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Optimal control of a fractional diffusion equation with state constraints

Gisèle M. Mophou [a,*], Gaston M. N'Guérékata [b]

[a] Laboratoire C.E.R.E.G.M.I.A., Université des Antilles et de la Guyane, Campus Fouillole 97159 Pointe-à-Pitre, (FWI), Guadeloupe

[b] Department of Mathematics, Morgan State University, 1700 East Cold Spring Lane Baltimore, MD 21251, USA

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ABSTRACT

This paper is concerned with the state constrained optimal control problems of a fractional diffusion equation in a bounded domain. The fractional time derivative is considered in the Riemann–Liouville sense. Under a Slater type condition we prove the existence a Lagrange multiplier and a decoupled optimality system.

1. Introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we consider the fractional diffusion equation:

$$\begin{cases} D^{\alpha}_{RL} y - \Delta y = h + v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y(0^+) = 0 & \text{in } \Omega \end{cases} \tag{1}$$

where $0 < \alpha < 1$, the control v and the function h belong to $L^2(Q)$, the fractional integral $I^{1-\alpha}$ and derivative D^{α}_{RL} are understood here in the Riemann–Liouville sense, $I^{1-\alpha}y(0^+) = \lim_{t\to 0^+} I^{1-\alpha}y(t)$.

Fractional-order models seem to be more adequate than integer-order models because fractional derivatives provide an excellent tool for the description of memory and heredity effects of various materials and processes, including gas diffusion and heat conduction, in fractal porous media [1,2]. Sokolov et al. [3], proved that fractional diffusion equations generalize Fick's second law and the Fokker–Planck equation by taking into account memory effects such as the stretching of polymers under external fields and the occupation of deep traps by charge carriers in amorphous semiconductors. Oldham and Spanier [4] discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. Mainardi [5] and Mainardi et al. [6,7] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2.

Optimal control problems with integer order have been widely studied and many techniques have been developed for solving such problems [8–11]. Also, state constrained optimal control problems have attracted several authors in the last three decades, mostly for their importance in various applications in optimal control partial differential equations with an integer time derivative. For such problems, it is well-known that one can derive optimality conditions if one can prove the existence of a Lagrange multiplier associated with the constraint in the state (see for instance [12,13]). For instance, considering a quadratic control for elliptic equations with pointwise constraints, Casas [14] proved the existence of a Lagrange multiplier and derived an optimality condition using results of convex analysis. Barbu and Precupanu [9] and Lasiecka [15] derived the existence of a Lagrange multiplier for some optimal control with integral state constraints. Considering a parabolic system controlled by Neumann conditions and subject to pointwise state constraints on the final

* Corresponding author.
E-mail addresses: gmophou@univ-ag.fr (G.M. Mophou), Gaston.N'Guerekata@morgan.edu (G.M. N'Guérékata).

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state, Mackenroth [16] prove the existence of a multiplier as a solution of a dual problem. By a penalization method, Bergounioux [10] and Bergounioux and Tiba [17] proved the existence of a multiplier and derived optimal conditions for elliptic and parabolic equations with state constraints respectively.

In the area of calculus of variations and optimal control of fractional differential equations, little has been done since that problem has only been recently considered. The first record of the formulation of the fractional optimal control problem was given by Agrawal in [18] where he presented a general formulation and proposed a numerical method to solve such problems. In that paper, the fractional derivative was defined in the Riemann–Liouville sense and the formulation was obtained by means of fractional variation principle [19] and the Lagrange multiplier technique. Following the same technique, Frederico et al. [20] obtained a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. Recently, Agrawal [21] presented an eigenfunction expansion approach for a class of distributed system whose dynamics are defined in the Caputo sense. Following the same approach as Agrawal, in [22] Özdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the Riemann–Liouville sense. In [23], Jelicic et al. formulated necessary conditions for optimal control problems with dynamics described by differential equations of fractional order. Using an expansion formula for the fractional derivative, they proposed optimality conditions and a new solution scheme, using an expansion formula for the fractional derivative. In [24], Baleanu et al. described a formulation for fractional optimal control problems defined in multi-dimensions when the dimensions of the state and control variables are unlike each other. The problem is formulated with the Riemann–Liouville fractional derivatives and the fractional differential equations involving the state and control variables are solved using Grünwald–Letnikov approximation. Zhou [25] considered the following Lagrange problem:

Find $(x_0, u_0) \in C([0, T], \mathbb{X}) \times U_{ad}$ solution of

$$\min_{u \in U_{ad}} \int_0^T \mathcal{L}(t, x^u(t), u(t))\mathrm{d}t$$

where $\mathbb{X}$ is a Banach space, $T > 0$, $C([0, T], \mathbb{X})$ denotes the space of all $\mathbb{X}$-value functions defined and continue on $[0, T]$ and x^u denotes the solution of system $D^\alpha x(t) = -Ax(t) + f(t, x(t)) + C(t)u(t),\ t \in [0, T]; x(0) = x_0$. Under a suitable condition on $\mathcal{L}$, he proved that the Lagrange problem has at least one optimal pair. In [26] Mophou considered the following fractional optimal control problem: find the control $u = u(x, t) \in L^2(Q)$ that minimizes the cost function

$$J(v) = \|y(v) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \quad z_d \in L^2(Q) \text{ and } N > 0$$

subject to the system (1) with $h \equiv 0$. The author proved that the optimal control problem has a unique solution and derived an optimality system. We also refer to [27] where boundary fractional optimal control with finite observation expressed in terms of a Riemann–Liouville integral of order α is studied.

In this paper, we are concerned with a fractional optimal control with constraints on the state. More precisely, we first prove that under the above assumptions on the data, Problem (1) has a unique solution in $L^2(0, T; H^2 \cap H^1_0(\Omega))$ (see Theorem 2.10). Then we define the affine application $\mathbb{T}$, from $L^2(Q)$ to $L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega))$ such that $y = \mathbb{T}(v)$ is the unique solution of (1). We also define the functional $J : L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(Q) \to R_+$ by

$$J(y, v) = \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)} \tag{2}$$

where $z_d \in L^2(Q)$ and $N > 0$.

Finally, we consider the following optimal control problem with constraint on the state:

$$\begin{cases} \min J(y, v), \\ y = \mathbb{T}(v), \\ y \in K \text{ and } v \in \mathcal{U}_{ad} \end{cases} \tag{3}$$

where K and $\mathcal{U}_{ad}$ are two nonempty closed convex subsets of $L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega))$ and $L^2(Q)$ respectively. Using a penalization method, we prove the existence of a Lagrange multiplier and a decoupled optimality condition for the fractional diffusion (1). To the best of our knowledge, the fractional optimal control problem (3) is new since most fractional optimal control problems in the literature are considered for a performance index subject to the system dynamic constraints and the initial condition.

The rest of the paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3 we show that our optimal control problem holds and under a Slater type condition we prove the existence of a Lagrange multiplier and a decoupled optimality system. Concluding remarks are presented in Section 4.

2. Preliminaries

Definition 2.1. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}_+$ and $\alpha > 0$. Then the expression

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\,\mathrm{d}s, \quad t > 0$$

is called the Riemann–Liouville integral of the function f order α.

