On the rationality of moduli spaces of vector bundles on Fano surfaces

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Communicated by L. Robbiano; received 28 September 1996; received in revised form 28 July 1997

Abstract

In this paper we prove the rationality of the moduli space of rank two, H-stable vector bundles on Fano surfaces. © 1999 Elsevier Science B.V. All rights reserved.

AMS Classification: 14F05; 14D20; 14J99

1. Introduction

Let X be a smooth algebraic surface over the complex field and let $\overline{M}_L(c_1,c_2)$ be the moduli space of rank two torsion free sheaves $E$ on $X$ semistable with respect to a polarization $L$ (in the sense of Gieseker–Maruyama) with $\det(E) = c_1 \in \text{Pic}(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$ and let $M_L(c_1,c_2)$ be the open subscheme parameterizing $L$-stable (in the sense of Mumford–Takemoto) locally free sheaves. Moduli spaces for stable vector bundles on algebraic surfaces were constructed in the 1970s. Since then, many mathematicians have studied their structure, from the point of view of algebraic geometry, topology and of differential geometry, giving very pleasant connections between these areas. In this paper we will take a strictly algebraic point of view.

It is well known that $M_L(c_1,c_2)$ is a quasi-projective variety and for $c_2$ sufficiently large it is non-empty (see [7, 14]), generically smooth of dimension $4c_2 - c_1^2 - 3\chi(O_X)$ [5,26] and irreducible (see [8,20]).

This paper is concerned about the rationality of $M_L(c_1,c_2)$. To be more precise, we are interested in the following question:

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1 Partially supported by DGICYT PB94-0850 and DGICYT PB93-0034.
2 Partially supported by DGICYT PB94-0850.
Question. Let $X$ be a smooth, rational, projective surface. Fix a polarization $L$, $c_1 \in \text{Pic}(X)$ and $0 \ll c_2 \in \mathbb{Z}$. Is $M_L(c_1,c_2)$ rational?

For $X = \mathbb{P}^2$, Maruyama (resp. Ellingsrud and Stromme) proved that if $c_1^2 - 4c_2 \not\equiv 0 \pmod{8}$, then the moduli space $M(c_1,c_2)$ of stable rank 2 vector bundles on $\mathbb{P}^2$ with Chern classes $c_1$ and $c_2$ is rational [6, 17]. Later on, Maeda proved that the rationality of the moduli space $M(c_1,c_2)$ holds for all $(c_1,c_2) \in \mathbb{Z}^2$ provided $M(c_1,c_2)$ is non-empty [13].

The goal of this paper is to give an affirmative answer to the above question when $X$ is a smooth Fano surface.

As a main tool we use the birational properties of moduli spaces of rank 2 stable vector bundles on algebraic surfaces. In [21–23], Qin studies the change of $M_L(c_1,c_2)$ when $L$ varies. It turns out that the ample cone of $X$ has a chamber structure such that $M_L(c_1,c_2)$ only depends on the chamber of $L$ and, in general, $M_L(c_1,c_2)$ changes when $L$ passes through the wall between two chambers (see [23] and [21]).

We say that an irreducible component $M$ of a moduli space $M_L(c_1,c_2)$ is trivial if for any polarization $H$, there exists a sheaf in $M$ which is also $H$-stable. A polarization $L$ is trivial of type $(c_1,c_2)$ if every irreducible component of the moduli space $M_L(c_1,c_2)$ is trivial. In [21], Qin states the following conjecture:

Conjecture (Qin [21]). Trivial polarizations of type $(c_1,c_2)$ exist when $4c_2 - c_1^2$ is larger than some constant $c = c(X)$ depending on $X$.

The first goal of this paper is to prove Qin’s conjecture for smooth projective anticanonical rational surfaces, i.e. rational surfaces $X$ whose anticanonical divisor $-K_X$ is effective. To be more precise, we prove that if $X$ is an anticanonical rational surface then any polarization $L$ is trivial of type $(c_1,c_2)$ provided $M_L(c_1,c_2)$ is non-empty and $4c_2 - c_1^2 > 2 - 3K_X^2/2$; and for any two ample divisors $L_1$ and $L_2$ the moduli spaces $M_{L_1}(c_1,c_2)$ and $M_{L_2}(c_1,c_2)$ are birational whenever non-empty and $4c_2 - c_1^2 > 2 - 3K_X^2/2$ (Theorem 3.9). Therefore, for many purposes we can fix the polarization $L$ and this is what we will always do.

Smooth projective rational surfaces $X$ whose anticanonical divisor $-K_X$ is effective constitute an interesting class of surfaces with Kodaira dimension $\kappa(X) \leq 0$. For instance, they include all Del Pezzo surfaces, all blowing up of relatively minimal models of rational surfaces at 8 or fewer points, and all smooth complete toric surfaces, but also include surfaces for which there is an effective but highly non-reduced anticanonical divisor.

The structure of this paper is as follows: In Section 2, we review the birational properties of moduli spaces $M_L(c_1,c_2)$ of rank two $L$-stable vector bundles on a smooth, projective surface $X$ needed in the sequel. In Section 3, we compute the invariant $d(\xi)$ (Corollary 3.4) and we prove Theorem 3.9, which fully solves Qin’s conjecture for smooth, projective, anticanonical rational surfaces. Even more, we give explicitly the constant $c = c(X)$ which only depends on $X$. As application, we give sufficient conditions on $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$ in order to assure, for any polarization $L$. 
the rationality of the moduli space \( M_L(c_1, c_2) \) of rank two \( L \)-stable vector bundles with Chern classes \((c_1, c_2)\) on a smooth, projective, anticanonical, rational surface (Theorem 3.12). In Section 4, we prove the rationality of the moduli space \( M_L(c_1, c_2) \) of rank two \( L \)-stable vector bundles with Chern classes \((c_1, c_2), c_2 \gg 0\), on a smooth Fano surface \( X \) (Theorem A). According to the classification of smooth Fano surfaces, we distinguish two cases: (a) \( X \) is a quadric surface (Theorem 4.2.4) and (b) \( X \) is a Del Pezzo surface obtained blowing up \( s, 1 \leq s \leq 8 \), different points of \( \mathbb{P}^2 \) (Theorem 4.3.9). In both cases, we analyze separately all possible values of the first Chern class and we prove the rationality using either the criterion given in Theorem 3.12 or elementary transformations or constructing suitable families of rank 2 vector bundles on \( X \).

This paper was written in the context of the group "Vector bundles on higher-dimensional varieties" of Europroj.

**Notation.** Let \( X \) be a smooth algebraic surface over the complex number field. We denote by \( \text{Num}(X) \) the group of divisors modulo numerical equivalence, and by \( C_X \) the cone in \( \text{Num}(X) \otimes \mathbb{R} \) generated by all ample divisors. A polarization on \( X \) is an element in \( C_X \). We will identify \( H^4(X, \mathbb{Z}) \) with \( \mathbb{Z} \).

2. Background material

2.1. Chern classes and Euler–Poincaré characteristic

First of all, we recall the formulas for the Chern classes and the Euler–Poincaré characteristic for vector bundles on non-singular projective surfaces with canonical line bundle \( K = K_X \).

2.1.1. Let \( E \) be a rank \( r \) vector bundle on a non-singular projective variety of dimension \( n \) and let \( L \) be a line bundle on \( X \). Then,

\[
c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E)c_1(L)^{k-i}.
\]

2.1.2. Let \( E \) be a rank \( r \) vector bundle on a non-singular projective surface. Let \( c_1 \) and \( c_2 \) be the Chern classes of \( E \). Then,

\[
\chi(E) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, E) = r(1 + p_a(X)) + c_1(-K/2) + (c_1^2 - 2c_2)/2.
\]

2.1.3. Let \( X \) be a smooth algebraic surface with canonical line bundle \( K \) and let \( V \) be a rank 2 vector bundle on \( X \). It holds

1. If \( V \) is \( H \)-stable and \( c_1(V)H < 0 \) then \( H^0(V) = 0 \).
2. If \( V \) is \( H \)-stable, \( \chi(V) > 0 \) and \( c_1(V^* \otimes K)H < 0 \) then \( H^0(V) \neq 0 \).
2.2. Walls and Chambers structures

We now recall some results about walls and chambers from [21–23].

