. Furthermore, we have

\[
(\tilde{a} \oplus \tilde{b})_a = [\tilde{a}_L / \tilde{a}_R, \tilde{a}_R / \tilde{a}_L].
\]

We say that \( \tilde{a} \) is a crisp number with value \( m \) if its membership function is

\[
\xi_{\tilde{a}}(r) = \begin{cases} 1 & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}
\]

\( \tilde{a} \) is denoted by \( \tilde{1}_{(m)} \).
**Theorem 4.1** (The Strong Law of Large Numbers for Fuzzy Random Variables). Let $\tilde{X}_n$ be independent and identically distributed fuzzy random variables and $\tilde{\mu}$ be the (fuzzy) expectation of $\tilde{X}_n$. Let $\tilde{Y}_n = \tilde{1}_{(1/n)} \otimes (\tilde{X}_1 \oplus \cdots \oplus \tilde{X}_n)$. Then $\{\tilde{Y}_n\}$ converges to $\tilde{\mu}$ with probability one.

**Proof.** From (1), we have $\tilde{\mu}_L = E[\tilde{X}_n]_L$ and $\tilde{\mu}_R = E[\tilde{X}_n]_R$. The result follows from the usual strong law of large numbers for random variables and Proposition 4.1.

**5. MONTE CARLO METHOD**

Let $\tilde{f}$ be a closed fuzzy-valued measurable function defined on $[0, 1]$, and let $\check{f}$ be fuzzy Riemann-integrable on $[0, 1]$. We consider the fuzzy Riemann integral

$$\int_0^1 \check{f}(x) \, dx.$$  

From Theorem 2.1, we have

$$\left( \int_0^1 \check{f}(x) \, dx \right)_a = \left[ \int_0^1 \check{f}_L(x) \, dx, \int_0^1 \check{f}_R(x) \, dx \right].$$

Let $U$ be a $U(0, 1)$ random variable. We set $\tilde{X} = \check{f}(U)$. Then $\tilde{X}_L = \check{f}_L(U)$ and $\tilde{X}_R = \check{f}_R(U)$ are random variables for all $\alpha \in [0, 1]$ since $\check{f}_L$ and $\check{f}_R$ are measurable functions for all $\alpha \in [0, 1]$. From Proposition 3.2, we conclude that $\tilde{X} = \check{f}(U)$ is a fuzzy random variable.

Let $\{U_n\}$ be a sequence of independent $U(0, 1)$ random variables. From Definition 3.2, we see that $\check{f}(U_n)$ are independent and identically distributed fuzzy random variables. From the strong law of large numbers for random variables, we have

$$\frac{1}{n} (\check{f}_L(U_1) + \cdots + \check{f}_L(U_n)) \to E[\check{f}_L(U)] = \int_0^1 \check{f}_L(x) \, dx = \left( \int_0^1 \check{f}(x) \, dx \right)_a^{L}$$

and

$$\frac{1}{n} (\check{f}_R(U_1) + \cdots + \check{f}_R(U_n)) \to E[\check{f}_R(U)] = \int_0^1 \check{f}_R(x) \, dx = \left( \int_0^1 \check{f}(x) \, dx \right)_a^{R}$$

with probability one. Therefore, from Theorem 4.1, we have

$$\tilde{1}_{(1/n)} \otimes (\check{f}(U_1) \oplus \cdots \oplus \check{f}(U_n)) \to E[\check{f}(U)] = \int_0^1 \check{f}(x) \, dx$$
with probability one. Hence we can generate a large number of random numbers \( \{u_i\} \) and take
\[
\bar{I}_{1/n} \otimes (\tilde{f}(u_1) \oplus \cdots \oplus \tilde{f}(u_n))
\]
as the approximation of the fuzzy Riemann integral \( \int_0^1 \tilde{f}(x) \, dx \). This approach to approximating the fuzzy Riemann integrals is called the Monte Carlo approach.

