A unified treatment for stability preservation in computer simulations of impulsive BAM networks

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Abstract

This paper is concerned with the stability preservation in computer simulations of an impulsive bidirectional associative memory (BAM) network. The simulations are provided by difference equations formulated from a semi-discretisation technique and impulsive maps as discrete-time representations of the nonlinear impulses which attempt to destabilise the BAM network at fixed moments of time. Prior to producing the computer simulations, the analogue is analysed for its exponential convergence towards a unique equilibrium state. The analysis exploits the method of Lyapunov sequences to derive several sufficient conditions that govern the network parameters and the impulse magnitude and frequency. As special cases, one can obtain from our results, those corresponding to the non-impulsive discrete-time BAM networks and also those corresponding to continuous-time (impulsive and non-impulsive) systems. The treatment of the analysis leads us to a relation between the Lyapunov exponent of the non-impulsive system and that of the impulsive system involving the size of the impulses and the inter-impulse intervals.

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1. Introduction

Studies concerning the computer simulations of Hopfield neural networks (HNNs), cellular neural networks (CNNs) and bidirectional associative memory (BAM) networks have grown steadily over the past few years. These simulations are based on implementing discrete-time analogues which are formulated by semi-discretising the continuous-time neural networks whose dynamics are modelled by nonlinear differential equations, delayed differential equations and integrodifferential equations. One of the special features exhibited by these computer simulations is that the convergence dynamics of the continuous-time neural networks are preserved without requiring to control the value of the time-step parameter \cite{1–13}.

The applications of these analogues were extended recently by Akça et al. \cite{14}, Covachev et al. \cite{15}, and Gopalsamy \cite{16} for obtaining numerical solutions of impulsive neural networks. It was found that the stability...
2. Impulsive BAM networks

The impulsive BAM network consists of \( m \) processing units on the \( I \)-layer and \( q \) units on the \( J \)-layer, whose neural states \( x_i(\cdot), i \in \mathcal{I} = \{1, 2, \ldots, m\} \) and \( y_j(\cdot), j \in \mathcal{J} = \{1, 2, \ldots, q\} \) are governed by the following differential equation system

\[
\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{q} b_{ij} f_j(y_j(t)) + I_i, \quad i \in \mathcal{I}, \tag{2.1a}
\]

\[
\frac{dy_j(t)}{dt} = -c_j y_j(t) + \sum_{i=1}^{m} d_{ij} g_i(x_i(t)) + J_j, \quad j \in \mathcal{J} \tag{2.1b}
\]

for \( t > t_0, t \neq t_k \), that is subject to initial values

\[
x(t_0) = x_0 \in \mathbb{R}^m, \quad y(t_0) = y_0 \in \mathbb{R}^q \tag{2.1c}
\]

and impulsive state displacements characterised by

\[
\Delta x_i|_{t=t_k} \equiv x_i(t^+_k) - x_i(t^-_k) = r_{ik}(x_i(t^-_k)),
\]

\[
\Delta y_j|_{t=t_k} \equiv y_j(t^+_k) - y_j(t^-_k) = s_{jk}(y_j(t^-_k))
\]

at fixed instants of time \( t = t_k, k \in \mathbb{N} = \{1, 2, 3, \ldots\} \), wherein \( x_i(t_k^+) = \lim_{t \to t_k^+} x_i(t), x_i(t_k^-) = \lim_{t \to t_k^-} x_i(t) \), \( y_j(t_k^+) = \lim_{t \to t_k^+} y_j(t) \), and \( y_j(t_k^-) = \lim_{t \to t_k^-} y_j(t) \) with the sequence of times \( \{t_k\}_{k=1}^{\infty} \) satisfying \( t_0 < t_1 < t_2 < \cdots < t_k \to \infty \) as \( k \to \infty \) and \( \Delta t_k = t_k - t_{k-1} \geq \theta \), where the value \( \theta > 0 \) denotes the minimum time-interval between successive impulses. The value \( \theta > 0 \) ensures that impulses do not occur too frequently, but \( \theta \to \infty \) means that the network becomes impulse free.

It is assumed in the impulsive BAM network (2.1) that

- \((H_1)\) the self-regulating parameters \( a_i, c_j \), the connection weights \( b_{ij}, d_{ji} \), and the exogenous inputs \( I_i, J_j \) satisfy \( a_i, c_j > 0, b_{ij}, d_{ji}, I_i, J_j \in \mathbb{R} \);
- \((H_2)\) the activation functions \( f_j, g_i : \mathbb{R} \to \mathbb{R} \) with \( f_j(0) = g_i(0) = 0 \) satisfy

\[
\sup_{u \neq v} \left| \frac{f_j(u) - f_j(v)}{u - v} \right| = L_j, \quad j \in \mathcal{J}; \quad \sup_{u \neq v} \left| \frac{g_i(u) - g_i(v)}{u - v} \right| = M_i, \quad i \in \mathcal{I} \tag{2.2}
\]

where \( L_j, M_i > 0 \); and
- \((H_3)\) the impulse functions \( r_{ik}, s_{jk} : \mathbb{R} \to \mathbb{R} \) with \( r_{ik}(x^*_i) = s_{jk}(y^*_j) = 0 \) for all \( i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathbb{N} \), where \( x^*_i, y^*_j \) denote the components of an equilibrium \((x^*, y^*)^T \in \mathbb{R}^{m+q}\) of the impulsive BAM network (2.1), satisfy

\[
\sup_{u \neq v} \left| \frac{r_{ik}(u) - r_{ik}(v)}{u - v} \right| = \gamma_{ik}, \quad i \in \mathcal{I}; \quad \sup_{u \neq v} \left| \frac{s_{jk}(u) - s_{jk}(v)}{u - v} \right| = \delta_{jk}, \quad j \in \mathcal{J} \tag{2.3}
\]

for \( k \in \mathbb{N} \), where \( \gamma_{ik}, \delta_{jk} > 0 \).
We remark that the functions satisfying the conditions (2.2) and (2.3) do not necessarily be bounded, differentiable and monotonically increasing. Furthermore, the usual Lipschitz continuous conditions (locally or globally) can be obtained from (2.2) and (2.3).