Definition 2.2 ([28]). Let $f : \mathbb{R}_+ \to \mathbb{R}$. The Riemann–Liouville fractional derivative of order α of f is defined by

$$D_{RL}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s, \quad t > 0,$$

where $\alpha \in (n-1, n),\ n \in \mathbb{N}$.

Definition 2.3 ([28]). Let $f : \mathbb{R}_+ \to \mathbb{R}$. The Caputo fractional derivative of order α of f is defined by

$$D_{C}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \quad t > 0,$$

where $\alpha \in (n-1, n),\ n \in \mathbb{N}$.

Lemma 2.4 ([29,28]). *Let $T > 0, u \in C^m([0;T]), p \in (m-1;m), m \in \mathbb{N}$ and $v \in C^1([0;T])$. Then for $t \in [0;T]$, the following properties hold:*

$$D_{RL}{}^{p} v(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-p} v(t), \quad m = 1, \tag{4}$$

$$D_{RL}^{p} I^{p} v(t) = v(t); \quad D_{C}^{p} I^{p} v(t) = v(t) \tag{5}$$

$$I^{p} D_{RL}^{p} u(t) = u(t) - \sum_{k=1}^{m} \frac{t^{p-k}}{\Gamma(p-k+1)} (I^{k-p} u)^{(m-k)}(0); \tag{6}$$

$$I^{p} D_{C}^{p} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^{k}}{k!} u^{(k)}(0); \tag{7}$$

$$I^{p} D_{RL}^{p} u(t) = u(t) - \frac{t^{p-1}}{\Gamma(p)} (I^{1-p} u)(0) \quad \text{if } m = 1; \tag{8}$$

$$I^{p} D_{C}^{p} u(t) = u(t) - u(0) \quad \text{if } m = 1. \tag{9}$$

From now on we set:

$$\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s)\,\mathrm{d}s. \tag{10}$$

Remark 2.5. $-\mathcal{D}^{\alpha} f(t)$ is the so-called right fractional Caputo derivative. It represents the future state of $f(t)$. For more details on this derivative we refer to [28,29]. Note also that when $T = +\infty$, $\mathcal{D}^{\alpha} f(t)$ is the Weyl fractional integral of order α of $f'(t)$ [30].

Lemma 2.6 ([29,31]). *Let $0 < \alpha < 1$. Let $g \in L^p(0,T)$, $1 \le p \le \infty$ and $\phi :]0,T] \to \mathbb{R}_+$ be the function defined by:*

$$\phi(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Then for almost every $t \in [0,T]$, the function $s \mapsto \phi(t-s)g(s)$ is integrable on $[0,T]$. Set

$$(\phi \star g)(t) = \int_0^t \phi(t-s) g(s)\,\mathrm{d}s.$$

Then $\phi \star g \in L^p(0,T)$ and

$$\|\phi \star g\|_{L^p(0,T)} \le \|\phi\|_{L^1(0,T)} \|g\|_{L^p(0,T)}. \tag{11}$$

We need the following lemmas which assure the integration by parts for a fractional diffusion equation with a Riemann–Liouville derivative for the resolution of the optimal control problem associated with (1).

Lemma 2.7 ([26]). *Let $0 < \alpha < 1$. Then for any $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$, we have*

$$\int_0^T \int_{\Omega} \left(D_{RL}^{\alpha} y(x,t) - \Delta y(x,t) \right) \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega} \varphi(x,T) I^{1-\alpha} y(x,T)\,\mathrm{d}x - \int_{\Omega} \varphi(x,0) I^{1-\alpha} y(x,0^+)\,\mathrm{d}x$$
$$+ \int_0^T \int_{\partial\Omega} y \frac{\partial \varphi}{\partial \nu}\,\mathrm{d}\sigma\,\mathrm{d}t - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu} \varphi\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T \int_{\Omega} y(x,t)(-\mathcal{D}^{\alpha} \varphi(x,t) - \Delta \varphi(x,t))\,\mathrm{d}x\,\mathrm{d}t.$$

From Lemma 2.7, we deduce the following result.

Lemma 2.8. *Let* $0 < \alpha < 1$. *Then for any* $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$ *such that* $\varphi(x, T) = 0$ *in* Ω *and* $\varphi = 0$ *on* Σ, *we have*

$$\int_0^T \int_\Omega \left(D^\alpha_{RL} y(x,t) - \Delta y(x,t)\right)\varphi(x,t)\,dx\,dt = -\int_\Omega \varphi(x,0) I^{1-\alpha} y(x,0^+)\,dx + \int_0^T \int_{\partial\Omega} y\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt + \int_0^T\int_\Omega y(x,t)\,(-\mathcal{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))\,dx\,dt.$$

The following results will be useful to prove that problem (1) as well as the adjoint system of our optimal control problem has a unique solution.

Theorem 2.9 (*Theorem 4.2 [32]*)**.** *Let* $f \in L^2(Q)$. *Then the following fractional diffusion equation with Caputo derivative:*

$$\begin{cases} D^\alpha_C y = \Delta y + f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega \end{cases} \tag{12}$$

has a unique solution $y \in L^2((0,T); H^2(\Omega) \cap H^1_0(\Omega))$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|y\|_{L^2((0,T);H^2(\Omega))} + \|D^\alpha_C y\|_{L^2(Q)} \le C\|f\|_{L^2(Q)}. \tag{13}$$

Theorem 2.10. *Let* $0 < \alpha < 1$ *and* $h, v \in L^2(Q)$. *Then problem* (1) *has a unique solution* $y \in L^2((0,T); H^2(\Omega) \cap H^1_0(\Omega))$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|y\|_{L^2((0,T);H^2(\Omega))} + \|D^\alpha_{RL} y\|_{L^2(Q)} \le C\|f\|_{L^2(Q)}. \tag{14}$$

Proof. Let y be the solution of (1). As

$$D^\alpha_C y(t) = D^\alpha_{RL} y(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0), \tag{15}$$

we have $I^\alpha D^\alpha_C y(t) = I^\alpha D^\alpha_{RL} y(t) + y(0)$ since $I^\alpha(t^{-\alpha}) = 1$. Therefore, using relation (8) and (9), it follows that $y(t) - y(0) = y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} y(0^+) + y(0)$. Thus $I^{1-\alpha}y(0^+) = 0$ implies $y(0) = 0$ and from (15), we get $D^\alpha_C y(t) = D^\alpha_{RL} y(t)$ for a.e. $t \in (0, T]$. We thus have proved that if y is solution of (1) then y satisfies (12).

Conversely, let y be the solution of (12). Then from (15), we obtain that $D^\alpha_C y(t) = D^\alpha_{RL} y(t)$ for a.e. $t \in (0, T]$ since $y(0) = 0$. Applying I^α to each side of the relation (15), we get,

$$y(t) = y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} y(0^+).$$

This means that $I^{1-\alpha}y(0^+) = 0$ since $t \in (0, T)$. Thus y is also a solution of (1). This means that system (1) is equivalent to system (12). From Theorem 2.9, it follows on the one hand that (1) has a unique solution $y \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$, and on the other hand that (14) holds. □

For more reading on fractional diffusion equations, we refer to [33,4,7,34–38] and the references therein.