Let \( \tilde{f} \) be a closed fuzzy-valued measurable function defined on \([a, b]\), and let \( \tilde{f} \) be fuzzy Riemann-integrable on \([a, b]\). We consider the fuzzy Riemann integral
\[
\int_a^b \tilde{f}(x) \, dx.
\]

From Theorem 2.1, we have
\[
\left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha = \left[ \int_a^b \tilde{f}_a^L(x) \, dx, \int_a^b \tilde{f}_a^R(x) \, dx \right].
\]

Then
\[
\int_a^b \tilde{f}_a^L(x) \, dx = \int_0^1 g_a^L(x) \, dx \quad \text{and} \quad \int_a^b \tilde{f}_a^R(x) \, dx = \int_0^1 g_a^R(x) \, dx,
\]
where \( g_a^L(x) = (b - a)\tilde{f}_a^L(a + (b - a)x) \) and \( g_a^R(x) = (b - a)\tilde{f}_a^R(a + (b - a)x) \).

From the strong law of large numbers for random variables, we have
\[
\frac{1}{n} \left( g_a^L(U_1) + \cdots + g_a^L(U_n) \right) \to E[g_a^L(U)] = \int_0^1 g_a^L(x) \, dx = \int_a^b \tilde{f}_a^L(x) \, dx \quad \left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha^L
\]
and
\[
\frac{1}{n} \left( g_a^R(U_1) + \cdots + g_a^R(U_n) \right) \to E[g_a^R(U)] = \int_0^1 g_a^R(x) \, dx = \int_a^b \tilde{f}_a^R(x) \, dx \quad \left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha^R
\]
with probability one. That is, we have
\[
\frac{b - a}{n} \left( \tilde{f}_a^L(a + (b - a)U_1) + \cdots + \tilde{f}_a^L(a + (b - a)U_n) \right) \to \left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha^L
\]
and
\[
\frac{b - a}{n} \left( \tilde{f}_a^R(a + (b - a)U_1) + \cdots + \tilde{f}_a^R(a + (b - a)U_n) \right) \to \left( \int_a^b \tilde{f}(x) \, dx \right)_\alpha^R
\]
with probability one. Therefore, from Theorem 4.1, we have
\[ \mathbb{1}_{(b-a)/n} \otimes \left( \tilde{f}(a + (b-a)U_1) \oplus \cdots \oplus \tilde{f}(a + (b-a)U_n) \right) \rightarrow \int_a^b \tilde{f}(x) \, dx \]
with probability one. Hence we can generate a large number of random numbers \( \{u_i\} \) and take
\[ \mathbb{1}_{(b-a)/n} \otimes \left( \tilde{f}(a + (b-a)u_1) \oplus \cdots \oplus \tilde{f}(a + (b-a)u_n) \right) \]
as the approximation of the fuzzy Riemann integral \( \int_a^b \tilde{f}(x) \, dx \).