The solution of the impulsive BAM network (2.1) is denoted by the vector \((x(t), y(t))^T \in \mathbb{R}^{m+q}\), in which the components of \(x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T \in \mathbb{R}^m\) and \(y(t) = (y_1(t), y_2(t), \ldots, y_q(t))^T \in \mathbb{R}^q\) are piecewise continuous on \((t_0, \beta)\) for some \(\beta > t_0\) such that \(x_i(t_k^+), x_i(t_k^-), y_j(t_k^+), y_j(t_k^-)\) exist, and \(x_i(t), y_j(t)\) are differentiable on \((t_{k-1}, t_k) \subset (t_0, \beta)\).

3. Semi-discretisation

To provide the computer simulations of the impulsive BAM network (2.1), we discretise the time-domain of the network in accordance with the rules \(t_0 = n_0 h, t = n h, \) and \(t_k = n_k h\), where \(n_0 = \lfloor t_0/h \rfloor, n = \lfloor t/h \rfloor, \) and \(n_k = \lfloor t_k/h \rfloor\) in which \([r]\) denotes the integer part of the real number \(r\), and the parameter \(h\) whose value satisfies \(0 < h < \theta\) denotes the time-step of a uniform discretisation. On any interval \([nh, t)\), where \(t < (n+1)h, t \neq t_k\), we approximate (2.1a) by equations with piecewise constant arguments of the form

\[
\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{q} b_{ij} f_j(y_{j}(nh)) + I_i, \\
\frac{dy_j(t)}{dt} = -c_j y_j(t) + \sum_{i=1}^{m} d_{ij} g_{i}(x_{i}(nh)) + J_j.
\]

This system is then integrated over the interval \([nh, t)\). The result after applying the limit \(t \to (n+1)h\) yields the discrete-time system

\[
x_i(n+1) = e^{-a_i h} x_i(n) + \phi_i(h) \sum_{j=1}^{q} b_{ij} f_j(y_{j}(n)) + \phi_i(h) I_i, \quad i \in \mathcal{I}, \tag{3.1a}
\]

\[
y_j(n+1) = e^{-c_j h} y_j(n) + \psi_j(h) \sum_{i=1}^{m} d_{ij} g_{i}(x_{i}(n)) + \psi_j(h) J_j, \quad j \in \mathcal{J}
\]

for \(n \geq n_0, n \neq n_k\), where \(\phi_i(h) = (1 - e^{-a_i h})/a_i, \psi_j(h) = (1 - e^{-c_j h})/c_j\) and \(x_i(n) \equiv x_i(nh), y_j(n) \equiv y_j(nh)\); the system (3.1a) is accompanied with the initial values

\[
x(n_0) = x_0 \in \mathbb{R}^m, \quad y(n_0) = y_0 \in \mathbb{R}^q \tag{3.1b}
\]

and the impulsive state displacements characterised by

\[
x_i(n_k^+) = x_i(n_k^-) + r_{ik}(x_i(n_k^-)), \quad i \in \mathcal{I},
\]

\[
y_j(n_k^+) = y_j(n_k^-) + s_{jk}(y_j(n_k^-)), \quad j \in \mathcal{J} \tag{3.1c}
\]

for \(k \in \mathbb{N}\).

We interpret (3.1a) and (3.1c) as follows: The notations \(n_k^+\) and \(n_k^-\) denote the same integer \(n_k\) at which the values \(x_i(n_k^-), y_j(n_k^-)\) generated by the system (3.1a) at \(n = n_k - 1\) are mapped by (3.1c) to give the values \(x_i(n_k^+), y_j(n_k^+)\). These mapped values \(x_i(n_k^+), y_j(n_k^+)\) are then supplied back to the system (3.1a) as initial values required for the next successive iterations of \(x_i(n+1), y_j(n+1)\) for \(n = n_k, n_k+1, n_k+2, \ldots, n_k+1 - 1\). Thus, the impulsive analogue (3.1) is well-posed, and this justifies the existence of a unique solution \((x(n), y(n))^T \in \mathbb{R}^{m+q}\) of the impulsive analogue (3.1) for \(n \geq n_0\).

One observes for \(n = n_k, k \in \mathbb{N}\) that \(n_k \to t_k, x_i(n_k^-) \to x_i(t_k^+), x_i(n_k^+) \to x_i(t_k^-), y_j(n_k^-) \to y_j(t_k^+), y_j(n_k^+) \to y_j(t_k^-)\), and hence (3.1c) approaches (2.1c) as \(h \to 0\). For \(n \geq n_0, n \neq n_k\), we have \(n \to t, x_i(n) \to x_i(t), y_j(n) \to y_j(t),\) and the discrete-time system (3.1a) approaches the differential equation system (2.1a) as the value \(h\) approaches zero. This property if supplemented with the exponential stability results (obtained later) equips the
impulsive discrete-time analogue (3.1) with the necessary feature for producing the desired computer simulations of the exponentially convergent impulsive BAM network (2.1) in resisting nonlinear impulses with significant magnitude.

4. Exponential stability results

In this section, we analyse the exponential stability of an equilibrium state \((x^*, y^*)^T \in \mathbb{R}^{m+q}\) of the impulsive analogue (3.1) whose components \(x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \in \mathbb{R}^m\) and \(y^* = (y_1^*, y_2^*, \ldots, y_q^*)^T \in \mathbb{R}^q\) satisfy the algebraic system

\[
\begin{align*}
    a_i x_i^* &= \sum_{j=1}^{q} b_{ij} f_j(y_j^*) + I_i, \quad i \in \mathcal{I}, \\
    c_j y_j^* &= \sum_{i=1}^{m} d_{ji} g_i(x_i^*) + J_j, \quad j \in \mathcal{J},
\end{align*}
\]

(4.1)

with the equations in (3.1c) reduce to \(r_{ik}(x_i^*) = s_{j}(y_j^*) = 0\) for all \(i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathbb{N}\).