Definition 2.2.1 (see Qin [23], Definition 1.2.1.5). Let $C_X$ be the ample cone in $\mathbb{R} \otimes \text{Num}(X)$. For $\xi \in \text{Num}(X)$ let $W^\xi := \{x \in \text{Num}(X) \otimes \mathbb{R} \mid x \cdot \xi = 0\}$. $W^\xi$ is called the wall of type $(c_1, c_2)$ determined by $\xi$ if and only if there exists $G \in \text{Pic}(X)$ with $G \equiv \xi$ such that $G + c_1$ is divisible by 2 in $\text{Pic}(X)$ and $c_1^2 - 4c_2 \leq G^2 < 0$. $W^\xi$ is non-empty if there is a polarization $L$ with $L \xi = 0$. Let $W(c_1, c_2)$ be the union of the walls of type $(c_1, c_2)$. A chamber of type $(c_1, c_2)$ is a connected component of $C_X \setminus W(c_1, c_2)$. A face of type $(c_1, c_2)$ is the intersection between a chamber of type $(c_1, c_2)$ and a wall of the same type.

Remark 2.2.2. In [23], Qin proves that $M_L(c_1, c_2)$ only depends on the chamber of $L$ (Corollary 2.2.2) and that the study of moduli spaces of rank two vector bundles stable with respect to a polarization lying on walls may be reduced to the study of moduli spaces of rank two vector bundles stable with respect to a polarization lying in chambers (Remark 2.2.6). We will denote by $M_\Phi(c_1, c_2)$ (resp. $M_\mathcal{F}(c_1, c_2)$) the moduli space $M_L(c_1, c_2)$ where $L$ is a polarization lying in the chamber $\Phi$ (resp. face $\mathcal{F}$).

Definition 2.2.3. Let $\xi$ be a numerical equivalence class defining a wall of type $(c_1, c_2)$. We define $E_\xi(c_1, c_2)$ to be the quasi-projective variety parameterizing rank 2 vector bundles $E$ on $X$ given by an extension

$$0 \to O_X(F) \to E \to O_X(c_1 - F) \otimes I_Z \to 0,$$

where $F$ is a divisor with $2F - c_1 \equiv \xi$ and $Z$ is a locally complete intersection 0-cycle of length $c_2 + (c_2^2 - c_1^2)/4$. Moreover, we require that $F$ is not given by the trivial extension when $c_2^2 = c_1^2 - 4c_2$.

We define $D(\xi) := \dim E_\xi(c_1, c_2)$ and we put $d(\xi) := d(c_1, c_2) := D(\xi) = (4c_2 - c_1^2 - 3\chi(O_X))$; i.e., $d(\xi)$ is the difference between the dimension of $E_\xi(c_1, c_2)$ and the expected dimension of a non-empty moduli space $M_L(c_1, c_2)$.

The local structure of $M_L(c_1, c_2)$ is described precisely by the Kuranishi deformation theorem. In particular, if $\text{End}_0(V)$ denotes the trace-free endomorphisms of $E$ and $H^2(\text{End}_0(V)) = 0$, then $M_L(c_1, c_2)$ is smooth at $V$, the Zariski tangent space $T_{[V]} M_L(c_1, c_2)$ is identified with $H^1(\text{End}_0(V))$ and the Hirzebruch–Riemann–Roch theorem gives us:

$$\dim_{[V]} M_L(c_1, c_2) = -\chi(\text{End}_0(V)) = 4c_2 - c_1^2 - 3\chi(O_X).$$

By [23, Theorem 1.2.5], if $L_1$ and $L_2$ are two ample divisors on $X$ and $E$ is a rank 2 vector bundle on $X$ which is $L_1$-stable but $L_2$-unstable, then $E \in E_\xi(c_1, c_2)$ where $\xi$ defines a non-empty wall of type $(c_1, c_2)$ separating $L_1$ and $L_2$ (i.e. $\xi L_1 < 0 < \xi L_2$; moreover, we can consider the ample divisor $L := (\xi L_2) L_1 - (\xi L_1) L_2$ on $X$ and we have $L \xi = 0$).
Remark 2.2.4. So, if \(d(\xi) < 0\) for any \(\xi\) which defines a non-empty wall of type \((c_1, c_2)\), then any two moduli spaces \(M_{\xi}(c_1, c_2)\) and \(M_{\xi}(c_1, c_2)\) are birational whenever non-empty and any polarization \(L\) is trivial of type \((c_1, c_2)\) provided \(M_{\xi}(c_1, c_2)\) is non-empty.

3. Calculation of \(d(\xi)\) and comparison of moduli spaces

Throughout this section \(X\) will be a smooth, projective, anticanonical, rational surface, i.e. a rational surface whose anticanonical divisor \(-K_X\) is effective. In particular, \(p_a(X) = p_g(X) = q(X) = 0\). From now on we fix \(c_1 \in \text{Pic}(X)\), we assume that \(\xi\) determines a non-empty wall of type \((c_1, c_2)\) and \(F\) is any divisor such that \(\xi \geq 2F - c_1\). The first goal of this section is to calculate \(d(\xi)\).

Lemma 3.1. With the above notation it holds

(1) \(H^0(X, O_X(2F - c_1)) = 0\) and \(H^0(X, O_X(-2F + c_1)) = 0\); and

(2) \(H^0(X, O_X(K_X - (2F - c_1))) = 0\) and \(H^0(X, O_X(K_X + (2F - c_1))) = 0\).

Proof. (1) Since \(\xi\) defines a non-empty wall we have \(\xi L_1 > 0 > \xi L_2\) for some ample divisors \(L_1\) and \(L_2\). Thus, \(2F - c_1\) and \(c_1 - 2F\) are not effective and \(H^0(X, O_X(2F - c_1)) = H^0(X, O_X(-2F + c_1)) = 0\).

(2) Since \(X\) is a smooth projective anticanonical surface, the divisor \(-K_X\) is effective and \(-K_X L \geq 0\) for any ample divisor \(L\). If \(K_X + 2F - c_1\) or \(K_X - 2F + c_1\) are effective then \((K_X + 2F - c_1)L \geq 0\) or \((K_X - 2F + c_1)L \geq 0\) for any ample divisor \(L\); i.e., \((2F - c_1)L \geq -K_X L\) or \((-2F + c_1)L \geq -K_X L\) for any ample divisor \(L\). However, the inequalities \(0 > (2F - c_1)L_2\) and \(0 > (-2F + c_1)L_1\) give us \(-K_X L_1 < 0\) and \(-K_X L_2 < 0\) which contradicts the fact that \(-K_X\) is effective.

Remark 3.2. Assume that \(\xi\) determines a non-empty wall of type \((c_1, c_2)\). Then,

(1) \(-h^1(O_X(-\xi)) = \chi(O_X(-\xi)) = \xi(\xi + K_X)/2 + 1\); and

(2) \(-h^1(O_X(\xi)) = \chi(O_X(\xi)) = \xi(\xi - K_X)/2 + 1\).

Proof. It easily follows from Lemma 3.1 and the Riemann–Roch’s Theorem. In particular, we have:

(1) \(\xi(\xi + K_X)/2 + 1 \leq 0\); and

(2) \(\xi(\xi - K_X)/2 + 1 \leq 0\).

Lemma 3.3. Let \(Z \subset X\) be a locally complete intersection 0-cycle of length \(l(Z) = c_2 + (\xi^2 - c_1^2)/4\). Then, we have

\[\dim(Ext^1(I_Z, O_X(2F - c_1))) = (4c_2 - c_1^2 - \xi^2)/4 + (\xi K_X)/2 - 1.\]

Proof. We apply the functor \(Hom(., O_X(2F - c_1))\) to the exact sequence

\[0 \to I_Z \to O_X \to O_Z \to 0\]
and we get the exact sequence
\[
0 \rightarrow H^1(O_X(2F - c_1)) \rightarrow \text{Ext}^1(I_Z, O_X(2F - c_1)) \rightarrow H^0(O_Z) \\
\rightarrow H^2(O_X(2F - c_1)) \rightarrow \text{Ext}^2(I_Z, O_X(2F - c_1)) \rightarrow 0.
\]

Using Lemma 3.1, Serre’s duality and the Euler–Poincaré characteristic, we obtain
\[H^2(O_X(2F - c_1)) = 0\] and \[h^1(O_X(2F - c_1)) = -1 - (c^2 - 2c_1)/2.\] Therefore, we have
\[
\dim(\text{Ext}^1(I_Z, O_X(2F - c_1))) = h^0(O_Z) + h^1(O_X(2F - c_1))
\]
\[= (4c_2 - c^2_1 - c^2) / 4 + (\xi K_X) / 2 - 1. \]