3. Optimal control

In this section, we are concerned with the following optimal control problem.

$$\begin{cases} \min J(y, v), \\ y = \mathbb{T}(v), \\ y \in K \text{ and } v \in \mathcal{U}_{ad} \end{cases} \tag{16}$$

where J is defined by (2), K and $\mathcal{U}_{ad}$ are two nonempty closed convex subsets of $L^2((0,T); H^2(\Omega) \cap H^1_0(\Omega))$ and $L^2(Q)$ respectively.

We denote by $\mathcal{A} = \{(y, v) \in K \times \mathcal{U}_{ad}$ such that $y = \mathbb{T}(v)\}$ the admissible domain of (16) and we assume now and in the sequel that

$$\mathcal{A} \ne \emptyset. \tag{17}$$

Assumption (17) means that there exists $v_0 \in \mathcal{U}_{ad}$ such that $y_0 = \mathbb{T}(v_0) \in K$.

Since $K \times \mathcal{U}_{ad}$ is a closed convex subset of $L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(Q)$ and the application $\mathbb{T}$ is affine, $\mathcal{A}$ is a nonempty closed convex set. Moreover, J being strictly convex, one can prove as in [26] that problem (16) has a unique solution $(\bar{y}, \bar{v})$. Hence writing the Euler–Lagrange optimality which characterizes $(\bar{y}, \bar{v})$, we obtain

$$\int_0^T \int_\Omega (\bar{y} - z_d)(z - \bar{y}) \mathrm{d}x\, \mathrm{d}t + N \int_0^T \int_\Omega \bar{v}(\varphi - \bar{v}) \mathrm{d}x\, \mathrm{d}t \geq 0, \ \forall (z, \varphi) \in \mathcal{A}.$$

In this optimality condition the functions φ and z are linked by the relation $z = T(\varphi)$. To obtain optimality system with φ and z decoupled, we use the penalization method due to Lions [8].

So, let $\varepsilon > 0$. Let also $\mathcal{K}$ be defined by

$$\mathcal{K} = \{y \in L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega)),\ D_{RL}^\alpha y - \Delta y \in L^2(Q), y|_\Sigma = 0,\ I^{1-\alpha} y(x, 0^+) = 0 \text{ in } \Omega,\ y \in K\}. \tag{18}$$

Since K is a nonempty closed subset of $L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega))$, so is $\mathcal{K}$. For any $(y, v) \in \mathcal{K} \times \mathcal{U}_{ad}$, define the functional J_ε by:

$$J_\varepsilon(y, v) = \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|v\|^2_{L^2(Q)} + \frac{1}{2\varepsilon}\left\|D_{RL}^\alpha y - \Delta y - h + v\right\|^2_{L^2(Q)}. \tag{19}$$

Consider the penalized problem

$$\min_{(y,v) \in \mathcal{K} \times \mathcal{U}_{ad}} J_\varepsilon(y, v). \tag{20}$$

Before going further we need the following results.

Lemma 3.1. *Let $f \in L^2(Q)$ and $y \in L^2(Q)$ be such that $D_{RL}^\alpha y - \Delta y = f$. Then $(y|_\Sigma, I^{1-\alpha} y_\varepsilon(x, 0))$ exists and belongs to $\left(H^{-1}((0,T); H^{-1/2}(\partial\Omega)), H^{-1}(\Omega)\right)$.*

Proof. Let $y \in L^2(Q)$, then in view of Lemma 2.6, $I^{1-\alpha} y(x,t) \in L^2(Q)$. Therefore, on the one hand we have $D_{RL}^\alpha y(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} y(x,t) \in H^{-1}((0,T); L^2(\Omega))$ and then, $\Delta y \in H^{-1}((0,T); L^2(\Omega))$ since $D_{RL}^\alpha y - \Delta y = f$. Thus $y \in L^2(Q)$ and $\Delta y \in H^{-1}((0,T); L^2(\Omega))$. Hence, we deduce that $y|_\Sigma$ exists and belongs to $H^{-1}((0,T); H^{-1/2}(\partial\Omega))$ (see [39]).

On the other hand, we have $\Delta y \in L^2((0,T); H^{-2}(\Omega))$. And since $D_{RL}^\alpha y - \Delta y = f$, we obtain that $D_{RL}^\alpha y(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} y(x,t) \in L^2((0,T); H^{-2}(\Omega))$. Thus $I^{1-\alpha} y(x,t) \in L^2(Q)$ and $\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} y(x,t) \in L^2((0,T); H^{-2}(\Omega))$. Consequently $I^{1-\alpha} y$ belongs to $C([0,T], H^{-1}(\Omega))$ (see [8]). This means that $I^{1-\alpha} y(x, 0)$ exists and belongs to $H^{-1}(\Omega)$.

Proposition 3.2. *Assume that (17) holds. Let $\varepsilon > 0$. Then there exists a unique pair $(y_\varepsilon, v_\varepsilon) \in \mathcal{K} \times \mathcal{U}_{ad}$ which is an optimal solution to (20).*

Proof. Since $(\bar{y}, \bar{v})$ is the solution of (16) and $J_\varepsilon(y, v) \geq 0$, we can define the real d_ε such that

$$d_\varepsilon = \min\{J_\varepsilon(y, v) | (y, v) \in \mathcal{K} \times \mathcal{U}_{ad}\}.$$

Let $(y_n, v_n) \in \mathcal{K} \times \mathcal{U}_{ad}$ be a minimizing sequence such that

$$0 < d_\varepsilon \leq J_\varepsilon(y_n, v_n) < d_\varepsilon + \frac{1}{n} < d_\varepsilon + 1.$$

In particular,

$$0 < d_\varepsilon \leq J_\varepsilon(\bar{y}, \bar{v}) = \|\bar{y} - z_d\|^2_{L^2(Q)} + \|\bar{v}\|^2_{L^2(Q)} < \infty.$$

Therefore,

$$\|v_n\|_{L^2(Q)} \leq C, \tag{21a}$$

$$\|D_{RL}^\alpha y_n - \Delta y_n - h + v_n\|_{L^2(Q)} \leq C\sqrt{\varepsilon}, \tag{21b}$$

$$\|y_n\|_{L^2((0,T); H^2(\Omega))} \leq C, \tag{21c}$$

where C represents now and in the sequel various positive constants independent of n and ε.