**Lemma 4.1.** Let the hypotheses \(H_{1,2}\) be satisfied and \(p \geq 1\) be an integer. Suppose there exist positive numbers \(\alpha_\kappa (\kappa = 1, 2, \ldots, m + q)\) and real numbers \(\zeta_{l, i}, \xi_{l, i}, \zeta_{l, j}, \xi_{l, j} (i \in \mathcal{I}, j \in \mathcal{J}, l = 1, 2, \ldots, p)\) that satisfy

\[
\begin{align*}
    \sum_{l=1}^{p} \zeta_{l, i} &= 1, \quad \sum_{l=1}^{p} \xi_{l, j} = 1, \quad \sum_{l=1}^{p} \zeta_{l, j} = 1, \quad \sum_{l=1}^{p} \xi_{l, j} = 1
\end{align*}
\]

for which the conditions

\[
\begin{align*}
    a_i &> \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}|^p \zeta_{l, i} L_j^p \zeta_{l, j} + \cdots + |b_{ij}|^p \zeta_{l, j} L_j^p \zeta_{l, i} + \frac{\alpha_{m+j}}{\alpha_i} |d_{ji}|^p \zeta_{l, i} L_j^p \zeta_{l, j} + \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}|^p \zeta_{l, i} L_j^p \zeta_{l, j} \right), \quad i \in \mathcal{I}, \\
    c_j &> \frac{1}{p} \sum_{i=1}^{m} \left( |d_{ji}|^p \zeta_{l, i} M_i^p \zeta_{l, j} + \cdots + |d_{ji}|^p \zeta_{l, j} M_i^p \zeta_{l, i} + \frac{\alpha_{m+j}}{\alpha_i} |b_{ij}|^p \zeta_{l, i} L_j^p \zeta_{l, j} + \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}|^p \zeta_{l, i} L_j^p \zeta_{l, j} \right), \quad j \in \mathcal{J}
\end{align*}
\]

hold. Then the algebraic system (4.1) has a unique solution.

**Proof.** Let \(C^0\) denote the space of continuous functions defined on \(\mathbb{R}^{m+q}\) with values in \(\mathbb{R}^{m+q}\). The space is endowed with a general Euclidean norm \(\|(u, v)\|_p = \left( \sum_{i=1}^{m} |u_i|^p + \sum_{j=1}^{q} |v_j|^p \right)^{1/p}\), where \(p \geq 1\) is an integer and \((u, v)^T\) is a vector in \(\mathbb{R}^{m+q}\) with \(u = (u_1, u_2, \ldots, u_m)^T\) and \(v = (v_1, v_2, \ldots, v_q)^T\). Associated with the algebraic system (4.1), we construct a map \(H : \mathbb{R}^{m+q} \mapsto \mathbb{R}^{m+q}\) defined by

\[
H(x, y) = (H_1(x, y), H_2(x, y), \ldots, H_m(x, y), H_{m+1}(x, y), \ldots, H_{m+q}(x, y))^T,
\]

where

\[
\begin{align*}
    H_i(x, y) &= -a_i x_i + \sum_{j=1}^{q} b_{ij} f_j(y_j) + I_i, \quad i \in \mathcal{I}, \\
    H_{m+j}(x, y) &= -c_j y_j + \sum_{i=1}^{m} d_{ji} g_i(x_i) + J_j, \quad j \in \mathcal{J}.
\end{align*}
\]

One can see that \(H \in C^0\) by virtue of the hypothesis \(H_2\). It is known from the topology of continuous mappings that if the map \(H \in C^0\) is a homeomorphism on \(\mathbb{R}^{m+q}\), then there is a unique solution of the algebraic system (4.1). We invoke a result due to Forti and Tesi [18], namely that, if the map \(H \in C^0\) is injective on \(\mathbb{R}^{m+q}\) and satisfies \(\|H(x, y)\|_p \to \infty\) as \(\|(x, y)^T\|_p \to \infty\), then \(H \in C^0\) is a homeomorphic mapping on \(\mathbb{R}^{m+q}\).

It is clear that the mapping \(H : \mathbb{R}^{m+q} \mapsto \mathbb{R}^{m+q}\) is onto. Thus the injective part will be satisfied if one shows that \(H : \mathbb{R}^{m+q} \mapsto \mathbb{R}^{m+q}\) is one-to-one. That is, if \((x, y)^T\) and \((u, v)^T\) are arbitrary vectors in \(\mathbb{R}^{m+q}\), then \(H(x, y) = H(u, v)\) implies \((x, y)^T = (u, v)^T\), i.e., \(x_i = u_i (i \in \mathcal{I})\) and \(y_j = v_j (j \in \mathcal{J})\). We have from the definition of \(H(-)\) and the equation \(H(x, y) = H(u, v)\) that
\(-a_i x_i + \sum_{j=1}^{q} b_{ij} f_j(y_j) + I_i = -a_i u_i + \sum_{j=1}^{q} b_{ij} f_j(v_j) + I_i, \quad i \in \mathcal{I},\)

\(-c_j y_j + \sum_{i=1}^{m} d_{ji} g_i(x_i) + J_j = -c_j v_j + \sum_{i=1}^{m} d_{ji} g_i(u_i) + J_j, \quad j \in \mathcal{J}\)

which yield

\[a_i |x_i - u_i| \leq \sum_{j=1}^{q} |b_{ij}| L_j |y_j - v_j|, \quad i \in \mathcal{I},\]

\[c_j |y_j - v_j| \leq \sum_{i=1}^{m} |d_{ji}| M_i |x_i - u_i|, \quad j \in \mathcal{J}\]

under the given hypotheses. On applying the geometric–arithmetic mean inequality \(\eta_1 \cdot \eta_2 \cdots \eta_p \leq \frac{1}{p} (\eta_1^p + \eta_2^p + \cdots + \eta_p^p),\) where \(\eta_l (l = 1, 2, \ldots, p)\) denote nonnegative real numbers,