Corollary 3.4. With the above notation, we have
(1) \[D(\xi) = 3(4c_2 - c^2_1)/4 + c^2/4 + (\xi K_X)/2 - 2; \] and
(2) \[d(\xi) = (c^2_1 - 4c_2)/4 + c^2/4 + (\xi K_X)/2 + 1. \]

Proof. (1) Any vector bundle \(E\) in \(E_\xi(c_1, c_2)\) sits in a non-trivial extension
\[0 \rightarrow O_X(F) \rightarrow E \rightarrow O_X(c_1 - F) \otimes I_Z \rightarrow 0,
\]
where \(F\) is a divisor with \(2F - c_1 \equiv \xi\) and \(Z\) is a locally complete intersection 0-cycle of length \(c_2 + (\xi^2 - c^2_1)/4.\) Moreover, the invertible sheaf \(O_X(F)\) and the 0-cycle \(Z\) are uniquely determined by \(E.\) Therefore, \[\dim E_\xi(c_1, c_2) = \# \text{moduli } O_X(F) + \# \text{moduli } (Z) + \dim \text{Ext}^1(I_Z, O_X(2F - c_1)) - h^0E(-F).
\]
On the other hand, \[h^0E(-F) = h^0(O_X) = 1, \] \[\# \text{moduli } (Z) = 2 \text{ length}(Z) = (4c_2 + \xi^2 - c^2_1)/2, \] \[\# \text{moduli } O_X(F) = q(X) = 0 \] and \[
\dim(\text{Ext}^1(I_Z, O_X(2F - c_1))) = (4c_2 - c^2_1 - \xi^2) / 4 + (\xi K_X) / 2 - 1 (\text{Lemma 3.3}).
\] Thus, we have
\[D(\xi) = 3(4c_2 - c^2_1)/4 + c^2/4 + (\xi K_X)/2 - 2.
\]
(2) By definition we have \[d(\xi) := d_\xi(c_1, c_2) = D(\xi) - (4c_2 - c^2_1 - 3\chi(O_X)) = (c^2_1 - 4c_2) / 4 + c^2/4 + (\xi K_X)/2 + 1. \]

Remark 3.5. Notice that for any numerical equivalence class \(\xi\) which defines a non-empty wall of type \((c_1, c_2)\) we have \(d(\xi) \leq 0.\) In fact, it follows from Remark 3.2 and the inequality \(c^2_1 - 4c_2 \leq c^2\).

Proposition 3.6. Let \(X\) be a smooth, projective, anticanonical, rational surface, \(L\) a polarization and \(\xi\) a numerical equivalence class defining a wall of type \((c_1, c_2).\) Assume \(d(\xi) = 0.\) It holds
(a) If \(E \in M_\xi(c_1, c_2)\) then \(\chi(E((-c_1 - \xi)/2)) = 1.\)
(b) If \(E \in M_\xi(c_1, c_2)\) and \(4c_2 - c^2_1 > 2 - 3K^2_X / 2\) then \(h^0(E((-c_1 - \xi)/2)) > 0.\)

Remark 3.7. We point out that \(E((-c_1 - \xi)/2)\) has sense because \(\xi + c_1\) is divisible by 2 in \(\text{Pic}(X)\) (see Definition 2.2.1).

Proof. First of all notice that, by Remark 3.5, the hypothesis \(d(\xi) = 0\) is equivalent to \(c^2 = c^2_1 - 4c_2\) and \(c^2 + 2K_X + 2 = 0.\)
(a) Applying 2.1.1 and 2.1.2 we easily see that
\[ c_1(E((-c_1 - \zeta)/2))) = -\zeta, \]
\[ c_2(E((-c_1 - \zeta)/2))) = 0, \] and
\[ \chi(E((-c_1 - \zeta)/2))) = 1. \]

(b) First, we prove that the divisor \(- (2K + \zeta)\) is effective. Indeed, since \(\zeta\) is not effective and \(-K_X\) is effective we have \(h^0(\zeta + 3K_X) = 0\) and by Serre's duality \(h^2(- \zeta - 2K_X) = 0\). Therefore, applying Riemann–Roch's theorem we get
\[
h^0(- \zeta - 2K_X) = h^1(- \zeta - 2K_X) = \chi(- \zeta - 2K_X) = (- \zeta - 2K_X)(- \zeta - 3K_X)/2 + 1 = 2(4c_2 - c_1^2 - 2) + 3K_X^2 > 0
\]
which gives us \(h^0(- \zeta - 2K_X) > 0\) or, equivalent, \(- (\zeta + 2K_X)\) is effective. Hence, \(- (2K + \zeta)L \geq 0\) for any ample divisor \(L\) on \(X\) or, equivalently,
\[
c_1((E((-c_1 - \zeta)/2))^* \otimes K)L = (2K + \zeta)L \leq 0.
\]
If the last inequality is strict we obtain (Fact 2.1.3)
\[
h^0(E((-c_1 - \zeta)/2))) > 0.
\]
If \(c_1((E((-c_1 - \zeta)/2))^* \otimes K)L = 0\) we get
\[
h^0(E((-c_1 - \zeta)/2))) > 0 \quad \text{or} \quad h^2(E((-c_1 - \zeta)/2))) > 0
\]
and we will prove that the last inequality is not possible. Indeed, by Serre duality,
\[
0 < h^2(E((-c_1 - \zeta)/2))) = h^0(E^*((c_1 + \zeta)/2 + K_X)).
\]
A non-zero section of \(h^0(E^*((c_1 + \zeta)/2 + K_X))\) defines an injection
\[
O_X(c_1 - \zeta/2 - K_X) \hookrightarrow E
\]
and from the \(L\)-stability of \(E\) we have
\[
\left(\frac{c_1 - \zeta}{2} - K_X\right) L < \frac{c_1 L}{2}
\]
which contradicts the fact \((2K + \zeta)L = 0\). \(\square\)

The following corollary will be the key point for proving the main result of this section.
Corollary 3.8. Let $X$ be a smooth, projective, anticanonical, rational surface, $L$ a polarization and $\xi$ a numerical class defining a wall of type $(c_1, c_2)$. Assume $d(\xi) = 0$. If $\xi L \geq 0$ and $4c_2 - c_1^2 > 2 - 3K_X^2/2$ then $M_L(c_1, c_2) = \emptyset$.

Proof. Assume $M_L(c_1, c_2) \neq \emptyset$. For any $E \in M_L(c_1, c_2)$ we apply Proposition 3.7 and we take a non-zero section $s \in H^0(E((-c_1 - \xi)/2))$. It defines an injection

$$O_X\left(\frac{c_1 + \xi}{2}\right) \hookrightarrow E.$$ 

Since $E$ is $L$-stable, we have

$$\left(\frac{c_1 + \xi}{2}\right) L \leq \frac{c_1 L}{2},$$

i.e.; $\xi L < 0$ which contradicts the hypothesis $\xi L \geq 0$. $\square$

Theorem 3.9. Let $X$ be a smooth, projective, anticanonical, rational surface, $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - (3K_X^2/2)$. We have

(a) Any polarization $L$ is trivial of type $(c_1, c_2)$ provided $M_L(c_1, c_2)$ is non-empty.

(b) For any two ample divisors $L_1$ and $L_2$ on $X$ the moduli spaces $M_{L_1}(c_1, c_2)$ and $M_{L_2}(c_1, c_2)$ are birational whenever non-empty.

Proof. By [23, Remark 2.2.6] we may assume that $L_1$ and $L_2$ lie in chambers. Let $\mathcal{C}_1$ be the chamber containing $L_1$ and $\mathcal{C}_2$ the chamber containing $L_2$. If $\mathcal{C}_1 = \mathcal{C}_2$, then the moduli spaces can be naturally identified [23, Corollary 2.2.2]. Assume $\mathcal{C}_1 \neq \mathcal{C}_2$. Since the set of walls of type $(c_1, c_2)$ is locally finite [23, Proposition 2.1.6], we can choose finitely many ample divisors

$$L_1 = L^{(1)}, L^{(2)}, \ldots, L^{(r-1)}, L^{(r)} = L_2,$$

on the line segment connecting $L_1$ and $L_2$ in such a way that we have

1. $L^{(i)}$ lies in some chamber for all $i = 1, \ldots, r$; and
2. $L^{(i)}$ and $L^{(i+1)}$ are separated by a single wall for all $i = 1, \ldots, r - 1$.