Let \( \tilde{f} \) be a closed fuzzy-valued measurable function defined on \([a, +\infty)\), and let \( \tilde{f} \) be improper fuzzy Riemann-integrable on \([a, +\infty)\). We are going to consider the improper fuzzy Riemann integral
\[
\int_a^{+\infty} \tilde{f}(x) \, dx.
\]
From Theorem 2.3, we have
\[
\left( \int_a^{+\infty} \tilde{f}(x) \, dx \right)_a = \left[ \int_a^{+\infty} \tilde{f}_L^L(x) \, dx, \int_a^{+\infty} \tilde{f}_R^R(x) \right].
\]
Then
\[
\int_a^{+\infty} \tilde{f}_L^L(x) \, dx = \int_0^1 g_a^L(x) \, dx \quad \text{and} \quad \int_a^{+\infty} \tilde{f}_R^R(x) \, dx = \int_0^1 g_a^R(x) \, dx,
\]
where
\[
g_a^L(x) = \frac{\tilde{f}_L^L(\frac{1}{2} + (a-1))}{x^2} \quad \text{and} \quad g_a^R(x) = \frac{\tilde{f}_R^R(\frac{1}{2} + (a-1))}{x^2}.
\]
From the strong law of large numbers for random variables, we have
\[
\frac{1}{n} \left( g_a^L(U_1) + \cdots + g_a^L(U_n) \right) \rightarrow E[g_a^L(U)] = \int_0^1 g_a^L(x) \, dx = \int_a^{+\infty} \tilde{f}_L^L(x) \, dx
\]
\[
= \left( \int_a^{+\infty} \tilde{f}(x) \, dx \right)_a^L
\]
and
\[
\frac{1}{n} \left( g_a^R(U_1) + \cdots + g_a^R(U_n) \right) \rightarrow E[g_a^R(U)] = \int_0^1 g_a^R(x) \, dx = \int_a^{+\infty} \tilde{f}_R^R(x) \, dx
\]
\[
= \left( \int_a^{+\infty} \tilde{f}(x) \, dx \right)_a^R
\]
with probability one. That is, we have
\[
\frac{1}{n} \left( \frac{\tilde{f}_L^L(1/U_1 + (a-1))}{U_1^2} + \cdots + \frac{\tilde{f}_L^L(1/U_n + (a-1))}{U_n^2} \right) \rightarrow \left( \int_a^{+\infty} \tilde{f}(x) \, dx \right)_a^L
\]
and
\[
\frac{1}{n} \left( \frac{\tilde{f}_R^a(1/U_1 + (a-1))}{U_1^2} + \cdots + \frac{\tilde{f}_R^a(1/U_n + (a-1))}{U_n^2} \right) \rightarrow \left( \int_a^{+\infty} \tilde{f}(x) \, dx \right)_R
\]
with probability one. Therefore, from Theorem 4.1 and Proposition 4.1, we have
\[
\tilde{I}_{\{1/n\}} \otimes \left[ \left( \tilde{I}_{\{1/n\}} \otimes \tilde{f} \left( \frac{1}{U_1} + (a-1) \right) \right) \right.
\]
\[
\left. \oplus \cdots \oplus \left( \tilde{I}_{\{1/n\}} \otimes \tilde{f} \left( \frac{1}{U_n} + (a-1) \right) \right) \right] \rightarrow \int_a^{+\infty} \tilde{f}(x) \, dx
\]
with probability one. Hence we can generate a large number of random numbers \( \{u_i\} \) and take
\[
\tilde{I}_{\{1/n\}} \otimes \left[ \left( \tilde{I}_{\{1/n\}} \otimes \tilde{f} \left( \frac{1}{u_1} + (a-1) \right) \right) \oplus \cdots \oplus \left( \tilde{I}_{\{1/n\}} \otimes \tilde{f} \left( \frac{1}{u_n} + (a-1) \right) \right) \right]
\]
as the approximation of the improper fuzzy Riemann integral \( \int_a^{+\infty} \tilde{f}(x) \, dx \).

The improper fuzzy Riemann integrals
\[
\int_a^{-\infty} \tilde{f}(x) \, dx \quad \text{and} \quad \int_a^{+\infty} \tilde{f}(x) \, dx
\]
can also be evaluated similarly using the Monte Carlo method.