\[
\sum_{i=1}^{m} \alpha_i |x_i - u_i|^p + \sum_{j=1}^{q} \alpha_{m+j} |y_j - v_j|^p \\
\leq \sum_{i=1}^{m} \alpha_i \left\{ \sum_{j=1}^{q} |b_{ij}| L_j |y_j - v_j||x_i - u_i|^p \right\} + \sum_{j=1}^{q} \alpha_{m+j} \left\{ \sum_{i=1}^{m} |d_{ji}| M_i |x_i - u_i||y_j - v_j|^p \right\} \\
= \sum_{i=1}^{m} \alpha_i \left\{ \sum_{j=1}^{q} (|b_{ij}|^{\varrho_{1,j}} L_j^{\xi_{1,j}} |x_i - u_i|) \times \cdots \times (|b_{ij}|^{\varrho_{p-1,j}} L_j^{\xi_{p-1,j}} |x_i - u_i|) \times (|b_{ij}|^{\varrho_{p,j}} L_j^{\xi_{p,j}} |y_j - v_j|) \right\} \\
+ \sum_{j=1}^{q} \alpha_{m+j} \left\{ \sum_{i=1}^{m} (|d_{ji}|^{\varphi_{1,i}} M_i^{\zeta_{1,i}} |y_j - v_j|) \times \cdots \times (|d_{ji}|^{\varphi_{p-1,i}} M_i^{\zeta_{p-1,i}} |y_j - v_j|) \times (|d_{ji}|^{\varphi_{p,i}} M_i^{\zeta_{p,i}} |x_i - u_i|) \right\} \\
\leq \sum_{i=1}^{m} \alpha_i \left\{ \frac{1}{p} \sum_{j=1}^{q} \left[ |b_{ij}|^{\varrho_{p,j}} L_j^{\xi_{p,j}} |x_i - u_i|^p + \cdots + |b_{ij}|^{\varrho_{p-1,j}} L_j^{\xi_{p-1,j}} |x_i - u_i|^p \right] \right\} \\
+ \sum_{j=1}^{q} \alpha_{m+j} \left\{ \frac{1}{p} \sum_{i=1}^{m} \left[ |d_{ji}|^{\varphi_{p,i}} M_i^{\zeta_{p,i}} |y_j - v_j|^p + \cdots + |d_{ji}|^{\varphi_{p-1,i}} M_i^{\zeta_{p-1,i}} |y_j - v_j|^p \right] \right\} \\
= \sum_{i=1}^{m} \alpha_i \left\{ \frac{1}{p} \sum_{j=1}^{q} \left[ |b_{ij}|^{\varrho_{p,j}} L_j^{\xi_{p,j}} + \cdots + |b_{ij}|^{\varrho_{p-1,j}} L_j^{\xi_{p-1,j}} \right] + \frac{\alpha_{m+j}}{\alpha_i} \sum_{j=1}^{q} |d_{ji}|^{\varphi_{p,j}} M_i^{\zeta_{p,j}} \right\} |x_i - u_i|^p \\
+ \sum_{j=1}^{q} \alpha_{m+j} \left\{ \frac{1}{p} \sum_{i=1}^{m} \left[ |d_{ji}|^{\varphi_{p,i}} M_i^{\zeta_{p,i}} + \cdots + |d_{ji}|^{\varphi_{p-1,i}} M_i^{\zeta_{p-1,i}} \right] \right\} |y_j - v_j|^p
which yields
\[
\sum_{i=1}^{m} \alpha_i \left\{ a_i - \frac{1}{p} \sum_{j=1}^{q} \left[ |b_{ij}|^{p\xi_{i,j}} L_j^{p\xi_{i,j}} + \cdots + |b_{ij}|^{p\xi_{i,j}} L_j^{p\xi_{i,j}} \right] \right\} |x_i - u_i|^p
\]
\[
+ \sum_{j=1}^{q} \alpha_{m+j} \left\{ c_j - \frac{1}{p} \sum_{i=1}^{m} \left[ |d_{ji}|^{p\xi_{i,j}} M_i^{p\xi_{i,j}} + \cdots + |d_{ji}|^{p\xi_{i,j}} M_i^{p\xi_{i,j}} \right] \right\} |y_j - v_j|^p \leq 0.
\]
This inequality under the given conditions can be satisfied if \( x_i = u_i \ (i \in \mathcal{I}) \) and \( y_j = v_j \ (j \in \mathcal{J}) \) implying that \( (x, y)^T = (u, v)^T \). Thus, \( H : \mathbb{R}^{m+q} \to \mathbb{R}^{m+q} \) is one-to-one.

Let us now consider a corresponding map \( \hat{H}(x, y) = H(x, y) - H(\theta, 0) \) whose components are given by
\[
\hat{H}_i(x, y) = -a_i x_i + \sum_{j=1}^{q} b_{ij} f_j(y_j), \quad i \in \mathcal{I},
\]
\[
\hat{H}_{m+j}(x, y) = -c_j y_j + \sum_{i=1}^{m} d_{ji} g_i(x_i), \quad j \in \mathcal{J}.
\]
It is clear that if \( \| \hat{H}(x, y) \|_p \to \infty \) as \( \| (x, y)^T \|_p \to \infty \), then \( \| H(x, y) \|_p \to \infty \) as \( \| (x, y)^T \|_p \to \infty \).