So, without loss of generality, we may suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common wall $W$ of type $(c_1, c_2)$. Take $\xi \in \text{Num}(X)$ such that $W^\xi = W$. Since $L \xi \geq 0$ implies $M_L(c_1, c_2) = \emptyset$ (Corollary 3.8) and the moduli spaces $M_{L_1}(c_1, c_2)$ and $M_{L_2}(c_1, c_2)$ are non-empty we deduce $d(\xi) \neq 0$ and, hence, $d(\xi) < 0$. Therefore, we have [23, Theorem 1.3.3]

$$M_{L_1}(c_1, c_2) = \left( M_{L_1}(c_1, c_2) \bigcup_{\eta} E_{-\eta}(c_1, c_2) \right) \bigcup_{\eta} \left( \bigcup_{\eta} E_{\eta}(c_1, c_2) \right),$$

where $\eta$ satisfies $\eta L < 0$ for some $L \in \mathcal{C}_1$ and runs over all numerical equivalence classes which define the common wall $W = W^\xi$. Moreover, $d(\eta) < 0$ (Remark 3.5 and Corollary 3.8) and we conclude that $M_{L_1}(c_1, c_2)$ and $M_{L_2}(c_1, c_2)$ are birationally equivalent. $\square$
Corollary 3.10. Let $X$ be a smooth Fano surface, $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. We have

(a) Any polarization $L$ is trivial of type $(c_1,c_2)$ provided $M_L(c_1,c_2)$ is non-empty.

(b) For any two ample divisors $L_1$ and $L_2$ on $X$ the moduli spaces $M_{L_1}(c_1,c_2)$ and $M_{L_2}(c_1,c_2)$ are birational whenever non-empty.

Proof. Any smooth Fano surface $X$ is rational and anticanonical. So, the result follows from Theorem 3.9 because if the moduli space $M_L(c_1,c_2)$ is non-empty then $4c_2 - c_1^2 > 0 > 2 - (3K_X^2/2)$. □

We will end this section proving that under some extra conditions the moduli spaces $M_L(c_1,c_2)$ are either empty or smooth, irreducible, rational quasi-projective varieties of dimension $4c_2 - c_1^2 - 3$.

Proposition 3.11. Let $X$ be a smooth, projective, anticanonical, rational surface, $L$ a polarization, $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Then, the moduli space $M_L(c_1,c_2)$ is empty or a smooth, irreducible quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.

Proof. Assume $M_L(c_1,c_2) \neq \emptyset$. Since $-K_X$ is effective, for any vector bundle $E \in M_L(c_1,c_2)$, we have $H^2(\text{End}_0 E) = 0$. Hence, $M_L(c_1,c_2)$ is smooth at $E$ and $\dim[E] M_L(c_1,c_2) = 4c_2 - c_1^2 - 3$. The irreducibility of $M_L(c_1,c_2)$ follows from [2, Theorem 2.2] and Theorem 3.9. □

Theorem 3.12. Let $X$ be a smooth, projective, anticanonical, rational surface, $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - (3K_X^2/2)$ and that there exists a numerical equivalence class $\xi$ which defines a non-empty wall of type $(c_1,c_2)$ such that $d(\xi) = 0$ (i.e. $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + 4K_X + 2 = 0$). Then, for any polarization $L$ on $X$, the moduli space $M_L(c_1,c_2)$ is a smooth, irreducible, rational quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ whenever non-empty.

Proof. By Theorem 3.9(b), it is enough to see that if $L$ is an ample divisor such that $\xi L < 0$ and $L \in \mathcal{C}$ with $W^\xi \cap \text{cloures}(\mathcal{C}) \neq \emptyset$ then $M_L(c_1,c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$. For such $L$ and $\mathcal{C}$ we have [23, Proposition 1.3.1]:

$$M_L(c_1,c_2) = M_{\mathcal{F}}(c_1,c_2) \bigcup \left( \bigcup_{\mu} E_{\mu}(c_1,c_2) \right),$$

where $\mathcal{F}$ is one of the faces of $\mathcal{C}$ contained in $W^\xi$, $\mu L < 0$ for some $L \in \mathcal{C}$ and $\mu$ runs over all numerical equivalence classes which define the wall $W^\xi$. For any $L' \in \mathcal{F}, L' \xi = 0$. So $M_{\mathcal{F}}(c_1,c_2) = \emptyset$ (Corollary 3.8). Moreover, $W^\mu = W^\eta$ if only if $\mu = \lambda \eta$, for some $\lambda \in \mathbb{R}$. Therefore, we conclude

$$M_L(c_1,c_2) \cong E_{\xi}(c_1,c_2).$$

Let us see that $E_{\xi}(c_1,c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$. By definition, for any $E$ in $E_{\xi}(c_1,c_2)$, we have the exact sequence

$$0 \to O_X(F) \to E \to O_X(c_1 - F) \otimes I_Z \to 0.$$
where $F$ is a divisor with $2F - c_1 \equiv \xi$ and $Z$ is a locally complete intersection 0-cycle with $i(Z) = c_2 + (\xi^2 - c_1^2)/4$. By hypothesis $d(\xi) = 0$. Thus, $\xi^2 = c_1^2 - 4c_2$ and $Z = 0$. Therefore, $E$ is given by a non-trivial extension

$$0 \rightarrow O_X(F) \rightarrow E \rightarrow O_X(c_1 - F) \rightarrow 0,$$

where $F \equiv (\xi + c_1)/2$ i.e; $E \in \mathbb{P}(H^1(\xi))$. Finally, using Riemann–Roch’s Theorem, we get $\mathbb{P}(H^1(\xi)) \cong \mathbb{P}^{c_2 - c_1^2 - 3}$. □

Remark 3.13. The results of this section also works for smooth projective surfaces with anticanonical bundle $-K_X$ numerically effective (a divisor is said to be numerically effective if its intersection number with any effective divisor is non-negative) and arithmetic genus $p_a = 0$. See [3] for a complete classification of smooth projective surfaces with anticanonical bundle numerically effective.

We would like to end this section pointing out that if $L_1$ and $L_2$ are two polarization lying in different chambers then the birational map between $M_{L_1}(c_1,c_2)$ and $M_{L_2}(c_1,c_2)$ is not, in general, an isomorphism. The structure of these birational maps will be studied in [4].

4. Rationality of moduli spaces on Fano surfaces

The goal of this last section is to prove the rationality of the moduli space $M_L(c_1,c_2)$ of rank two $L$-stable vector bundles $E$ on Fano surfaces with Chern classes $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. So from now on $X$ will be a smooth Fano surface.

4.1. Generalities

First of all, we review some facts on families of rank 2 vector bundles on $X$ needed in the sequel. To this end we need to fix some more notations.

Let $H_l := \text{Hilb}_l(X)$ be the Hilbert scheme of zero-dimensional subschemes of length $l$ on $X$ and let $\mathcal{F}_l$ be the ideal sheaf of the universal subscheme $\mathcal{F}_l$ in $X \times H_l$. Let $\pi$ and $\pi_X$ be the projections of $X \times H_l$ to $H_l$ and $X$ respectively. For any $D,c_1 \in \text{Pic}(X)$, we define $G_1 := p_X^*(D)$ and $G_2 := \mathcal{F}_l \otimes p_X^*(D + c_1)$. We put

$$\delta_{D,c_1} := \text{Ext}^1(G_2,G_1),$$

where $\text{Ext}^1(G_2,\cdot)$ is the right derived functor of $\text{Hom}_X(G_2,\cdot) := \pi_* \text{Hom}(G_2,\cdot)$.