6. COMPUTATIONAL PROCEDURE AND EXAMPLES

Let \( \tilde{f}(x) \) be a closed and bounded fuzzy-valued function on \([a, b]\). Suppose that \( \tilde{f}_L^a(x) \) and \( \tilde{f}_R^a(x) \) are continuous on \([a, b]\) for all \( \alpha \in [0, 1] \); then, from Theorem 2.2, \( \tilde{f}(x) \) is fuzzy Riemann-integrable on \([a, b]\). Suppose that we have generated a sequence of random numbers \( \{u_i\} \). Let
\[
A_{\alpha} = \left[ \frac{b-a}{n} \left( \tilde{f}_L^a(a+(b-a)u_1) + \cdots + \tilde{f}_L^a(a+(b-a)u_n) \right) \right.
\]
\[
\left. + \frac{b-a}{n} \left( \tilde{f}_R^a(a+(b-a)u_1) + \cdots + \tilde{f}_R^a(a+(b-a)u_n) \right) \right]
\]
\[
= [\eta(\alpha), \xi(\alpha)].
\]
We write
\[
\tilde{y}_{\alpha}(a, b) = \tilde{I}_{\{b-a/n\}} \otimes \left( \tilde{f}(a+(b-a)u_1) \oplus \cdots \oplus \tilde{f}(a+(b-a)u_n) \right).
\]
Then, from Proposition 2.3, the membership function of \( \tilde{y}_{\alpha}(a, b) \) can be written as
\[
\xi_{\tilde{y}_{\alpha}(a, b)}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_{\alpha}}.
\]
For any fixed $a, b$ and the sequence $\{u_i\}$, the family of closed intervals $\{A_a\}$ is continuously shrinking with respect to $\alpha$, by Proposition 3.3. Therefore, given $r$, there exists an $\alpha_0$ (depends on $r$) such that $r \in A_a$ for $\alpha \leq \alpha_0$ and $r \notin A_a$ for $\alpha > \alpha_0$. Then

$$\psi(\alpha) = \alpha \cdot 1_{A_a}(r) = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq \alpha_0, \\ 0 & \text{if } 1 \geq \alpha > \alpha_0. \end{cases}$$

Thus $\{\alpha : f(\alpha) \geq y\} = [y, \alpha_0]$ is a closed set. This says that $\psi(\alpha) = \alpha \cdot 1_{A_a}(r)$ is upper semicontinuous for any fixed $r$. From Bazarra and Shetty [2], the upper semicontinuous function $\psi(\alpha)$ assumes a maximum over a compact set $[0, 1]$. Therefore, we have

$$\xi_{\tilde{y}_a(a, b)}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_a} = \max_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_a}$$

$$= \max \{\alpha : 0 \leq \alpha \leq 1, r \in A_a = [\eta(\alpha), \zeta(\alpha)]\}.$$

Then we can solve the following mathematical programming problem in order to get the membership value for any fixed $r$.

$$\xi_{\tilde{y}_a(a, b)}(r) = \max \begin{array}{c} \alpha \\ \text{subject to} \\ 0 \leq \alpha \leq 1 \\ \eta(\alpha) \leq r \\ \zeta(\alpha) \geq r \end{array}$$

The membership value of $\xi_{\tilde{y}_a(a, b)}(r)$ for any given $r$ is equal to zero for $r \notin [\eta(0), \zeta(0)]$. That is, the above nonlinear program is infeasible for $r \notin [\eta(0), \zeta(0)]$ since $\eta(\alpha)$ is increasing and $\zeta(\alpha)$ is decreasing. For further analysis, we can discard one of the constraints in the ways described below.

(i) If $\eta(1) \leq r \leq \zeta(1)$ then $\xi_{\tilde{y}_a(a, b)}(r) = 1$.

(ii) If $r < \eta(1)$ then the constraint $\zeta(\alpha) \geq r$ is redundant since $\zeta(\alpha)$ is decreasing. That is, $\zeta(\alpha) \geq \zeta(1) \geq \eta(1) > r$ for $\alpha \in [0, 1]$. Thus we solve the relaxed nonlinear program

$$\xi_{\tilde{y}_a(a, b)}(r) = \max \begin{array}{c} \alpha \\ \text{subject to} \\ 0 \leq \alpha \leq 1 \\ \eta(\alpha) \leq r \end{array}$$