Since y_n satisfies (18), we have

$$D_{RL}^\alpha y_n - \Delta y_n \in L^2(Q), \tag{22a}$$

$$y_n = 0 \quad \text{on } \Sigma, \tag{22b}$$

$$I^{1-\alpha} y_n(x, 0) = 0 \quad \text{in } \Omega, \tag{22c}$$

$$y_n \in K; \tag{22d}$$

and it follows from (21a) and (21b) that

$$\|D^{\alpha}_{RL}y_n - \Delta y_n\|_{L^2(Q)} \leq C(1+\sqrt{\varepsilon}). \tag{23}$$

Hence there exist $y_\varepsilon \in L^2((0,T);H^2(\Omega))$, v_ε and δ_ε in $L^2(Q)$ and subsequences extracted from (v_n) and (y_n) (still called (v_n) and (y_n)) such that

$$v_n \rightharpoonup v_\varepsilon \quad \text{weakly in} \quad L^2(Q), \tag{24a}$$

$$D^{\alpha}_{RL}y_n - \Delta y_n \rightharpoonup \delta_\varepsilon \quad \text{weakly in } L^2(Q), \tag{24b}$$

$$y_n \rightharpoonup y_\varepsilon \quad \text{weakly in } L^2((0,T);H^2(\Omega)). \tag{24c}$$

Since K and $\mathcal{U}_{ad}$ are closed convex sets and $y_n \in K,\ v_n \in \mathcal{U}_{ad}$ using (24c) and (24a), we get

$$y_\varepsilon \in K \quad \text{and } v_\varepsilon \in \mathcal{U}_{ad}. \tag{25}$$

We set

$$\mathbb{D}(Q) = \{\varphi \in \mathcal{C}^\infty(Q) \text{ such that } \varphi|_{\partial\Omega} = 0, \varphi(x,0) = \varphi(x,T) = 0 \text{ in } \Omega\}$$

and we denote by $\mathbb{D}'(Q)$ the dual of $\mathbb{D}(Q)$.

In view of Lemma 2.8, we have

$$\int_0^T \int_\Omega (D^{\alpha}_{RL}y_n(x,t) - \Delta y_n(x,t))\varphi(x,t)\,dx\,dt = \int_0^T \int_\Omega y_n(x,t)(-\mathcal{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))\,dx\,dt, \quad \forall\varphi \in \mathbb{D}(Q).$$

Therefore in view of (24c), we obtain for $\varphi \in \mathbb{D}(Q)$,

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \int_\Omega (D^{\alpha}_{RL}y_n(x,t) - \Delta y_n(x,t))\varphi(x,t)\,dx\,dt &= \int_0^T \int_\Omega y_\varepsilon(x,t)(-\mathcal{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))\,dx\,dt \\ &= \int_0^T \int_\Omega (D^{\alpha}_{RL}y_\varepsilon(x,t) - \Delta y_\varepsilon(x,t))\varphi(x,t)\,dx\,dt.\end{aligned}$$

This means that

$$D^{\alpha}_{RL}y_n - \Delta y_n \rightharpoonup D^{\alpha}_{RL}y_\varepsilon - \Delta y_\varepsilon \quad \text{weakly in} \quad \mathbb{D}'(Q).$$

Then using (24b), we get

$$D^{\alpha}_{RL}y_\varepsilon - \Delta y_\varepsilon = \delta_\varepsilon \in L^2(Q). \tag{26}$$

And in view of (21b), (24b), (24a) and (26), we deduce that

$$D^{\alpha}_{RL}y_n - \Delta y_n - h - v_n \rightharpoonup D^{\alpha}_{RL}y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon \quad \text{weakly in} \quad L^2(Q). \tag{27}$$

Since $y_\varepsilon \in L^2(Q)$ and $D^{\alpha}_{RL}y_\varepsilon - \Delta y_\varepsilon = \delta_\varepsilon \in L^2(Q)$, in view of Lemma 3.1, $y_\varepsilon|_\Sigma$ and $I^{1-\alpha}y_\varepsilon(x,0)$ exist and belong respectively to $H^{-1}((0,T);H^{-1/2}(\partial\Omega))$ and $H^{-1}(\Omega)$.

So, multiplying $D^{\alpha}_{RL}y_n - \Delta y_n - h - v_n$ by $\varphi \in \mathcal{C}^\infty(\overline{Q})$ with $\varphi|_{\partial\Omega} = 0$ and $\varphi(T,x) = 0$ on Ω, and integrating by parts over Q, we obtain by using Lemma 2.8,

$$\begin{aligned}&\int_0^T \int_\Omega (D^{\alpha}_{RL}y_n(x,t) - \Delta y_n(x,t) - h(x,t) - v_n(x,t))\varphi(x,t)\,dx\,dt \\ &\quad = -\int_0^T \int_\Omega (h(x,t) + v_n(x,t))\,\varphi(x,t)\,dx\,dt + \int_0^T \int_\Omega y_n(x,t)(-\mathcal{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))\,dx\,dt.\end{aligned}$$

Passing this latter identity to the limit when $n \to \infty$ while using (27) and (24c),

$$\begin{aligned}&\int_0^T \int_\Omega \left(D^{\alpha}_{RL}y_\varepsilon(x,t) - \Delta y_\varepsilon(x,t) - h(x,t) - v_\varepsilon(x,t)\right)\varphi(x,t)\,dx\,dt \\ &\quad = \int_0^T \int_\Omega y_\varepsilon(x,t)\ (-\mathcal{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))\ dx\,dt - \int_0^T \int_\Omega (h(x,t) + v_\varepsilon(x,t))\,\varphi(x,t)\,dx\,dt.\end{aligned} \tag{28}$$

Integrating by parts the right side of (28) while using Lemma 2.7, we obtain

$$\int_0^T \int_\Omega \left(D^\alpha_{RL} y_\varepsilon(x,t) - \Delta y_\varepsilon(x,t) - h(x,t) - v_\varepsilon(x,t)\right)\varphi(x,t)\,dx\,dt$$
$$= +\langle \varphi(x,0),\ I^{1-\alpha} y_\varepsilon(x,0^+)\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} - \left\langle y_\varepsilon, \frac{\partial\varphi}{\partial\nu}\right\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}$$
$$+ \int_0^T \int_\Omega (D^\alpha_{RL} y_\varepsilon(x,t) - \Delta y_\varepsilon(x,t) - h(x,t) - v_\varepsilon(x,t))\varphi(x,t)\,dx\,dt,$$
$$\text{for all } \varphi \in \mathcal{C}^\infty(\overline{Q}) \text{ with } \varphi|_{\partial\Omega} = 0 \text{ and } \varphi(x,T) = 0 \text{ on } \Omega, \tag{29}$$

where $\langle\cdot,\ \cdot\rangle_{Y,Y'}$ represents the duality bracket between the spaces Y and Y'.

Hence, (29) yields

$$0 = \langle \varphi(x,0),\ I^{1-\alpha} y_\varepsilon(x,0^+)\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} - \int_0^T \langle y_\varepsilon, \frac{\partial\varphi}{\partial\nu}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\,dt,$$
$$\text{for all } \varphi \in \mathcal{C}^\infty(\overline{Q}) \text{ with } \varphi|_{\partial\Omega} = 0 \text{ and } \varphi(x,T) = 0 \text{ on } \Omega.$$

Therefore taking φ such that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ in this latter identity, we obtain

$$I^{1-\alpha} y_\varepsilon(x,0^+) = 0 \quad \text{in } \Omega \tag{30}$$

and then,

$$y_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{31}$$

In view of (25)–(27), (30) and (31), we deduce that $(y_\varepsilon, v_\varepsilon) \in \mathcal{K} \times \mathcal{U}_{ad}$. From weak lower semi-continuity of the function $v \to J(v)$ we deduce

$$\liminf_{n\to\infty} J_\varepsilon(y_n, v_n) \geq J_\varepsilon(y_\varepsilon, v_\varepsilon) = d_\varepsilon.$$