We derive the followings:
\[
\sum_{i=1}^{m} \alpha_i |x_i|^{p-1} \text{sgn}(x_i) \hat{H}_i(x, y) + \sum_{j=1}^{q} \alpha_{m+j} |y_j|^{p-1} \text{sgn}(y_j) \hat{H}_{m+j}(x, y)
\]
\[
= \sum_{i=1}^{m} \alpha_i \left\{ -a_i |x_i|^p + \sum_{j=1}^{q} b_{ij} \text{sgn}(x_i) |x_i|^{p-1} f_j(y_j) \right\}
\]
\[
+ \sum_{j=1}^{q} \alpha_{m+j} \left\{ -c_j |y_j|^p + \sum_{i=1}^{m} d_{ji} \text{sgn}(y_j) |y_j|^{p-1} g_i(x_i) \right\}
\]
\[
\leq \sum_{i=1}^{m} \alpha_i \left\{ -a_i |x_i|^p + \sum_{j=1}^{q} |b_{ij}| L_j |x_i|^{p-1} |y_j|^p \right\} + \sum_{j=1}^{q} \alpha_{m+j} \left\{ -c_j |y_j|^p + \sum_{i=1}^{m} |d_{ji}| M_i |y_j|^{p-1} |x_i|^p \right\}.
\]
By applying the geometric–arithmetic mean inequality and the subsequent steps like before,
\[
\sum_{i=1}^{m} \alpha_i |x_i|^{p-1} \text{sgn}(x_i) \hat{H}_i(x, y) + \sum_{j=1}^{q} \alpha_{m+j} |y_j|^{p-1} \text{sgn}(y_j) \hat{H}_{m+j}(x, y)
\]
\[
\leq -\varepsilon \left( \sum_{i=1}^{m} \alpha_i |x_i|^p + \sum_{j=1}^{q} \alpha_{m+j} |y_j|^p \right),
\]
where the positive number \( \varepsilon \) is determined as \( \varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \) in which the numbers
\[
\varepsilon_1 = \min_{i \in \mathcal{I}} \left\{ a_i - \frac{1}{p} \sum_{j=1}^{q} \left[ |b_{ij}|^{p\xi_{i,j}} L_j^{p\xi_{i,j}} + \cdots + |b_{ij}|^{p\xi_{i,j}} L_j^{p\xi_{i,j}} \right] \right\}.
\]
\[
\varepsilon_2 = \min_{j \in \mathcal{J}} \left\{ c_j - \frac{1}{p} \sum_{i=1}^{m} \left[ |d_{ji}|^{p\xi_{i,j}} M_i^{p\xi_{i,j}} + \cdots + |d_{ji}|^{p\xi_{i,j}} M_i^{p\xi_{i,j}} \right] \right\}
\]
are positive by virtue of the given conditions. It will follow that
\[
\varepsilon \left( \sum_{i=1}^{m} \alpha_i |x_i|^p + \sum_{j=1}^{q} \alpha_{m+j} |y_j|^p \right) \leq \sum_{i=1}^{m} \alpha_i |x_i|^{p-1} \hat{H}_i(x, y) + \sum_{j=1}^{q} \alpha_{m+j} |y_j|^{p-1} \hat{H}_{m+j}(x, y).
\]
which we rewrite as
\[ \varepsilon \sum_{k=1}^{m+q} \alpha_k |u_k|^p \leq \sum_{k=1}^{m+q} \alpha_k |u_k|^{p-1} |\hat{H}_k(\cdot)|, \]
where \( u_k = x_i \), \( \hat{H}_k(\cdot) = \hat{H}_k(x,y) \) for \( k = 1, 2, \ldots, m \) and \( u_k = y_j \), \( \hat{H}_k(\cdot) = \hat{H}_{m+j}(x,y) \) for \( k = m + j, j = 1, 2, \ldots, q \). It is now possible to apply a H"older inequality in the above so as to obtain
\[ \varepsilon \alpha \sum_{k=1}^{m+q} |u_k|^p \leq \alpha \left( \sum_{k=1}^{m+q} |u_k|^{(p-1)r} \right)^{1/r} \left( \sum_{k=1}^{m+q} |\hat{H}_k(\cdot)|^p \right)^{1/p}, \]
where \( 1/r + 1/p = 1, \alpha = \min_{k=1,2,\ldots,m+q} \{ \alpha_k \}, \bar{\alpha} = \max_{k=1,2,\ldots,m+q} \{ \alpha_k \} \). This gives
\[ \left( \sum_{k=1}^{m+q} |u_k|^p \right)^{1/p} \leq \frac{\bar{\alpha}}{\varepsilon} \left( \sum_{k=1}^{m+q} |\hat{H}_k(\cdot)|^p \right)^{1/p} \]
or equivalently,
\[ \left( \sum_{i=1}^{m} |x_i|^p + \sum_{j=1}^{q} |y_j|^p \right)^{1/p} \leq \frac{\bar{\alpha}}{\varepsilon} \left( \sum_{i=1}^{m} |\hat{H}_i(x,y)|^p + \sum_{j=1}^{q} |\hat{H}_{m+j}(x,y)|^p \right)^{1/p}. \]

One can see that \( \| \hat{H}(x,y) \|_p \to \infty \) as \( \| (x,y)^T \|_p \to \infty \), and hence \( \| H(x,y) \|_p \to \infty \) as \( \| (x,y)^T \|_p \to \infty \). The existence of a unique solution \( (x,y)^T \) whose components satisfy the algebraic system (4.1) therefore follows. The proof is complete. \( \square \)

**Lemma 4.2.** Let the hypotheses \( H_{1,2} \) be satisfied. Suppose there exist positive numbers \( \beta_k \) (\( k = 1, 2, \ldots, m + q \)) for which the conditions
\[ a_i > \sum_{j=1}^{q} \beta_{m+j} |b_{ij}| L_j, \quad i \in \mathcal{I}; \quad c_j > \sum_{i=1}^{m} \beta_i |d_{ji}| M_i, \quad j \in \mathcal{J} \]
hold. Then there is a unique solution of the algebraic system (4.1).

**Proof.** It follows from the given conditions that there is positive number \( \rho \) satisfying \( 0 < \rho < 1 \) such that
\[ \max_{i \in \mathcal{I}} \left\{ \frac{1}{a_i} \sum_{j=1}^{q} \beta_{m+j} |b_{ij}| L_j \right\} \leq \rho, \quad \max_{j \in \mathcal{J}} \left\{ \frac{1}{c_j} \sum_{i=1}^{m} \beta_i |d_{ji}| M_i \right\} \leq \rho. \]

Let \( X^*_i = \beta_i^{-1} x^*_i \) and \( Y^*_j = \beta_{m+j}^{-1} y^*_j \) into the system (4.1) so as to obtain
\[ X^*_i = \frac{1}{a_i \beta_i} \sum_{j=1}^{q} b_{ij} f_j(\beta_{m+j} Y^*_j) + \frac{l_i}{a_i \beta_i}, \quad i \in \mathcal{I}, \quad Y^*_j = \frac{1}{c_j \beta_{m+j}} \sum_{i=1}^{m} d_{ji} g_i(\beta_i X^*_i) + \frac{j_j}{c_j \beta_{m+j}}, \quad j \in \mathcal{J}. \]