(iii) If $r > \zeta(1)$ then the constraint $\eta(\alpha) \leq r$ is redundant since $\eta(\alpha)$ is increasing. That is, $\eta(\alpha) \leq \eta(1) \leq \zeta(1) < r$ for $\alpha \in [0, 1]$. Thus we solve the relaxed nonlinear program

$$\xi_{\tilde{y}_a(a, b)}(r) = \max \begin{array}{c} \alpha \\ \text{subject to} \\ 0 \leq \alpha \leq 1 \\ \zeta(\alpha) \geq r \end{array}.$$
We say that \( \tilde{a} \) is a triangular fuzzy number if its membership function is

\[
\tilde{\xi}_a(r) = \begin{cases} 
(r - a_1)/(a_2 - a_1) & \text{if } a_1 \leq r \leq a_2 \\
(a_3 - r)/(a_3 - a_2) & \text{if } a_2 < r \leq a_3 \\
0 & \text{otherwise.}
\end{cases}
\]

\( \tilde{a} \) is denoted by \((a_1, a_2, a_3)\). We also have

\[
\tilde{a}_\alpha = [(1 - \alpha)a_1 + \alpha a_2, (1 - \alpha)a_3 + \alpha a_2].
\]

**Example 6.1.** Let \( \tilde{f}(x) = (\tilde{x} \otimes \bar{x}) \oplus (\tilde{2} \otimes \bar{x}) \oplus \bar{x} \) be a closed fuzzy-valued function, where \( \tilde{x} = (x - 1, x, x + 1) \), \( \tilde{1} = (0, 1, 2) \), and \( \tilde{2} = (1, 2, 3) \) are positive triangular fuzzy numbers and \( x \geq 1 \). Then we want to evaluate the fuzzy Riemann integral \( \int_{\tilde{a}}^{\bar{a}} \tilde{f}(x) \, dx \).

First of all, we have

\[
\tilde{x}_L^n = x + \alpha - 1, \quad \tilde{x}_R^n = x - \alpha + 1; \\
\tilde{5}_L^n = 1 + \alpha, \quad \tilde{5}_R^n = 3 - \alpha; \\
\tilde{1}_L^n = \alpha, \quad \tilde{1}_R^n = 2 - \alpha.
\]

Then, from Proposition 4.1, we have

\[
\tilde{f}_L^\alpha(x) = 2\alpha^2 + \alpha(3x - 1) + (x^2 - x)
\]

and

\[
\tilde{f}_R^\alpha(x) = 2\alpha^2 - \alpha(3x + 7) + (x^2 + 5x + 6).
\]

Therefore \( \eta(\alpha) = 4\alpha^2 + B\alpha + C \), where

\[
B = \frac{2}{n} \sum_{i=1}^{n}(2 + 6u_i) \quad \text{and} \quad C = \frac{4}{n} \sum_{i=1}^{n} u_i + \frac{8}{n} \sum_{i=1}^{n} u_i^2,
\]

and \( \xi(\alpha) = 4\alpha^2 - D\alpha + E \), where

\[
D = \frac{2}{n} \sum_{i=1}^{n}(10 + 6u_i) \quad \text{and} \quad E = 24 + \frac{28}{n} \sum_{i=1}^{n} u_i + \frac{8}{n} \sum_{i=1}^{n} u_i^2.
\]

Let \( U \) be a \( U(1, 0) \) random variable. Then \( E[U] = 1/2 \) and \( Var[U] = 1/12 \). We know that

\[
\bar{U} = \frac{1}{n} \sum_{i=1}^{n} U_i \quad \text{and} \quad \frac{1}{n-1} \sum_{i=1}^{n} (U_i - \bar{U})^2
\]

are unbiased estimators of \( E[U] \) and \( Var[U] \), respectively. Since \( u_i \) are random numbers generated from \( U(0, 1) \), we see that

\[
\tilde{u} = \frac{1}{n} \sum_{i=1}^{n} u_i \approx E[U] = 1/2 \quad \text{and} \quad \frac{1}{n-1} \sum_{i=1}^{n} \left( u_i - \frac{1}{2} \right)^2 \approx Var[U] = 1/12.
\]

Then we get

\[
\sum_{i=1}^{n} u_i^2 \approx \frac{4n - 1}{12}.
\]
We also have
\[ B = 4 + 12\bar{u}, \quad C = 4\bar{u} + \frac{8}{n} \sum_{i=1}^{n} u_i^2, \quad D = 20 + 12\bar{u}, \quad E = 24 + 28\bar{u} + \frac{8}{n} \sum_{i=1}^{n} u_i^2. \]