Lemma 4.1.1. Let $U$ (resp. $\mathcal{F}$) be the family of rank 2 torsion free sheaves (resp. vector bundles) $E$ on $X$ given by a non-trivial extension

$$0 \rightarrow O_X(-D) \rightarrow E \rightarrow O_X(D + c_1) \otimes I_2 \rightarrow 0,$$

where $Z \subset X$ is a 0-cycle of length $l$. Assume that $\pm(2D + c_1)$ and $2D + c_1 + K$ are non effective divisors. Then, $U$ (resp. $\mathcal{F}$) is an irreducible, rational, projective (resp. quasi-projective) variety.
Proof. We apply \cite[Corollary 11.44]{24}, to the functors $F = \pi_*$ and $G = \mathcal{H}om(G_2, \cdot)$ and we get the long exact sequence

$$
0 \rightarrow R^1\pi_*(\mathcal{H}om(G_2, G_1)) \rightarrow R^1(\pi_*\mathcal{H}om(G_7, \cdot))(G_1) \rightarrow \pi_*(R^4\mathcal{H}om(G_2, G_1)) \\
\rightarrow R^2\pi_*(\mathcal{H}om(G_2, G_1)) \rightarrow R^2(\pi_*\mathcal{H}om(G_2, \cdot))(G_1).
$$

By base-change theorem, we can see, arguing as in \cite[Lemma 3.21]{9} (see also \cite{11}), that $\mathcal{E}_{D,c_1}$ is a locally free sheaf of rank $r = \dim \mathcal{E}xt^1(O_X(D + c_1) \otimes I_Z, O_X(-D))$ and there is a natural bijective morphism

$$
\psi : \mathbb{P}(\mathcal{E}_{D,c_1}) \rightarrow U.
$$

Therefore, $U$ (resp. $\mathcal{F}$) is an irreducible, rational, projective (resp. quasi-projective) variety. \hfill $\Box$

Let us recall the concept of elementary transformation. These transformations were introduced first in the case of vector bundles over curves by Tyurin \cite{25}, and their general definition is due to Maruyama \cite{15,16}.

**Definition 4.1.2.** Let $E$ be a rank $r$ vector bundle over a smooth algebraic variety $X$ and $Z \subset X$ a hypersurface. Denote by $i : Z \rightarrow X$ the embedding. Suppose that we have chosen some quotient bundle $F$ of the restriction $i^*E$. Then we have a surjective map of sheaves $E \rightarrow i_*F$ on $X$. We define the coherent sheaf $\text{Elm}_{E,F} := \text{Ker}(E \rightarrow i_*F)$. It is not difficult to see that $\text{Elm}_{E,F}$ is a rank $r$ vector bundle over $X$; it is called the elementary transformation of $E$ along $(Z,F)$.

Moreover, the vector bundle $E$ can be reconstructed from its elementary transformation by applying the "inverse" elementary transformation. Indeed, with the above notations let $E(Z)$ be the sheaf whose sections are sections of $E$ with simple poles along $Z$. Then, $\text{Elm}_{E,F}$ is a subsheaf of $E(Z)$ whose sections after multiplying by the local equation of $Z$ belong to $\text{Elm}_{E,F} = \text{Ker}(E \rightarrow i_*F)$. $\text{Elm}_{E,F}$ and $\text{Elm}_{E,F}^\perp$ are mutually inverse operations.

To end with the generalities, let us to review the classification of smooth Fano surfaces. By definition a smooth Fano surface is a smooth irreducible projective surface with ample anticanonical line bundle.

Let $X$ be a smooth Fano surface. Set $d = K_X \cdot K_X$. We have \cite{1}:

(i) $1 \leq d \leq 9$.

(ii) • If $d = 9$ then $X \cong \mathbb{P}^2$.

• If $d = 8$ then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X$ is the blow up of $\mathbb{P}^2$ in a point.

• If $1 \leq d \leq 7$ then $X$ is the blow up of $\mathbb{P}^2$ in $9 - d$ different points.

The case $X \cong \mathbb{P}^2$ has been studied by several authors (see \cite{6,13,17}). In the next two subsections we will consider the other cases.
4.2. The quadric surface

Let \( X \cong \mathbb{P}^1 \times \mathbb{P}^1 \) be a nonsingular quadric surface in \( \mathbb{P}^3 \). Then \( \text{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z} \) ([10]; II, Example 6.6.1) and we can take as generators lines \( l \) and \( m \) one from each family. Then \( l^2 = 0 \), \( m^2 = 0 \) and \( lm = 1 \). With this notation, the canonical divisor on \( X \) takes the form \( K_X = -2l - 2m \).

It is well known that a divisor \( L = al + bm \) on \( X \) is ample if and only if \( L \) is very ample, if only if \( a > 0 \) and \( b > 0 \), and that \( D = a'l + b'm \) is effective if and only if \( a' \geq 0 \) and \( b' \geq 0 \) ([10], II, Example 7.6.2).

Since a rank 2 vector bundle \( E \) on \( X \) is \( H \)-stable if and only if \( E \otimes O_X(L) \) is \( H \)-stable for any divisor \( L \in \text{Pic}(X) \), we may assume that \( c_1(E) \) is one of the following: \( 0, l, m, l + m \).

**Proposition 4.2.1.** Let \( X \subset \mathbb{P}^3 \) be a smooth quadric surface and \( L \) an ample divisor on \( X \). Then the moduli space \( M_l(l,c_2) \) (resp. \( M_l(m,c_2) \)) is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension \( 4c_2 - 3 \).

**Proof.** In fact, we take the numerical equivalence class \( \xi = l - 2c_2m \) (resp. \( \xi = -2c_2l + m \)).

**Claim.** \( \xi \) defines a non-empty wall of type \( (l,c_2) \) (resp. \( (m,c_2) \)) and \( d(\xi) = 0 \).

**Proof.** We will only prove that \( \xi = l - 2c_2m \) defines a non-empty wall of type \( (l,c_2) \) and \( d(\xi) = 0 \). Similarly, we prove that \( \xi = -2c_2l + m \) defines a non-empty wall of type \( (m,c_2) \).

Since \( \xi + c_1 = 2l - 2c_2m, d(\xi) = 0 \) and \( c_2^2 - 4c_2 = \xi^2 < 0 \), we only have to check (see Definition 2.2.1) that there exist ample divisors \( L \) and \( L' \) such that

\[
\xi L < 0 < \xi L'.
\]

We know that \( L = al + bm \) is an ample divisor if only if \( a > 0, b > 0 \). Thus, for some convenient positive integers we have

\[
\xi L = (l - 2c_2m)(al + bm) = (-2c_2)a + b < 0,
\]

and

\[
\xi L' - (l - 2c_2m)(a'l + b'm) = (-2c_2)a' + b' > 0.
\]

Thus, we can apply Proposition 3.11 and Theorem 3.12 and we get the required result. \( \square \)

**Proposition 4.2.2.** Let \( X \subset \mathbb{P}^3 \) be a smooth quadric surface and \( H \) an ample divisor on \( X \). Then the moduli space \( M_H(l + m,c_2) \) is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension \( 4c_2 - 5 \).
Proof. In fact, we take the numerical equivalence class \( \xi = l + (1 - 2c_2)m \). Arguing as in Proposition 4.2.1 we can see that \( \xi \) defines a non-empty wall of type \((l + m, c_2)\) and \( d(\xi) = 0 \). So, we apply Proposition 3.11 and Theorem 3.12 and we get what we want. \( \square \)

For \( c_1 = 0 \), there is no a numerical equivalence class \( \xi \) satisfying the hypothesis of the criterion stated in Theorem 3.12. We will prove the rationality of this remaining case using elementary transformations.

**Proposition 4.2.3.** Let \( X \subset \mathbb{P}^3 \) be a smooth quadric surface, \( H \) an ample divisor on \( X \), and \( c_2 \in \mathbb{N} \). Then, the moduli space \( M_H(0, c_2) \) is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension \( 4c_2 - 3 \).

**Proof.** By Proposition 3.11 and Theorem 3.9 we only need to check the rationality of \( A_{4c_2}(0, c_2) \) for a suitable ample divisor \( L \) on \( X \).

We take the ample divisor on \( X \): \( L = (c_2 + 2)l + m \). Fix \( l_0 \in |l| \) and denote by \( N \subset M_L(l, c_2) \) the open subset parameterizing rank two vector bundles \( E \in M_L(l, c_2) \) such that

\[
E|_{l_0} \cong O_{l_0} \oplus O_{l_0}.
\]

**Claim.** \( N \) is non-empty.

**Proof.** Let \( E \) be a rank two vector bundle on \( X \) given by a non-trivial extension

\[
0 \to O_X(-c_2l) \to E \to O_X((c_2 + 1)l) \otimes I_Z \to 0,
\]

where \( Z \subset X \) is a locally complete intersection 0-cycle of length \( c_2 \) such that \( Z \cap l_0 = \emptyset \) and \( H^0(X, O_X((2c_2 + 1)l) \otimes I_Z) = 0 \). It is easy to check that \( c_1(E) = l \) and \( c_2(E) = c_2 \).

Let us see that for any rank 1 subbundle \( F \) of \( E \) we have: \( c_1(F) < 0 \). In particular,

\[
c_1(F)L < \frac{c_1(E)L}{2} = \frac{l((c_2 + 2)l + m)}{2} = \frac{1}{2}
\]

and \( E \) is \( L \)-stable.