In other words, $(y_\varepsilon, v_\varepsilon)$ is the optimal control. The uniqueness of $(y_\varepsilon, v_\varepsilon)$ is the immediate consequence of the strict convexity of J_ε. □

Theorem 3.3. *Assume that (17) holds. Let $\varepsilon > 0$ and $(y_\varepsilon, v_\varepsilon)$ be the solution of (20). Then there exist $p_\varepsilon \in L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega))$ and $\rho_\varepsilon \in L^2(Q)$ such that $(y_\varepsilon, v_\varepsilon, \rho_\varepsilon, p_\varepsilon)$ satisfies the following optimality system:*

$$\begin{cases} D^\alpha_{RL} y_\varepsilon - \Delta y_\varepsilon = h + v_\varepsilon + \varepsilon\rho_\varepsilon & \text{in } Q, \\ y_\varepsilon = 0, & \text{on } \Sigma, \\ I^{1-\alpha} y_\varepsilon(x,0^+) = 0 & \text{in } \Omega, \\ (y_\varepsilon, v_\varepsilon) \in K \times \mathcal{U}_{ad}. \end{cases} \tag{32}$$

$$\begin{cases} -\mathcal{D}^\alpha p_\varepsilon - \Delta p_\varepsilon = y_\varepsilon - z_d & \text{in } Q, \\ p_\varepsilon = 0 & \text{on } \Sigma, \\ p_\varepsilon(T) = 0 & \text{in } \Omega \end{cases} \tag{33}$$

$$\int_Q \left(D^\alpha_{RL}(z - y_\varepsilon) - \Delta(z - y_\varepsilon)\right)(p_\varepsilon + \rho_\varepsilon)\,dx\,dt \geq 0, \quad \forall z \in \mathcal{K}, \tag{34}$$

$$\int_Q (Nv_\varepsilon - \rho_\varepsilon)(\varphi - v_\varepsilon)dxdt \geq 0 \quad \forall \varphi \in \mathcal{U}_{ad}. \tag{35}$$

Proof. We express the Euler–Lagrange optimality conditions which characterize the optimal control $(y_\varepsilon, v_\varepsilon)$:

$$\frac{d}{d\mu} J(y_\varepsilon + \mu\,(z - y_\varepsilon), v_\varepsilon)|_{\mu=0} \geq 0, \quad \forall z \in \mathcal{K} \tag{36}$$

and

$$\frac{d}{d\mu} J(y_\varepsilon v_\varepsilon + \mu\,(\varphi - v_\varepsilon))|_{\mu=0} \geq 0, \quad \forall \varphi \in \mathcal{U}_{ad}. \tag{37}$$

After calculations, (36) and (37) give respectively

$$\frac{1}{\varepsilon}\int_Q (D^\alpha_{RL} y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon)(D^\alpha_{RL}(z - y_\varepsilon) - \Delta(z - y_\varepsilon))\,dx\,dt + \int_Q (y_\varepsilon - z_d)(z - y_\varepsilon)\,dx\,dt \geq 0, \quad \forall z \in \mathcal{K} \tag{38}$$

and

$$\frac{-1}{\varepsilon}\int_Q (D^\alpha_{RL}y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon)(\varphi - v_\varepsilon)\mathrm{d}x\mathrm{d}t + \int_Q Nv_\varepsilon(\varphi - v_\varepsilon)\mathrm{d}x\mathrm{d}t \geq 0 \quad \forall\varphi \in \mathcal{U}_{ad}. \tag{39}$$

Set

$$\rho_\varepsilon = \frac{1}{\varepsilon}(D^\alpha_{RL}y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon). \tag{40}$$

Then on the one hand, we have $\rho_\varepsilon \in L^2(Q)$ according to (27), and on the other hand,

$$D^\alpha_{RL}y_\varepsilon - \Delta y_\varepsilon = h + v_\varepsilon + \varepsilon\rho_\varepsilon. \tag{41}$$

Therefore (41), (30), (31) and (25) give (32).

Replacing $\frac{1}{\varepsilon}(D^\alpha_{RL}y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon)$ by ρ_ε in (38) and (39), we have respectively

$$\int_Q \rho_\varepsilon\left(D^\alpha_{RL}(z - y_\varepsilon) - \Delta(z - y_\varepsilon)\right)\,\mathrm{d}x\,\mathrm{d}t + \int_Q (y_\varepsilon - z_d)(z - y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall z \in \mathcal{K} \tag{42}$$

and

$$\int_Q (Nv_\varepsilon - \rho_\varepsilon)(\varphi - v_\varepsilon)\mathrm{d}x\mathrm{d}t \geq 0 \quad \forall\varphi \in \mathcal{U}_{ad}. \tag{43}$$

Now, we consider the adjoint state equation:

$$\begin{cases} -\mathcal{D}^\alpha p_\varepsilon - \Delta p_\varepsilon = y_\varepsilon - z_d & \text{in } Q, \\ p_\varepsilon = 0 & \text{on } \Sigma, \\ p_\varepsilon(T) = 0 & \text{in } \Omega. \end{cases} \tag{44}$$

Let

$$\mathcal{T}_T p_\varepsilon(t) = p_\varepsilon(T - t), \quad t \in [0, T]. \tag{45}$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_T p_\varepsilon(t) = -p'_\varepsilon(T - t)$.

Next, making the change of variable $t \to T - t$ in

$$\mathcal{D}^\alpha p_\varepsilon(t) = \frac{1}{\Gamma(1-\alpha)}\int_t^T (s - t)^{-\alpha}p'_\varepsilon(s)\,\mathrm{d}s,$$

we obtain

$$\begin{aligned} \mathcal{D}^\alpha p_\varepsilon(T - t) &= \frac{1}{\Gamma(1-\alpha)}\int_{T-t}^T (s - (T - t))^{-\alpha}p'_\varepsilon(s)\,\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t - u)^{-\alpha}p'_\varepsilon(T - u)\,\mathrm{d}u \end{aligned}$$

which, according to the notations (45), can be rewritten as

$$\mathcal{D}^\alpha\mathcal{T}_T p_\varepsilon(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t (t - u)^{-\alpha}(\mathcal{T}_T p_\varepsilon)'(u)\,\mathrm{d}u.$$

This means that

$$\mathcal{D}^\alpha\mathcal{T}_T p_\varepsilon(t) = -D^\alpha_C\mathcal{T}_T p_\varepsilon(t).$$

Finally, making the change of variable $t \to T - t$ in (44), we obtain

$$\begin{cases} D^\alpha_C\mathcal{T}_T p_\varepsilon - \Delta\mathcal{T}_T p_\varepsilon = \mathcal{T}_T y_\varepsilon - \mathcal{T}_T z_d & \text{in } Q, \\ \mathcal{T}_T p_\varepsilon = 0 & \text{on } \Sigma, \\ \mathcal{T}_T p_\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