Therefore \( B \approx 10, \quad C \approx 2 + 32/12 = 14/3, \quad D \approx 26, \quad \text{and} \quad E \approx 38 + 32/12 = 122/3. \) This also says that \( \eta(\alpha) \approx 4\alpha^2 + 10\alpha + 14/3 \) and \( \zeta(\alpha) \approx 4\alpha^2 - 26\alpha + 122/3 \) when \( n \) is sufficiently large. We see that \( n(1) = \zeta(1) \approx 56/3. \) This value 56/3 is just the integral \( \int_{1}^{\alpha}(x^2 + 2x + 1) \, dx. \)

Since \( \eta(\alpha) \) is increasing and \( \zeta(\alpha) \) is decreasing, for any given \( r \), we can just consider the roots of the equation \( \eta(\alpha) = r \) if \( r < 56/3 \) and the roots of the equation \( \zeta(\alpha) = r \) if \( r > 56/3 \) instead of solving the mathematical programming problems. We conclude the result as follows.

<table>
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<tr>
<th>( r )</th>
<th>Membership value ( \xi(r) )</th>
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</thead>
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<tr>
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<td>0.723786547</td>
</tr>
<tr>
<td>15.00000000</td>
<td>0.786131954</td>
</tr>
<tr>
<td>16.00000000</td>
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</tr>
<tr>
<td>17.00000000</td>
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</tr>
<tr>
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<td>0.962653008</td>
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<tr>
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<tr>
<td>19.00000000</td>
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</tr>
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<td>21.00000000</td>
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<td>22.00000000</td>
<td>0.821866286</td>
</tr>
<tr>
<td>23.00000000</td>
<td>0.770920870</td>
</tr>
</tbody>
</table>

Now that the fuzzy Riemann integral is a fuzzy number, we can just say that this integral is around 56/3. If we are comfortable in saying that this integral will lie in a closed interval with belief degree \( \alpha \), then this closed interval can be taken as \( [\eta(\alpha), \zeta(\alpha)] \). For example, if we are comfortable in saying that this integral will lie in a closed interval with belief degree 0.98, then this closed interval can be taken as \( [\eta(0.98), \zeta(0.98)] = [18.30826667, 19.02826667] \). In this example, the random numbers \( \{u_i\} \) are not really generated. We can still get the approximate results by using the estimation theory in statistical analysis.

We consider the improper fuzzy Riemann integral \( \int_{a}^{+\infty} \tilde{f}(x) \, dx \). Let
\[
A_{\alpha} = \left[ \frac{1}{n} \left( \tilde{f}_{\alpha}^L \left( \frac{1}{u_1} + (a - 1) \right) / u_1^2 + \cdots + \tilde{f}_{\alpha}^L \left( \frac{1}{u_n} + (a - 1) \right) / u_n^2 \right) + \frac{1}{n} \left( \tilde{f}_{\alpha}^R \left( \frac{1}{u_1} + (a - 1) \right) / u_1^2 + \cdots + \tilde{f}_{\alpha}^R \left( \frac{1}{u_n} + (a - 1) \right) / u_n^2 \right) \right],
\]
\[
\approx [\eta(\alpha), \zeta(\alpha)].
\]
We write
\[ \tilde{y}_n(a) = \tilde{1}_{(1/n)} \otimes \left( \left( \tilde{1}_{(1/n)} \otimes \tilde{f}\left( \frac{1}{u_1} + (a - 1) \right) \right) \right. \]
\[ \left. \oplus \cdots \oplus \left( \tilde{1}_{(1/n)} \otimes \tilde{f}\left( \frac{1}{u_n} + (a - 1) \right) \right) \right] \].

Then, from Proposition 2.3, the membership function of \( \tilde{y}_n(a) \) can be written as
\[ \xi_{\tilde{y}_n(a)}(r) = \sup_{0 \leq \alpha \leq 1} \alpha \cdot 1_{A_r}. \]

The computational procedure proposed above is still applicable in this case.