Since \( E \) sits in an extension

\[
0 \to O_X(-c_2l) \to E \to O_X((c_2 + 1)l) \otimes I_Z \to 0
\]

we have \( O_X(F) \hookrightarrow O_X(-c_2l) \) or \( O_X(F) \hookrightarrow O_X((c_2 + 1)l) \otimes I_Z \). In the first case, \(-F - c_2l\) is an effective divisor. Since \( L \) is an ample divisor we have \((-F - c_2l)L \geq 0 \) and

\[
c_1(O_X(F))L = FL \leq -c_2lL < 0 < \frac{c_1(E)L}{2}.
\]

If \( O_X(F) \hookrightarrow O_X((c_2 + 1)l) \otimes I_Z \), \((c_2 + 1)l - F \) is an effective divisor. On the other hand, we have \( H^0(X, O_X(F + c_2l)) \subset H^0(X, O_X((2c_2 + 1)l) \otimes I_Z) = 0 \). So \( F + c_2l \) is not an effective divisor. Write \( F = \alpha l + \beta m \), we have \( \alpha + c_2 < 0 \) or \( \beta < 0 \).
Assume that \( \alpha + c_2 < 0 \) (in particular, \( \alpha < 0 \)). Since \((c_2 + 1)l - F\) is an effective divisor \( \beta \leq 0 \), \( c_2 + 1 - \alpha \geq 0 \) and we have
\[
    c_1(O_X(F))L = FL = \alpha + \beta (c_2 + 2) < 0 < \frac{IL}{2} = \frac{c_1(E)L}{2}.
\]

Assume that \( \beta < 0 \) and \( \alpha + c_2 \geq 0 \). Since \((n + 1)l - F\) is an effective divisor \( \beta \leq 0 \), \( c_2 + 1 - \alpha > 0 \) and we have
\[
    c_1(O_X(F))L = FL = \alpha + \beta (c_2 + 2) \leq c_2 + 1 + \beta (c_2 + 2) < 0 < \frac{IL}{2} = \frac{c_1(E)L}{2},
\]
which proves the \( L \)-stability of \( E \). Therefore, \( E \in M_L(l, c_2) \). Moreover, twisting the exact sequence
\[
    0 \to O_X(-c_2l) \to E \to O_X((c_2 + 1)l) \otimes I_Z \to 0
\]
by \( O_{l_0} \) we obtain an epimorphism
\[
    E_{|l_0} \to O_X((c_2 + 1)l) \otimes I_Z \otimes O_{l_0} \cong O_{l_0},
\]
whose kernel is isomorphic to \( O_{l_0} \) and we deduce that \( E_{|l_0} \cong O_{l_0} \oplus O_{l_0} \) which proves our claim.

Now, we will see that the map which associates to any vector bundle \( E \in N \) the elementary transformation of \( E \) along \( l_0 \) defines an injective morphism from \( N \) to \( M_L(0, c_2) \). Indeed, for any \( E \in N \) we have an epimorphism
\[
    E_{|l_0} \to O_{l_0}
\]
which composed with the natural restriction map \( E \to E_{|l_0} \), gives us an epimorphism \( \alpha : E \to O_{l_0} \). Let \( E' \) be the kernel of the epimorphism \( \alpha \). \( E' \) is a rank 2 vector bundle on \( X \) so-called the elementary transformation of \( E \) along \( (l_0, O_{l_0}) \) (see Definition 4.1.2). Using the exact sequence
\[
    0 \to E' \to E \to O_{l_0} \to 0
\]
we easily see that \( c_1(E') = 0 \) and \( c_2(E') = c_2 \). Let us to see that \( E' \) is \( L \)-stable. Let \( F \) be any rank 1 subbundle of \( E' \). In particular, \( F \) is a subbundle of \( E \) and we have \( FL < 0 \). Therefore, we obtain
\[
    c_1(O_X(F))L = FL < 0 = \frac{c_1(E')L}{2}.
\]
So, \( E' \) is \( L \)-stable and we have constructed an injective morphism
\[
    \rho : N \to M_L(0, c_2),
\]
\[
    E \to E',
\]
where \( E' \) is the elementary transformation of \( E \) along \( (l_0, O_{l_0}) \).

We know that \( M_L(l, c_2) \) is a rational, smooth, irreducible quasi-projective variety of dimension \( 4c_2 - 3 \) (Proposition 4.2.1) and that \( M_L(0, c_2) \) is a smooth, irreducible quasi-projective variety of dimension \( 4c_2 - 3 \). We have an open dense subset \( N \subset M_L(l, c_2) \).
and an injective morphism from $N$ to $M_L(0, c_2)$. Hence, we can conclude that $M_L(0, c_2)$ is rational. □

**Remark.** For an alternative proof of Proposition 4.2.3 without using elementary transformations see [4].

**Theorem 4.2.4.** Let $X \subset \mathbb{P}^3$ be a smooth quadric surface, $c_1 \in \text{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Then for any polarization $L$ on $X$, the moduli space $M_L(c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ whenever non-empty.

**Proof.** It easily follows from Propositions 4.2.1–3. □

4.3. **Blowing up points in $\mathbb{P}^2$**

Let $Z = \{P_1, \ldots, P_s\}$ be a set of $s$, $1 \leq s \leq 8$, distinct points in $\mathbb{P}^2 = \mathbb{P}^2 \subset \mathbb{P}^3$, let $X := \mathbb{P}^2(Z)$ be a Del Pezzo surface obtained from $\mathbb{P}^2$ by blowing up the points of $Z$ and let $\pi : X := \mathbb{P}^2(Z) \to \mathbb{P}^2$ be the blow up. Let $E_1, \ldots, E_s$ be the divisor classes in $\text{Pic}(X)$ which contain the exceptional lines corresponding to the blow ups of the points $P_1, \ldots, P_s$, respectively; and let $E_0$ be the divisor class in $\text{Pic}(X)$ which contains the transform of a line in $\mathbb{P}^2$ which misses all the points of $Z$. Then $\text{Pic}(X) \cong \mathbb{Z}^{s+1} \cong \langle E_0, E_1, \ldots, E_s \rangle$, $E_0^2 = 1 = -E_i^2 = \cdots = -E_s^2$, and $E_i E_j = 0$ if $i \neq j$. Once having obtained $X$, there may be other such morphisms $X \to \mathbb{P}^2$ and any such morphism factors into a sequence of blowings-up at points giving rise, as above, to a basis of $\text{Pic}(X)$. Such a basis, arising from a morphism $X \to \mathbb{P}^2$, is called an exceptional configuration.

**Remark 4.3.1.** Since a rank 2 vector bundle $E$ on $X$ is $H$-stable if and only if $E \otimes O_X(L)$ is $H$-stable for any divisor $L \in \text{Pic}(X)$, we may assume that $c_1(E)$ is one of the following: $0$, $E_0$, $E_i$ with $i = 1 \div s$, $\sum_{j=2}^{s} E_i$, with $2 \leq \rho \leq s$ or $E_0 + \sum_{j=1}^{p} E_i$, with $1 \leq \rho \leq s$; and $c_2(E) \geq 1$ ([18], Theorems 2.1 and 2.2).

**Proposition 4.3.2.** Let $X$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$, $L$ any ample divisor on $X$. Then the moduli space $M_L(0, c_2)$, is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$.

**Proof.** For $t \gg 0$, the divisor $H = tE_0 - \sum_{i=1}^{s} E_i$ is ample on $X$ and there is an open immersion ([19, Theorem 2.3])

$$M_{D^2}(0, c_2) \to M_H(0, c_2).$$

Furthermore, $M_{D^2}(0, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$ and $M_H(0, c_2)$ is a smooth, irreducible, quasi-projective variety of dimension $4c_2 - 3$. Hence, $M_H(0, c_2)$ is rational and the moduli space $M_L(0, c_2)$ is empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$ (Theorem 3.9 and Proposition 3.11). □
Proposition 4.3.3. Let $X$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$, $L$ any polarization on $X$. Then, the moduli space $M_L(E_0, c_2)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 4$.