Accordingly, we construct a mapping \( \mathbf{G} : \mathbb{R}^{m+q} \to \mathbb{R}^{m+q} \) defined by
\[ \mathbf{G}(X,Y) = (G_1(X,Y), G_2(X,Y), \ldots, G_m(X,Y), G_{m+1}(X,Y), \ldots, G_{m+q}(X,Y))^T, \]
where
\[ G_i(X,Y) = \frac{1}{a_i \beta_i} \sum_{j=1}^{q} b_{ij} f_j(\beta_{m+j} Y_j) + \frac{l_i}{a_i \beta_i}, \quad i \in \mathcal{I}, \]
\[ G_{m+j}(X,Y) = \frac{1}{c_j \beta_{m+j}} \sum_{i=1}^{m} d_{ji} g_i(\beta_i X_i) + \frac{j_j}{c_j \beta_{m+j}}, \quad j \in \mathcal{J}. \]
The mapping is endowed with the norm $\| \cdot \|_\infty$ given as
\[
\| G(X, Y) \|_\infty = \max \left\{ \max_{i \in I} \{ |G_i(X, Y)| \}, \max_{j \in J} \{ |G_{m+j}(X, Y)| \} \right\}.
\]
Let $(X, Y)^T$ and $(U, V)^T$ be arbitrary vectors in $\mathbb{R}^{m+q}$. It follows from the definition of $G(\cdot)$ that
\[
G_i(X, Y) - G_i(U, V) = \frac{1}{a_i} \sum_{j=1}^q b_{ij} \left[ f_j(\beta_{m+j}Y_j) - f_j(\beta_{m+j}U_j) \right], \quad i \in I,
\]
\[
G_{m+j}(X, Y) - G_{m+j}(X, Y) = \frac{1}{c_j} \sum_{i=1}^m d_{ji} \left[ g_i(\beta_iX_i) - g_i(\beta_iU_i) \right], \quad j \in J.
\]
On applying the hypotheses,
\[
\| G(X, Y) - G(U, V) \|_\infty = \max \left\{ \max_{i \in I} \{ |G_i(X, Y) - G_i(U, V)| \}, \max_{j \in J} \{ |G_{m+j}(X, Y) - G_{m+j}(U, V)| \} \right\}
\leq \max \left\{ \max_{i \in I} \left\{ \frac{1}{a_i} \sum_{j=1}^q |b_{ij}| |L_j \beta_{m+j}Y_j - V_j| \right\}, \max_{j \in J} \left\{ \frac{1}{c_j} \sum_{i=1}^m |d_{ji}| |M_i \beta_iX_i - U_i| \right\} \right\}
\leq \max \left\{ \max_{i \in I} \left\{ \frac{1}{a_i} \sum_{j=1}^q \frac{|\beta_{m+j}| |L_j|}{|\beta_i|} |b_{ij}| \right\} \left\| (X, Y)^T - (U, V)^T \right\|_\infty \right\},
\]
\[
\max_{j \in J} \left\{ \frac{1}{c_j} \sum_{i=1}^m \frac{|\beta_i| |M_i|}{|\beta_{m+j}|} |d_{ji}| \right\} \left\| (X, Y)^T - (U, V)^T \right\|_\infty \right\},
\]
\[
\leq \max \left\{ \rho \left\| (X, Y)^T - (U, V)^T \right\|_\infty \right\}, \rho \left\| (X, Y)^T - (U, V)^T \right\|_\infty \right\}.
\]

By the well-known contraction mapping principle, there is a unique fixed point $(X^*, Y^*)^T$ of the mapping $G : \mathbb{R}^{m+q} \mapsto \mathbb{R}^{m+q}$ such that $(X^*, Y^*)^T = G(X^*, Y^*)$, i.e., $X_i^* = G_i(X^*, Y^*)$, $i \in I$ and $Y_j^* = G_{m+j}(X^*, Y^*)$, $j \in J$. The existence of a unique solution of the algebraic system (4.1) will follow. The proof is complete. □

Now we are ready to establish the exponential stability of the equilibrium state $(x^*, y^*)^T$ of the impulsive analogue (3.1). For convenience, let
\[
u_i(n) = x_i(n) - x_i^*, \quad \nu_j(n) = y_j(n) - y_j^*;
\]
\[
G_i(u_i(n)) = g_i(u_i(n) + x_i^*) - g_i(x_i^*), \quad F_j(v_j(n)) = f_j(v_j(n) + y_j^*) - f_j(y_j^*),
\]
\[
R_{ik}(u_i(n^-)) = r_{ik}(u_i(n^-) + x_i^*) + u_i(n^-), \quad S_{jk}(v_j(n^-)) = s_{jk}(v_j(n^-) + y_j^*) + v_j(n^-)
\]
so that the impulsive analogue (3.1) can be transformed into
\[
u_i(n+1) = e^{-a_ih}u_i(n) + \phi_i(h) \sum_{j=1}^q b_{ij} F_j(v_j(n)), \quad i \in I,
\]
\[
u_j(n+1) = e^{-c_jh}v_j(n) + \psi_j(h) \sum_{i=1}^m d_{ji} G_i(u_i(n)), \quad j \in J
\]
for $n \geq n_0$, $n \neq n_k$, that is subject to initial values given by
\[
u_i(n_0) = x(n_0) - x^*, \quad \nu_j(n_0) = y(n_0) - y^*
\]
and impulsive state displacements characterised by
\[
u_i(n_k^+) = R_{ik}(u_i(n_k^-)), \quad i \in I; \quad \nu_j(n_k^+) = S_{jk}(v_j(n_k^-)), \quad j \in J
\]
for $k \in \mathbb{N}$. Here the impulse functions $R_{ik}, S_{jk} : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $R_{ik}(0) = S_{jk}(0) = 0$ and

$$
|R_{ik}(u)| \leq \gamma_{ik}|u| + |u| \quad \text{def} = \eta_{ik}|u|, \quad i \in \mathcal{I}, \ k \in \mathbb{N},
$$

and

$$
|S_{jk}(v)| \leq \delta_{jk}|v| + |v| \quad \text{def} = \sigma_{jk}|v|, \quad j \in \mathcal{J}, \ k \in \mathbb{N}
$$

for all $u, v \in \mathbb{R}$, and the activation functions $F_j(\cdot), G_i(\cdot)$ with $F_j(0) = G_i(0) = 0$ satisfy

$$
|F_j(v)| \leq L_j|v|, \quad j \in \mathcal{J}; \quad |G_i(u)| \leq M_i|u|, \quad i \in \mathcal{I}
$$

for all $u, v \in \mathbb{R}$. Given the relationships between the two analogues (3.1) and (4.2), it suffices to establish the exponential stability of the trivial equilibrium state $(u^*, v^*)^T = (0, 0)^T$ of the impulsive analogue (4.2)—that is, the state satisfies

$$
a_i u_i^* = \sum_{j=1}^{q} b_{ij} F_j(v_j^*), \quad i \in \mathcal{I}; \quad c_j v_j^* = \sum_{i=1}^{m} d_{ij} G_i(u_i^*), \quad j \in \mathcal{J}.
$$