**Example 6.2.** Let \( \tilde{f}(x) = \tilde{1} \otimes [(\tilde{1} \otimes \tilde{x}) \otimes (\tilde{1} \otimes \tilde{x})] \) be a closed fuzzy-valued function, where \( \tilde{x} = (x-1, x, x+1) \) and \( \tilde{1} = (0, 1, 2) \) are positive triangular fuzzy numbers and \( x \geq 1 \). We will evaluate the improper fuzzy Riemann integral \( \int_{1}^{\infty} \tilde{f}(x) \, dx \). First of all, we have
\[ \tilde{x}_n^L = x + \alpha - 1, \quad \tilde{x}_n^R = x - \alpha + 1; \]
\[ \tilde{1}_n^L = \alpha, \quad \tilde{1}_n^R = 2 - \alpha. \]

Then, from Proposition 4.1, we have
\[ \tilde{f}_n^L(x) = \frac{\alpha}{(x-2\alpha+3)^2} \quad \text{and} \quad \tilde{f}_n^R(x) = \frac{2-\alpha}{(x+2\alpha-1)^2}. \]

Therefore
\[ \eta(\alpha) = \frac{\alpha}{n} \sum_{i=1}^{n} \frac{1}{[1+(3-2\alpha)u_i]^2} \quad \text{and} \quad \zeta(\alpha) = \frac{2-\alpha}{n} \sum_{i=1}^{n} \frac{1}{[1+(2\alpha-1)u_i]^2}. \]

Next we want to obtain the approximate closed form of \( \eta(\alpha) \) and \( \zeta(\alpha) \). We first consider the summand of \( \eta(\alpha) \) by looking at the distribution \( X = \frac{1}{[1+(3-2\alpha)U]^2} \), where \( U \) is a \( U(0, 1) \) random variable. Then we have
\[ F_X(x) = P\{X \leq x\} = P\left\{ \frac{1}{[1+(3-2\alpha)U]^2} \leq x \right\} \]
\[ = P\left\{ U \geq \left( \frac{1}{3-2\alpha} \right) \left( \frac{1}{\sqrt{x}} - 1 \right) \right\} + P\left\{ U \leq \left( \frac{1}{3-2\alpha} \right) \left( -\frac{1}{\sqrt{x}} - 1 \right) \right\} \]
\[ = 1 - \left( \frac{1}{3-2\alpha} \right) \left( \frac{1}{\sqrt{x}} - 1 \right) \quad \text{(since} \left( \frac{1}{3-2\alpha} \right) \left( -\frac{1}{\sqrt{x}} - 1 \right) < 0 \text{).} \]
Therefore, the p.d.f. of $X$ is given by

$$f_X(x) = \left(\frac{1}{6 - 4\alpha}\right)\left(\frac{1}{x\sqrt{x}}\right) \quad \text{for} \quad \frac{1}{(4-2\alpha)^2} \leq x \leq 1.$$ 

The expectation of $X$ can be easily calculated as

$$E[X] = \frac{1}{4 - 2\alpha}.$$ 

Then $\eta(\alpha)$ can be rewritten as

$$\eta(\alpha) = \frac{\alpha}{n} \sum_{i=1}^{n} \frac{1}{1 + (3 - 2\alpha)u_i} = \frac{\alpha}{n} \sum_{i=1}^{n} x_i = \alpha \cdot \bar{x} \approx \alpha \cdot E[X] = \frac{\alpha}{4 - 2\alpha}$$

since $\bar{x}$ is an unbiased estimate of $E[X].$ Now we consider $\zeta(\alpha)$ by looking at the distribution

$$Y = \frac{1}{1 + (2\alpha - 1)U}.$$ 

We have two cases to be discussed.