Proof. We take the numerical equivalence class $\xi = (1 - 2c_2)E_0 + 2c_2E_1$. For all $c_2 > 0$, $\xi \equiv (1 - 2c_2)E_0 + 2c_2E_1$ defines a non-empty wall of type $(E_0, c_2)$ and $d(\xi) = 0$. Indeed, since $c_1^2 - 4c_2 = \xi^2 = 1 - 4c_2 < 0$, $\xi + c_1 = (2 - 2c_2)E_0 + 2c_2E_1$ and $d(\xi) = 0$, we only have to check that there exist ample divisors $L_t$ and $L'_r$ such that

$$\xi L_t < 0 < \xi L'_r.$$ 

It is well known that the divisor $L_t = tE_0 - \sum_{i=1}^{s}E_i$ is ample for $t \geq 3$ and $L_t \xi < 0$ for $t \gg 0$. On the other hand, we take the divisor $L'_r = n((4c_2 + 1)E_0 - (4c_2 - 1)E_1) - \sum_{j=2}^{s}E_j$. For $n \gg 0$, $L'_r$ is ample. Furthermore, we have

$$\xi L'_r = ((1 - 2c_2)E_0 + 2c_2E_1)(n(4c_2 + 1)E_0 - n(4c_2 - 1)E_1) = n > 0$$

which proves what we want.

Thus, we can apply Proposition 3.11 and Theorem 3.12 and we deduce that the moduli space $M_L(E_0, c_2)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 4$. \qed

Proposition 4.3.4. Let $X$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and let $L$ be any ample divisor on $X$. Then the moduli space $M_L(\sum_{i=1}^{s}E_i, n)$, $1 \leq \rho \leq s$, is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4n + \rho - 3$.

Proof. We take the numerical equivalence class $\xi = 2c_2E_0 + (2c_2 + 1)E_1 \cdots E_s$, $1 \leq \rho \leq s$. Once more applying Proposition 3.11 and Theorem 3.12 we get the required result. \qed

For the remaining values of $c_1$, there is no a numerical class $\xi$ satisfying the hypothesis of the criterion stated in Theorem 3.12. In these cases, we will prove the rationality constructing a suitable family over a big rational variety and using elementary transformations.

Remark 4.3.5. Let $X$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and $L$ any polarization on $X$. A rank two vector bundle $E$ on $X$ is $L$-stable if only if for all $F \in \text{Pic}(X)$ such that $FL \geq c_1(E)L/2$, we have $h^0(E(-F)) = 0$. It easily follows from the definition of $L$-stability.

Proposition 4.3.6. Let $X$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$ and $L$ any polarization on $X$. For $c_2 \gg 0$ the moduli space $M_L(E_0 + \sum_{i=1}^{s}E_i, c_2)$, $2 \leq \rho \leq s$, is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 4$. 
Proof. By Corollary 3.10 we only need to check the rationality of $M_l(E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ for a suitable ample divisor $L$ on $X$ and by ([19, Theorem 2.3]) we can assume $s = p$. We take $L = 3\alpha E_0 - 3(x - 1)E_1 - \sum_{i=2}^{\rho} E_i$ with $\alpha \gg 0$; it follows from [12] that $L$ is an ample divisor on $X$.

If $c_2 = 2n$ (resp. $c_2 = 2n + 1$), we consider the irreducible family $\mathcal{F}_n$ (resp. $\mathcal{H}_n$) of rank 2 vector bundles $E$ on $X$ given by a non-trivial extension

$$0 \to O_X \to E(D) \to O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right) \otimes I_Z \to 0,$$

where $D = (n+1)E_0 - (n+2)E_1 - 2E_2$ (resp. $D = nE_0 - (n+1)E_1 - E_2$) and $Z$ is a locally complete intersection 0-cycle of length $2n - 2$ (resp. $2n + 1$).

Let us show

(a) $dim \mathcal{F}_n = dim \mathcal{H}_n = 4c_2 + p - 4$.

(b) A general vector bundle $E$ of $\mathcal{F}_n$ (resp. $\mathcal{H}_n$) is $L$-stable and has Chern classes $(E_0 + \sum_{i=1}^{\rho} E_i, c_2)$.

(a) We only compute the dimension of $\mathcal{F}_n$. The dimension of $\mathcal{H}_n$ can be calculated similarly.

By definition we have

$$dim \mathcal{F}_n = \# \text{moduli}(Z) + dim Ext^1 \left(I_Z, O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right)\right) \otimes I_Z - h^0 E(D)$$

$$= 2 \text{length}(Z) + dim Ext^1 \left(I_Z, O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right)\right) \otimes I_Z - h^0 E(D).$$

From the exact cohomology sequence

$$0 \to H^0(O_X) \to H^0(E(D)) \to H^0 \left(O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right) \otimes I_Z\right) \to 0$$

associated to the exact sequence

$$0 \to O_X \to E(D) \to O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right) \otimes I_Z \to 0$$

we get $h^0(E(D)) = h^0(O_X) + h^0(O_X(2D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z) = 1$; because the divisor $2D + E_0 + \sum_{i=1}^{\rho} E_i = (2n + 3)E_0 - (2n + 3)E_1 - 3E_2 + \sum_{i=3}^{\rho} E_i$ is not effective. On the other hand, if we apply the functor $\text{Hom}(., O_X)$ to the exact sequence

$$0 \to I_Z \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right) \to O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right)$$

$$\to O_X \left(2D + E_0 + \sum_{i=1}^{\rho} E_i\right) \to 0$$
we get the exact sequence

\[ 0 \to H^1 \left( O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ \to \text{Ext}^1 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ \to H^0(O_Z) \to H^2 \left( O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ \to \text{Ext}^2 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \to 0. \]

But the divisor \(-2D - E_0 - \sum_{i=1}^{\rho} E_i\) is not effective, so we obtain

\[ \dim \text{Ext}^1 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ = \dim \text{Ext}^2 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ + h^0(O_Z) - \chi \left( O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right). \]

By Riemann–Roch’s Theorem we have

\[ \chi \left( O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ = \frac{\left( -(2D + E_0 + \sum_{i=1}^{\rho} E_i) \right)^{\left( 2 + (2D + E_0 + \sum_{i=1}^{\rho} E_i) + 3E_0 - \sum_{i=1}^{\rho} E_i \right)} + 1}{2} \]

\[ = \frac{(2n + 3)^2 - (2n + 3)^2 - 9 - (\rho - 2) - 3(2n + 3) + (2n + 3) + 3 - (\rho - 2)}{2} + 1 = -2n - 3 - \rho. \]

By Serre’s duality, we have

\[ \text{Ext}^2 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right) \]

\[ = \text{Hom} \left( O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right), I_Z \otimes K_X \right) \]
\begin{equation}
= H^0 \left( O_X \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i + K_X \right) \otimes I_Z \right)
\end{equation}

\begin{equation}
= H^0 \left( I_Z \left( 2nE_0 - (2n + 2)E_1 - 2E_2 + 2 \sum_{i=3}^{s} E_i \right) \right) = 0.
\end{equation}

The last equality follows from the fact that the divisor $2nE_0 - (2n + 2)E_1 - 2E_2$ is not effective. Putting all together we obtain

\begin{equation}
dim Ext^1 \left( I_Z, O_X \left( - \left( 2D + E_0 + \sum_{i=1}^{\rho} E_i \right) \right) \right)
= \text{length}(Z) - (-2n - 3 - \rho) = 4n + 1 + \rho
\end{equation}

and

\begin{equation}
dim \mathcal{F}_n = 2(2n - 2) + (4n + 1 + \rho) - 1 = 8n - 4 + \rho = 4c_2 + \rho - 4.
\end{equation}

(b) We will only check that a general vector bundle $E$ of $\mathcal{F}_n$ is L-stable. Similarly, we can prove the other case.

It is easy to see that for any $F \in \mathcal{F}_n$, $c_1(F) = F_0 + \sum_{i=1}^{\rho} F_i$ and $c_2(E) = c_2$. Let us see that for any rank 1 subbundle $F$ of a general $E \in \mathcal{F}_n$ we have: $c_1(F)L < c_1(E)L/2$. Therefore, $E$ is L-stable.