That is

$$\begin{cases} D^\alpha_C q_\varepsilon - \Delta q_\varepsilon = g_\varepsilon & \text{in } Q, \\ q_\varepsilon = 0 & \text{on } \Sigma, \\ q_\varepsilon(0) = 0 & \text{in } \Omega \end{cases} \tag{46}$$

where $q_\varepsilon(t) = \mathcal{T}_T p_\varepsilon(t) = p_\varepsilon(T - t)$ and $g_\varepsilon(t) = \mathcal{T}_T y_\varepsilon - \mathcal{T}_T z_d$. Observing that $T - t \in [0, T]$ for $t \in [0, T]$, we deduce that $g_\varepsilon \in L^2(Q)$ since y_ε and z_d belong to $L^2(Q)$. Therefore Theorem 2.9 allows us to say that there exists a unique $q_\varepsilon \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ which is a solution to (46). Moreover there exists a positive constant C such that $\|q_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \leq C\|g_\varepsilon\|_{L^2(Q)}$. This means that (44) has a unique solution, $p_\varepsilon \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. Moreover there exists a positive constant C such that

$$\|p_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \leq C\|y_\varepsilon - zd\|_{L^2(Q)}. \tag{47}$$

Thus, multiplying (44) by $z - y_\varepsilon$ and integrating by parts over Q, we obtain by using Lemma 2.8,

$$\begin{aligned}\int_0^T\int_\Omega (D_{RL}^\alpha(z-y_\varepsilon) - \Delta(z-y_\varepsilon))p_\varepsilon \,dx\,dt &= \int_0^T\int_\Omega (-\mathcal{D}^\alpha p_\varepsilon - \Delta p_\varepsilon)(z-y_\varepsilon)\,dx\,dt\\ &= \int_0^T\int_\Omega (y_\varepsilon - z_d)(z-y_\varepsilon)\,dx\,dt, \quad \forall z \in \mathcal{K}.\end{aligned}$$

Hence, in view of (42), we deduce that

$$\int_0^T\int_\Omega (D_{RL}^\alpha(z-y_\varepsilon) - \Delta(z-y_\varepsilon))(p_\varepsilon + \rho_\varepsilon)\,dx\,dt \geq 0, \quad \forall z \in \mathcal{K}.$$

Proposition 3.4. *Let (v_ε), y_ε and p_ε be defined as in Theorem 3.3. Then*

$$\|v_\varepsilon\|_{L^2(Q)} \leq C, \tag{48a}$$

$$\|y_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \leq C, \tag{48b}$$

$$\left\|D_{RL}^\alpha y_\varepsilon - \Delta y_\varepsilon - h + v_\varepsilon\right\|_{L^2(Q)} \leq C\sqrt{\varepsilon}, \tag{48c}$$

$$\|p_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \leq C, \tag{48d}$$

where $C > 0$ represents various constant independent of ε.

Proof. Estimates (48a)–(48c) result from (21) and the weak convergence (24). To obtain (48d), we use (47) and (48b). □

To pass to the limit in the optimality system (32)–(35) we need an estimate of the multiplier ρ_ε. To this end we need a stronger assumption than (17). So, we denote by $\mathbb{B}_2(u_0,\gamma) = \{u \in L^2(Q)$ such that $\|u-u_0\|_{L^2(Q)} \leq \gamma\}$ and we make the following assumption

$$\begin{aligned}&\exists u_0 \in \mathcal{U}_{ad}, \exists r > 0, \exists \tau > 0 \text{ such that } \forall k \in \mathbb{B}_2(0,1),\ \exists v_k \in \mathbb{B}_2(u_0,\tau)\cap\mathcal{U}_{ad},\ y_k\\ &\quad = \mathbb{T}(h + v_k - rk) \in K.\end{aligned} \tag{49}$$

Proposition 3.5. *Assume that (49) holds. Then there exists $C > 0$ such that*

$$\|\rho_\varepsilon\|_{L^2(Q)} \leq C. \tag{50}$$

Proof. Let $k \in \mathbb{B}_2(0,1)$. Then adding (34) to (35) with $z = y_k$ and $\varphi = v_k$, we get

$$\int_Q (D_{RL}^\alpha(y_k - y_\varepsilon) - \Delta(y_k - y_\varepsilon))(p_\varepsilon + \rho_\varepsilon)\,dx\,dt + \int_Q (Nv_\varepsilon - \rho_\varepsilon)(v_k - v_\varepsilon)dxdt, \quad \geq 0$$

which according to (49) gives

$$\int_Q p_\varepsilon(v_k - v_\varepsilon - rk - \varepsilon\rho_\varepsilon)\,dx\,dt + \int_Q \rho_\varepsilon(-rk - \varepsilon\rho_\varepsilon)\,dx\,dt + \int_Q Nv_\varepsilon(v_k - v_\varepsilon)dxdt \geq 0.$$

Hence we deduce that

$$\int_Q \rho_\varepsilon rk\,dx\,dt \leq \int_Q p_\varepsilon(v_k - v_\varepsilon - rk)\,dx\,dt - \int_Q p_\varepsilon\varepsilon\rho_\varepsilon\,dx\,dt - \int_Q \varepsilon\rho_\varepsilon^2\,dx\,dt + \int_Q Nv_\varepsilon(v_k - v_\varepsilon)dxdt.$$

Consequently,

$$\int_Q \rho_\varepsilon rk\,dx\,dt \leq \|p_\varepsilon\|_{L^2(Q)}(\|v_k\|_{L^2(Q)} + \|v_\varepsilon\|_{L^2(Q)} + r\|k\|_{L^2(Q)} + \varepsilon\|\rho_\varepsilon\|_{L^2(Q)}) + N\|v_\varepsilon\|_{L^2(Q)}(\|v_k\|_{L^2(Q)} + \|v_\varepsilon\|_{L^2(Q)}).$$

Observing that $\|\varepsilon\rho_\varepsilon\|_{L^2(Q)} \leq C\sqrt{\varepsilon}$ since (48c) and (40) hold, using (48a) and (48d) and (49) we have

$$\forall k \in \mathbb{B}_2(0,1),\ \int_Q \rho_\varepsilon k\,dx\,dt \leq \frac{1}{r}[C(\|u_0\|_{L^2(Q)} + \tau + C + r + \varepsilon\sqrt{\varepsilon}) + NC(\|u_0\|_{L^2(Q)} + \tau + C)].$$

Because u_0 does not depend on ε, we obtain $\|\rho_\varepsilon\|_{L^2(Q)} \leq C$ where

$$C = \frac{1}{r}[C(\|u_0\|_{L^2(Q)} + \tau + C + r + \varepsilon\sqrt{\varepsilon}) + NC(\|u_0\|_{L^2(Q)} + \tau + C)] > 0. \quad \square$$

Proposition 3.6. *Let $(\bar{y}, \bar{v})$ be the solution of (16). Then*

$$v_\varepsilon \to \bar{v} \quad \textit{strongly in } L^2(Q), \tag{51}$$

$$y_\varepsilon \to \bar{y} \quad \textit{strongly in } L^2((0,T); H^2(\Omega)). \tag{52}$$

Proof. In view of (48a) and (48b), there exist $v_0 \in L^2(Q)$ and $y_0 \in L^2((0,T); H^2(\Omega))$ and subsequences extracted from (v_ε) and (y_ε) (still called (v_ε) and (y_ε)) such that

$$v_\varepsilon \rightharpoonup v_0 \quad \text{weakly in } L^2(Q), \tag{53}$$

$$y_\varepsilon \rightharpoonup y_0 \quad \text{weakly in } L^2((0,T); H^2(\Omega)), \tag{54}$$