(i) If $0 \leq \alpha \leq 1/2$, then $2\alpha - 1 \leq 0.$ We have

$$F_Y(y) = P\{Y \leq y\} = P\left\{\frac{1}{1 + (2\alpha - 1)U} \leq y\right\}$$

$$= P\left\{U \leq \left(\frac{1}{2\alpha - 1}\right)\left(\frac{1}{\sqrt{y}} - 1\right)\right\} + P\left\{U \geq \left(\frac{1}{2\alpha - 1}\right)\left(-\frac{1}{\sqrt{y}} - 1\right)\right\}$$

$$= \left(\frac{1}{2\alpha - 1}\right)\left(\frac{1}{\sqrt{y}} - 1\right) \quad \text{(since} \quad \left(\frac{1}{2\alpha - 1}\right)\left(-\frac{1}{\sqrt{y}} - 1\right) > 1).$$

Then p.d.f. of $Y$ is given by

$$f_Y(y) = \left(\frac{1}{2 - 4\alpha}\right)\left(\frac{1}{y\sqrt{y}}\right) \quad \text{for} \quad 1 \leq y \leq \frac{1}{4\alpha^2}.$$ 

The expectation of $Y$ is $E[Y] = 1/2\alpha$.

(ii) If $1/2 \leq \alpha \leq 1$, then $2\alpha - 1 \geq 0$. We have

$$F_Y(y) = P\left\{U \geq \left(\frac{1}{2\alpha - 1}\right)\left(\frac{1}{\sqrt{y}} - 1\right)\right\} + P\left\{U \leq \left(\frac{1}{2\alpha - 1}\right)\left(-\frac{1}{\sqrt{y}} - 1\right)\right\}$$

$$= 1 - \left(\frac{1}{2\alpha - 1}\right)\left(\frac{1}{\sqrt{y}} - 1\right) \quad \text{(since} \quad \left(\frac{1}{2\alpha - 1}\right)\left(-\frac{1}{\sqrt{y}} - 1\right) < 0).$$

Then the p.d.f. of $Y$ is given by

$$f_Y(y) = \left(\frac{1}{4\alpha - 2}\right)\left(\frac{1}{y\sqrt{y}}\right) \quad \text{for} \quad \frac{1}{4\alpha^2} \leq y \leq 1.$$ 

The expectation of $Y$ is $E[Y] = 1/2\alpha.$
From (i) and (ii), we conclude that the expectation of $Y$ is

$$E[Y] = \frac{1}{2\alpha}.$$ 

Therefore $\zeta(\alpha)$ can be rewritten as

$$\zeta(\alpha) = \frac{2 - \alpha}{n} \sum_{i=1}^{n} \frac{1}{\left[1 + (2\alpha - 1)u_i\right]^2} = \frac{2 - \alpha}{n} \sum_{i=1}^{n} y_i = (2 - \alpha) \cdot \bar{y} \approx (2 - \alpha) \cdot E[Y] = \frac{2 - \alpha}{2\alpha}.$$ 

We see that $\eta(1) = \zeta(1) \approx 1/2$. This value of $1/2$ is just the improper integral $\int_{1}^{+\infty} \frac{1}{(x + 1)^2} \, dx$. Since $\eta(\alpha)$ is increasing and $\zeta(\alpha)$ is decreasing, for any given $r$, we can just consider the roots of the equation $\eta(\alpha) = r$ if $r < 1/2$ and the roots of the equation $\zeta(\alpha) = r$ if $r > 1/2$ instead of solving the mathematical programming problems. That is, we have $\alpha = 4r/(1 + 2r)$ if $r < 1/2$ and $\alpha = 2/(1 + 2r)$ if $r > 1/2$. We conclude the result as follows.

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If we are comfortable in saying that this integral will lie in a closed interval with belief degree 0.98, then this closed interval can be taken as $[\eta(0.98), \zeta(0.98)] = [0.480392157, 0.520408163]$. In this example, the random numbers $\{u_i\}$ are not really generated. We can still get the approximate results using the methods of the change of variables in probability theory and the estimation theory in statistical analysis.

REFERENCES