(4.5)

Let us assume for any integer $p \geq 1$ that $f^p(n) \rightarrow f^p(t)$ and

$$
\frac{1}{\phi(h)} \Delta f^p(n) = \frac{f^p(n + 1) - f^p(n)}{\phi(h)} \rightarrow \frac{df^p(t)}{dt} \quad \text{as } h \rightarrow 0,
$$

where $\phi(h) > 0$ is a denominator function associated with the function $f(n) \equiv f(nh)$ in which $n = \lfloor t/h \rfloor$. Here

$$
\frac{df^p(t)}{dt} = pf^{p-1}(t) \frac{df(t)}{dt}, \quad t \in \mathbb{R}
$$

for a differentiable function $f(t)$ defined for $t \in \mathbb{R}$. Corresponding to the derivative, we propose an analogue of the form

$$
\frac{1}{\phi(h)} \Delta f^p(n) = pf^{p-1}(n) \frac{1}{\phi(h)} \Delta f(n), \quad n \in \mathbb{Z}.
$$

(4.6)

**Theorem 4.1.** Let the value $0 < h < \theta$ be fixed and $p \geq 1$ be an integer. Assume that the hypotheses $H_{1,2,3}$ hold. Suppose there exist positive numbers $\alpha_k$ $(k = 1, 2, \ldots, m + q)$ and real numbers $\varrho_{l,ij}, \varphi_{l,ij}, \xi_{l,ji}, \zeta_{l,ji}$ $(l = 1, 2, \ldots, p)$ that satisfy $\sum_{l=1}^{p} \varrho_{l,ij} = 1$, $\sum_{l=1}^{p} \varphi_{l,ij} = 1$, $\sum_{j=1}^{p} \xi_{l,ji} = 1$, $\sum_{j=1}^{p} \zeta_{l,ji} = 1$ for which the conditions

$$
a_i > \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}| p^{\varrho_{l,ij}} L_{j}^{\varphi_{l,ij}} + \ldots + |b_{ij}| p^{\varphi_{p-1,ij}} L_{j}^{\varphi_{p-1,ij}} + \frac{\alpha_{m+j}}{\alpha_i} |d_{ij}| p^{\varrho_{p,ji}} M_{i}^{\varphi_{p,ji}} \right), \quad i \in \mathcal{I}, \quad (4.7)
$$

$$
c_j > \frac{1}{p} \sum_{i=1}^{m} \left( |d_{ij}| p^{\varphi_{l,ji}} M_{i}^{\varphi_{l,ji}} + \ldots + |d_{ij}| p^{\varphi_{p-1,ji}} M_{i}^{\varphi_{p-1,ji}} + \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}| p^{\varphi_{p,ij}} L_{j}^{\varphi_{p,ij}} \right), \quad j \in \mathcal{J}
$$

hold. Suppose, further, there exist positive numbers $\delta > 1$ and $\mu = \mu(h)$ satisfying $\mu > \frac{\ln \delta}{\theta}$ such that

$$
1 < \lambda_k \leq \delta, \quad k \in \mathbb{N},
$$

(4.8)

where $\lambda_k = \max\{\bar{\eta}_k, \bar{\sigma}_k\}$ with $\bar{\eta}_k = \max_{i \in \mathcal{I}} \{\eta_{ik}\} > 1$, $\bar{\sigma}_k = \max_{j \in \mathcal{J}} \{\sigma_{jk}\} > 1$. Then the trivial equilibrium state of the impulsive analogue (4.2) is globally exponentially stable in the sense of

$$
\left( \sum_{i=1}^{m} |u_i(n)|^p + \sum_{j=1}^{q} |v_j(n)|^p \right)^{1/p} \leq \sqrt{p} \beta e^{-(\mu - \frac{\ln \delta}{\theta})h(n-n_0)} \left( \sum_{i=1}^{m} |u_i(n_0)|^p + \sum_{j=1}^{q} |v_j(n_0)|^p \right)^{1/p}
$$

(4.9)

for $n \geq n_0$, where $\beta \geq 1$ is a real constant.
Proof. Firstly, we estimate the system (4.2a) by

\[ |u_i(n+1)| \leq e^{-a_i h} |u_i(n)| + \phi_i(h) \sum_{j=1}^{q} |b_{ij}| L_j |v_j(n)|, \quad i \in \mathcal{I}, \]

\[ |v_j(n+1)| \leq e^{-\varepsilon_j h} |v_j(n)| + \psi_j(h) \sum_{i=1}^{m} |d_{ji}| M_i |u_i(n)|, \quad j \in \mathcal{J} \tag{4.10} \]

for \( n \geq n_0, n \neq n_k \). Let us consider the functions \( G_i, F_j : \mathbb{R} \to \mathbb{R} \) for a fixed value \( 0 < h < \theta \) defined by

\[ G_i(\mu_i) = \frac{e^{(\mu_i-a_i)h} - 1}{\phi_i(h)} + \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}| p^{0 \xi_{1,ij}} L_j^{p \xi_{1,ij}} + \cdots + |b_{ij}| p^{0 \xi_{p-1,ij}} L_j^{p \xi_{p-1,ij}} \right) e^{\mu_i h} \]

\[ + \frac{1}{p} \sum_{j=1}^{q} \alpha_{m+j} |d_{ij}| p^{\xi_{p,ij}} M_i^{p \xi_{p,ij}} e^{\mu_i h}, \quad j \in \mathcal{J}, \]

\[ F_j(\psi_j) = \frac{e^{(\psi_j-c_j)h} - 1}{\psi_j(h)} + \frac{1}{p} \sum_{i=1}^{m} \left( |d_{ji}| p^{\xi_{1,ji}} M_i^{p \xi_{1,ji}} + \cdots + |d_{ji}| p^{\xi_{p-1,ji}} M_i^{p \xi_{p-1,ji}} \right) e^{\psi_j h} \]

\[ + \frac{1}{p} \sum_{i=1}^{m} \alpha_j |b_{ij}| p^{\xi_{p,ij}} L_j^{p \xi_{p,ij}} e^{\psi_j h}, \quad v_j \in \mathcal{J}, \]