Since K and $\mathcal{U}_{ad}$ are closed convex sets and $y_\varepsilon \in K,\ v_\varepsilon \in \mathcal{U}_{ad}$ using (54) and (53), we get

$$y_0 \in K \text{ and } v_0 \in \mathcal{U}_{ad}. \tag{55}$$

On the other hand, using Lemma 2.8, we have

$$\int_0^T \int_\Omega (D^\alpha_{RL} y_\varepsilon(x,t) - \Delta y_\varepsilon(x,t) - h - v_\varepsilon)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega y_\varepsilon(x,t)\,(-\mathcal{D}^\alpha \varphi(x,t) - \Delta\varphi(x,t))\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^T \int_\Omega (h + v_\varepsilon)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t, \quad \forall \varphi \in \mathbb{D}(Q).$$

Therefore passing this latter identity to the limit while using (54) and (53), we obtain

$$0 = \int_0^T \int_\Omega y_0(x,t)(-\mathcal{D}^\alpha \varphi(x,t) - \Delta\varphi(x,t))\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (h + v_0)\,\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega (D^\alpha_{RL} y_0(x,t) - \Delta y_0(x,t) - h - v_0)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t,\ \forall \varphi \in \mathbb{D}(Q).$$

This means that

$$D^\alpha_{RL} y_\varepsilon - \Delta y_\varepsilon - h - v_\varepsilon \rightharpoonup D^\alpha_{RL} y_0 - \Delta y_0 - h - v_0 \quad \text{weakly in } \mathbb{D}'(Q).$$

As according to (48c)

$$D^\alpha_{RL} y_\varepsilon - \Delta y_\varepsilon - h + v_\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(Q), \tag{56}$$

we deduce that

$$D^\alpha_{RL} y_0 - \Delta y_0 = h + v_0 \text{ in } Q. \tag{57}$$

Then proceeding as for y_ε on Pages 9 and 10, we prove on the one hand that $y_0|_\Sigma$ and $I^{1-\alpha} y_0(x, 0^+)$ exist and belong respectively to $H^{-1}((0,T); H^{-/2}(\partial\Omega))$ and $H^{-1}(\Omega)$, and on the other hand that

$$\begin{cases} y_0 = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y_0(x, 0^+) = 0 & \text{in } \Omega. \end{cases} \tag{58}$$

From (57), (58) and (55), we obtain that $y_0 = \mathbb{T}(v_0)$, $y_0 \in K$ and $v_0 \in \mathcal{U}_{ad}$.

Since

$$J_\varepsilon(y_\varepsilon, v_\varepsilon) \leq J_\varepsilon(\bar{y}, \bar{v}) = J(\bar{y}, \bar{v}),$$

we have

$$\liminf_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon, v_\varepsilon) \leq J(\bar{y}, \bar{v}).$$

This means that

$$J(y_0, v_0) \leq J(\bar{y}, \bar{v})$$

because $J_\varepsilon(y_0, v_0) = J(y_0, v_0)$. Hence, the uniqueness of the optimal control of (16) allows us to say that $J(y_0, v_0) = J(\bar{y}, \bar{v})$ and

$$y_0 = \bar{y}, \quad v_0 = \bar{v}. \tag{59}$$

Thus we have proved that

$$v_\varepsilon \rightharpoonup \bar{v} \quad \text{weakly in } L^2(Q), \tag{60}$$

$$y_\varepsilon \rightharpoonup \bar{y} \quad \text{weakly in } L^2(Q). \tag{61}$$

To prove the strong convergence, we first observe that according to the result above, we have $\lim_{\varepsilon\to} J_\varepsilon(y_\varepsilon, v_\varepsilon) = J(\bar{y}, \bar{v})$, which implies that

$$\lim_{\varepsilon\to}(\|y_\varepsilon - z_d\|^2_{L^2(Q)} + \|v_\varepsilon\|^2_{L^2(Q)}) = \|\bar{y} - z_d\|^2_{L^2(Q)} + \|\bar{v}\|^2_{L^2(Q)}. \tag{62}$$

Using (60) and (61), we get

$$\|\bar{y} - z_d\|^2_{L^2(Q)} \le \liminf_{\varepsilon\to 0} \|y_\varepsilon - z_d\|^2_{L^2(Q)},$$

$$\|\bar{v}\|^2_{L^2(Q)} \le \liminf_{\varepsilon\to 0} \|v_\varepsilon\|^2_{L^2(Q)},$$

which in view of (62) gives

$$\begin{aligned} \|\bar{y} - z_d\|^2_{L^2(Q)} &= \lim_{\varepsilon\to 0} \|y_\varepsilon - z_d\|^2_{L^2(Q)}, \\ \|\bar{v}\|^2_{L^2(Q)} &= \lim_{\varepsilon\to 0} \|v_\varepsilon\|^2_{L^2(Q)}. \end{aligned} \tag{63}$$

Therefore using the fact that

$$\|y_\varepsilon - \bar{y}\|^2_{L^2(Q)} = \|y_\varepsilon - z_d\|^2_{L^2(Q)} - 2\int_Q (y_\varepsilon - z_d)(\bar{y} - z_d)\mathrm{d}x\mathrm{d}t + \|\bar{y} - z_d\|^2_{L^2(Q)},$$

$$\|v_\varepsilon - \bar{v}\|^2_{L^2(Q)} = \|v_\varepsilon\|^2_{L^2(Q)} - 2\int_Q v_\varepsilon \bar{v}\mathrm{d}x\mathrm{d}t + \|\bar{v}\|^2_{L^2(Q)},$$

in view of (63), (60) and (61), we deduce that

$$\lim_{\varepsilon\to 0} \|y_\varepsilon - \bar{y}\|^2_{L^2(Q)} = 0,$$

$$\lim_{\varepsilon\to 0} \|v_\varepsilon - \bar{v}\|^2_{L^2(Q)} = 0.$$

Hence we obtain (51) and (52). □

Proposition 3.7. *As ε tends to 0,*

$$p_\varepsilon \to \bar{p} \quad \textit{weakly } L^2((0,T); H^2(\Omega)), \tag{64}$$

where $\bar{p}$ is a solution of

$$\begin{cases} -\mathcal{D}^\alpha \bar{p} - \Delta\bar{p} = \bar{y} - z_d & \textit{in } Q, \\ \bar{p} = 0 & \textit{on } \Sigma, \\ \bar{p}(T) = 0 & \textit{in } \Omega. \end{cases} \tag{65}$$

Proof. Let $q_\varepsilon(t) = \mathcal{T}_T p_\varepsilon(t) = p_\varepsilon(T - t)$. Then according to results obtained in Page 13, we have

$$\begin{cases} D^\alpha_C q_\varepsilon - \Delta q_\varepsilon = \mathcal{T}_T y_\varepsilon - \mathcal{T}_T z_d & \text{in } Q, \\ q_\varepsilon = 0 & \text{on } \Sigma, \\ q_\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

Therefore in view of Theorem 2.9, there exists $C > 0$ such that

$$\|q_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \le C\|\mathcal{T}_T y_\varepsilon - \mathcal{T}_T z_d\|_{L^2(Q)}.$$