Since $E$ sits in an extension

\begin{equation}
0 \rightarrow O_X(-D) \rightarrow E \rightarrow O_X \left( D + E_0 | \sum_{i=1}^{\rho} E_i \right) \otimes I_Z \rightarrow 0,
\end{equation}

where $Z$ is a locally complete intersection 0-cycle of length $2n - 2$; for any rank 1 subbundle $F$ of $E$ we have: (1) $O_X(F) \hookrightarrow O_X(-D)$ or (2) $O_X(F) \hookrightarrow O_X(D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z$. In the first case, $-F - D$ is an effective divisor. Since $L$ is an ample divisor we have $(-F - D)L \geq 0$ and

\begin{equation}
c_1(O_X(F))L = FL \leq -DI = 3n - 3n - 4 < 3n + \frac{\rho - 2}{2} = \frac{c_1(E)L}{2}.
\end{equation}

Assume $O_X(F) \hookrightarrow O_X(D + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z$ and recall that for a generic 0-cycle $Z$ in the Hilbert scheme $Hilb^{2n-2}(X)$, if $2n - 2 \geq h^0(O_X(D - F + E_0 + \sum_{i=1}^{\rho} E_i))$ then $h^0(O_X(D - F + E_0 + \sum_{i=1}^{\rho} E_i) \otimes I_Z) = 0$. Thus, we have $2n - 2 < h^0(O_X(D - F + E_0 + \sum_{i=1}^{\rho} E_i))$ and after an intricate computation we deduce that

\begin{equation}
c_1(O_X(F))L = FL < \frac{(E_0 + \sum_{i=1}^{\rho} E_i)L}{2} = \frac{c_1(E)L}{2};
\end{equation}

which proves the L-stability of $E$.

It follows from (b) that there is an injection from an open dense subset of $\mathcal{F}_n$ (resp. $\mathcal{K}_n$) to $M_{L}(E_0 + \sum_{i=1}^{\rho} E_i, c_2)$. Since for $c_2 > 0$, the moduli space $M_{L}(E_0 + \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible quasi-projective variety of dimension $4c_2 - \rho + 3$.
(Theorem 3.11), its rationality easily follows from (b), Lemma 4.1.1 and the fact that 
\( \dim \mathcal{Y}_n - \dim M_L(E_0 + \sum_{i=1}^{\rho} E_i, 2n) \) (resp. \( \dim \mathcal{X}_n - \dim M_L(E_0 + \sum_{i=1}^{\rho} E_i, 2n + 1) \)). □

**Proposition 4.3.7.** Let \( X \) be a Del Pezzo surface obtained blowing up \( s, 1 \leq s \leq 8 \), different points of \( \mathbb{P}^2 \) and \( L \) any ample divisor on \( X \). Then, the moduli space \( M_L(E_0 + E_1, c_2) \), is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension \( 4c_2 - 3 \).

**Proof.** By Proposition 3.11 and Theorem 3.9 we only need to check the rationality of 
\( M_L(E_0 + E_1, c_2) \) for a suitable ample divisor \( L \) on \( X \) and by [19, Theorem 2.3] we can assume that \( s = 1 \).

We fix the ample divisor \( L = \alpha E_0 - (\alpha - 1)E_1, \alpha \gg 0 \), on \( X \) and \( C \in |E_0 - E_1| \). Let us call \( N \subset M_L(0, c_2) \) the subset parameterizing rank two vector bundles \( E \in M_L(0, c_2) \) such that

\[ E|_C \cong O_C \oplus O_C. \]

**Claim.** \( N \) is an open dense subset of \( M_L(0, c_2) \).

**Proof.** We only need to see that \( N \) is non-empty or, equivalent, to construct a rank two vector bundle \( E \in M_L(0, c_2) \) such that \( E|_C \cong O_C \oplus O_C \). To this end, we consider the divisor \( D = ne_0 - ne_1 \) with \( n \in \mathbb{N} \) such that \( c_2 = 2n + 1 \) or \( c_2 = 2n \) and a “generic” locally complete intersection 0-dimensional subscheme \( Z \subset X \) of length \( c_2 \) such that \( Z \cap C = \emptyset \).

The rank 2 vector bundle \( E \) on \( X \) given by a non-trivial extension

\[ 0 \rightarrow O_X \rightarrow E(D) \rightarrow O_X(2D) \otimes I_Z \rightarrow 0 \]

has Chern classes \((0, c_2)\). Let us see that for any rank 1 subbundle \( F \) of \( E \) we have 
\( c_1(F)L < c_1(E)L/2 \). Therefore \( F \) is \( L \)-stable.

Since \( E \) sits in an extension

\[ 0 \rightarrow O_X \rightarrow E(D) \rightarrow O_X(2D) \otimes I_Z \rightarrow 0 \]

for any rank 1 subbundle \( F \) of \( E \) we have: (1) \( O_X(F) \hookrightarrow O_X(-D) \) or (2) \( O_X(F) \hookrightarrow O_X(D) \otimes I_Z \). In the first case, \( -F - D \) is an effective divisor. Since \( L \) is an ample divisor we have \( (-F - D)L \geq 0 \) and

\[ c_1(O_X(F))L = FL \leq -DL = n\alpha - n(\alpha - 1) < 0 = \frac{c_1(E)L}{2}. \]

Assume \( O_X(F) \hookrightarrow O_X(D) \otimes I_Z \) and write \( F = aE_0 - bE_1 \). Thus \( D - F = (n - a)E_0 - (n - b)E_1 \). Since \( Z \) is a generic 0-cycle of length \( c_2 \) and \( h^0(O_X(D - F) \otimes I_Z) \neq 0 \), we have \( c_2 < h^0(O_X(D - F)) \); which implies \( n - a > 0 \) and \( a < b \). Therefore, \( c_1(O_X(F))L = FL = a\alpha - b(\alpha - 1) < 0 = c_1(E)L/2 \) (It is enough to take \( \alpha > n \)) which proves the \( L \)-stability of \( E \).
Finally, twisting the above exact sequence by $O_C$ we obtain an epimorphism

$$E|_C \to O_C$$

and we easily conclude that

$$E|_C \cong O_C \oplus O_C.$$  

We will see that the map which associates to any vector bundle $E \in N$ the elementary transformation of $E$ along $C$ defines an injective morphism from $N$ to $M_L(-E_0 + E_1, c_2)$.

For any $E \in N$ there is an epimorphism

$$E|_C \to O_C.$$  

Let $E'$ be the kernel of the epimorphism $\pi$ obtained composing the above epimorphism with the natural restriction map $E \to E|_C$. $E'$ is a rank 2 vector bundle on $X$ so called the elementary transformation of $E$ along $(C, O_C)$. From the exact sequence

$$0 \to E' \to E \to O_C \to 0,$$

we obtain $c_1(E') = -E_0 + E_1$ and $c_2(E') = c_2$.

Let us see that $E'$ is $L$-stable. Let $F$ be any rank 1 subbundle of $E'$. In particular, $F$ is a subbundle of $E$ and since $E$ is $L$-stable we have $FL \leq -1$. Then we obtain

$$c_1(O_X(F)L) = FL \leq -1 < \frac{-1}{2} = \frac{c_1(E')L}{2}.$$  

So, $E'$ is $L$-stable and we have constructed an injective morphism

$$\rho : N \to M_L(-E_0 + E_1, c_2)$$

$$E \to E'$$

where $E'$ is the elementary transformation of $E$ along $(C, O_C)$.

Furthermore, we know that $M_L(0, c_2)$ is a rational, smooth, irreducible quasi-projective variety of dimension $4c_2 - 3$ (Proposition 4.2.1) and that $M_L(-E_0 + E_1, c_2)$ is a smooth, irreducible quasi-projective variety of dimension $4c_2 - 3$. We have an open dense subset $N \subset M_L(0, c_2)$ and an injective morphism from $N$ to $M_L(-E_0 + E_1, c_2)$. Hence, we can conclude that $M_L(-E_0 + E_1, c_2)$ is rational.

Finally, using the isomorphism $M_L(E_0 + E_1, c_2) \cong M_L(-E_0 + E_1, c_2)$ which sends $E$ to $E \otimes O_X(E_0)$, we obtain the rationality of $M_L(E_0 + E_1, c_2)$. □

**Remark.** For an alternative proof of Proposition 4.3.7 without using elementary transformations, see [4].

**Theorem 4.3.8.** Let $X \subset \mathbb{P}^3$ be a Del Pezzo surface obtained blowing up $s$, $1 \leq s \leq 8$, different points of $\mathbb{P}^2$, $c_1 \in \text{Pic}(X)$ and $0 < c_2 \in \mathbb{Z}$. Then, for any polarization $L$ on $X$, the moduli space $M_L(c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ whenever non-empty.
Proof. It easily follows from Propositions 4.3.2–7. □

Finally, glueing together Theorems 4.2.4 and 4.3.8, we get the main result of this work:

Theorem A. Let $X$ be a smooth Fano surface, $c_1 \in \text{Pic}(X)$ and $0 \ll c_2 \in \mathbb{Z}$. Then, for any polarization $L$ on $X$, the moduli space $M_L(c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ whenever non-empty.

References