Then, using the fact that $T - t \in [0, T]$ for $t \in [0, T]$ and (48b), we get

$$\|q_\varepsilon\|_{L^2((0,T);H^2(\Omega))} \le C.$$

Consequently, there exists $\bar{q}$ in $L^2((0,T);H^2(\Omega))$ and a subsequence extracted from (q_ε) (still called (q_ε)) such that

$$q_\varepsilon \rightharpoonup \bar{q} \quad \text{weakly in } L^2((0,T);H^2(\Omega)). \tag{66}$$

Because $q_\varepsilon(0)=0$, we obtain that

$$D_C^\alpha q_\varepsilon = D_{RL}^\alpha q_\varepsilon, \tag{67a}$$

$$I^{1-\alpha} q_\varepsilon(0) = 0 \tag{67b}$$

and that q_ε is also a solution of

$$\begin{cases} D_{RL}^\alpha q_\varepsilon - \Delta q_\varepsilon = \mathcal{T}_T y_\varepsilon - \mathcal{T}_T z_d & \text{in } Q, \\ q_\varepsilon = 0 & \text{on } \Sigma, \\ I^{1-\alpha} q_\varepsilon(0) = 0 & \text{in } \Omega. \end{cases}$$

So, proceeding as for y_ε on Pages 9 and 10, one can prove on the one hand that $\bar{q}|_\Sigma$ and $I^{1-\alpha}\bar{q}(x,0^+)$ exist and belong respectively to $H^{-1}((0,T);H^{-/2}(\partial\Omega))$ and $H^{-1}(\Omega)$, and on the other hand that

$$\begin{cases} D_{RL}^\alpha \bar{q} - \Delta \bar{q} = \mathcal{T}_T(\bar{y} - z_d) & \text{in } Q, \\ \bar{q} = 0 & \text{on } \Sigma, \\ I^{1-\alpha} \bar{q}(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{68}$$

Therefore using again (67), one can easily prove that $D_{RL}^\alpha \bar{q} = D_C^\alpha \bar{q}$. This implies that $I^{1-\alpha}\bar{q}(x,0^+) = \bar{q}(0) = 0$.

Thus, we have

$$\begin{cases} D_c^\alpha \bar{q} - \Delta \bar{q} = \mathcal{T}_T(\bar{y} - z_d) & \text{in } Q, \\ \bar{q} = 0 & \text{on } \Sigma, \\ \bar{q}(x,0^+) = 0 & \text{in } \Omega. \end{cases} \tag{69}$$

Now, in view (48d), there exists $\bar{p}$ in $L^2((0,T);H^2(\Omega))$ and a subsequence extracted from (p_ε) (still called (p_ε)) such that

$$p_\varepsilon \rightharpoonup \bar{p} \quad \text{weakly in } L^2((0,T);H^2(\Omega)), \tag{70}$$

and since $q_\varepsilon(t) = \mathcal{T}_T p_\varepsilon(t) = p_\varepsilon(T-t)$ we deduce that $\bar{q}(t) = \mathcal{T}_T \bar{p}(t) = \bar{p}(T-t)$. Hence make the change of variable $t \to T-t$ in (69), we deduce that (65). □

Theorem 3.8. *Assume that (49) hold. Let $(\bar{y},\bar{v}) \in \mathcal{A}$ and $\bar{p}$ be defined by (65). Then $(\bar{y},\bar{v})$ is an optimal solution to (16) if, and only if there exists $\bar{\rho} \in L^2(Q)$ such that $(\bar{y},\bar{v},\bar{p},\bar{\rho})$ satisfies:*

$$\int_Q (D_{RL}^\alpha(z-\bar{y}) - \Delta(z-\bar{y}))(\bar{p}+\bar{\rho})\,dx\,dt \ge 0, \quad \forall z \in \mathcal{K}, \tag{71}$$

$$\int_Q (N\bar{v} - \bar{\rho})(\varphi - \bar{v})\,dx\,dt \ge 0 \quad \forall \varphi \in \mathcal{U}_{ad}. \tag{72}$$

Proof. In view of (50), there exist $\bar{\rho}$ in $L^2(Q)$ and a subsequence extracted from (ρ_ε) (still called (ρ_ε)) such that

$$\rho_\varepsilon \rightharpoonup \bar{\rho} \quad \text{weakly in } L^2(Q). \tag{73}$$

So, passing to the limit in (35) while using (73) and (51), we get

$$\int_Q (N\bar{v} - \bar{\rho})(\varphi - \bar{v})\,dx\,dt \ge 0 \quad \forall \varphi \in \mathcal{U}_{ad}.$$

On the other hand, observing that (34) can be rewritten as

$$\int_Q (D_{RL}^\alpha(z-y_\varepsilon) - \Delta(z-y_\varepsilon))p_\varepsilon\,dx\,dt + \int_Q (D_{RL}^\alpha z - \Delta z)\rho_\varepsilon\,dx\,dt - \int_Q (D_{RL}^\alpha y_\varepsilon + \Delta y_\varepsilon)\rho_\varepsilon\,dx\,dt \ge 0, \quad \forall z \in \mathcal{K},$$

which in view of (41) is equivalent to

$$\int_Q (D_{RL}^\alpha(z-y_\varepsilon) - \Delta(z-y_\varepsilon))p_\varepsilon\,dx\,dt + \int_Q (D_{RL}^\alpha z - \Delta z)\rho_\varepsilon\,dx\,dt - \int_Q (h + v_\varepsilon + \varepsilon\rho_\varepsilon)\rho_\varepsilon\,dx\,dt \ge 0, \quad \forall z \in \mathcal{K},$$

using (73), (51), (52) and (64) while passing to the limit in this latter inequality, we deduce that

$$\int_Q (D^\alpha_{RL}(z-\bar{y}) - \Delta(z-\bar{y}))\bar{p}\,\mathrm{d}x\,\mathrm{d}t + \int_Q (D^\alpha_{RL}z - \Delta z)\bar{\rho}\,\mathrm{d}x\,\mathrm{d}t - \int_Q (h+\bar{v})\bar{\rho}\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \geq 0,\ \forall z \in \mathcal{K}.$$

Consequently, using the fact that $\bar{y} = \mathbb{T}(\bar{v})$, we obtain

$$\int_Q (D^\alpha_{RL}(z-\bar{y}) - \Delta(z-\bar{y}))\bar{p}\,\mathrm{d}x\,\mathrm{d}t + \int_Q (D^\alpha_{RL}(z-\bar{y}) - \Delta(z-\bar{y}))\bar{\rho}\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall z \in \mathcal{K}.$$

This means that

$$\int_Q (D^\alpha_{RL}(z-\bar{y}) - \Delta(z-\bar{y}))(\bar{p}+\bar{\rho})\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall z \in \mathcal{K}.$$

4. Concluding remarks

To obtain the estimate of the multiplier ρ_ε in $L^2(Q)$, we have taken k in a unit ball of $L^2(Q)$ in (49). Note that the same estimation holds if we choose k in any unit ball of a dense subset of $L^2(Q)$, say $\mathbb{D}(Q)$ for instance.

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