We see that \( G_i(a_i) > 0 \) and \( F_j(c_j) > 0 \). Also,

\[ G_i(0) = \frac{e^{-a_i h} - 1}{\phi_i(h)} + \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}| p^{0 \xi_{1,ij}} L_j^{p \xi_{1,ij}} + \cdots + |b_{ij}| p^{0 \xi_{p-1,ij}} L_j^{p \xi_{p-1,ij}} + \frac{\alpha_{m+j}}{\alpha_i} |d_{ij}| p^{\xi_{p,ij}} M_i^{p \xi_{p,ij}} \right) \]

satisfies \( G_i(0) < 0 \) in accordance with the condition (4.7). Similarly, \( F_j(0) < 0 \). Moreover, the functions \( G_i(\cdot), F_j(\cdot) \) increase monotonically with \( \mu_i, v_j \in \mathbb{R} \) since \( \frac{dG_i}{dv_j} > 0, \frac{dF_j}{dv_j} > 0 \) for \( \mu_i, v_j \in \mathbb{R} \). Thus there exist unique numbers \( \tilde{\mu}_i \in (0, a_i), \tilde{v}_j \in (0, c_j) \) which are roots of the functions \( G_i(\cdot), F_j(\cdot) \). Upon choosing \( \mu = \min\{\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_q\} > 0, \)

\[ G_i(\mu) = \frac{e^{(\mu-a_i)h} - 1}{\phi_i(h)} + \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}| p^{0 \xi_{1,ij}} L_j^{p \xi_{1,ij}} + \cdots + |b_{ij}| p^{0 \xi_{p-1,ij}} L_j^{p \xi_{p-1,ij}} \right) e^{\mu h} \]

\[ + \frac{1}{p} \sum_{j=1}^{q} \alpha_{m+j} |d_{ij}| p^{\xi_{p,ij}} M_i^{p \xi_{p,ij}} e^{\mu h} \leq 0 \quad \text{for all} \quad i \in \mathcal{I}, \]

\[ F_j(\mu) = \frac{e^{(\psi-c_j)h} - 1}{\psi_j(h)} + \frac{1}{p} \sum_{i=1}^{m} \left( |d_{ji}| p^{\xi_{1,ji}} M_i^{p \xi_{1,ji}} + \cdots + |d_{ji}| p^{\xi_{p-1,ji}} M_i^{p \xi_{p-1,ji}} \right) e^{\psi h} \]

\[ + \frac{1}{p} \sum_{i=1}^{m} \alpha_j |b_{ij}| p^{\xi_{p,ij}} L_j^{p \xi_{p,ij}} e^{\psi h} \leq 0 \quad \text{for all} \quad j \in \mathcal{J}. \tag{4.11} \]

We note that the number \( \mu = \mu(h) \) that satisfies (4.11) depends on the choice of \( 0 < h < \theta \). Correspondingly, we define the following sequences

\[ X_i(n) = e^{\mu h(n-n_0)} |u_i(n)|, \quad Y_j(n) = e^{\mu h(n-n_0)} |v_j(n)| \tag{4.12} \]

for \( i \in \mathcal{I}, j \in \mathcal{J}, n \geq n_0 \). On substituting (4.12) into (4.10),
\[ X_i(n + 1) \leq e^{(\mu - a_i)h} X_i(n) + \phi_i(h) \sum_{j=1}^{q} |b_{ij}| L_j e^{uh} Y_j(n), \]
\[ Y_j(n + 1) \leq e^{(\mu - c_j)h} Y_j(n) + \psi_j(h) \sum_{i=1}^{m} |d_{ji}| M_i e^{uh} X_i(n). \]

which can be rearranged as

\[
\frac{\Delta X_i(n)}{\phi_i(h)} \leq e^{(\mu - a_i)h} \frac{1}{\phi_i(h)} X_i(n) + \sum_{j=1}^{q} |b_{ij}| L_j e^{uh} Y_j(n), \quad i \in I,
\]

\[
\frac{\Delta Y_j(n)}{\psi_j(h)} \leq e^{(\mu - c_j)h} \frac{1}{\psi_j(h)} Y_j(n) + \sum_{i=1}^{m} |d_{ji}| M_i e^{uh} X_i(n), \quad j \in J.
\]

(4.13)

for \( n \geq n_0, n \neq n_k \), where \( \Delta X_i(n) = X_i(n + 1) - X_i(n), \Delta Y_j(n) = Y_j(n + 1) - Y_j(n) \).

We associate the system (4.13) with a Lyapunov sequence \( V(\cdot) \) defined by

\[
V(n) = \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) X_i^p(n) + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) Y_j^p(n) \quad \text{for} \quad n \geq n_0.
\]

(4.14)

By applying the analogue (4.6) and the geometric–arithmetic mean inequality (as mentioned before), we estimate the difference \( \Delta V(n) = V(n + 1) - V(n) \) for \( n \geq 0, n \neq n_k \) along the solutions of (4.13) as follows:

\[
\Delta V(n) = \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) \Delta X_i^p(n) + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) \Delta Y_j^p(n)
\]

\[
= \sum_{i=1}^{m} \alpha_i p X_i^{p-1}(n) \phi_i^{-1}(h) \Delta X_i(n) + \alpha_{m+j} p Y_j^{p-1}(n) \psi_j^{-1}(h) \Delta Y_j(n)
\]

\[
\leq \sum_{i=1}^{m} \alpha_i p X_i^{p-1}(n) \left\{ e^{(\mu - a_i)h} \frac{1}{\phi_i(h)} X_i(n) + \sum_{j=1}^{q} |b_{ij}| L_j e^{uh} Y_j(n) \right\}
\]

\[
+ \sum_{j=1}^{q} \alpha_{m+j} p Y_j^{p-1}(n) \left\{ e^{(\mu - c_j)h} \frac{1}{\psi_j(h)} Y_j(n) + \sum_{i=1}^{m} |d_{ji}| M_i e^{uh} X_i(n) \right\}
\]

\[
= \sum_{i=1}^{m} \alpha_i p \left\{ e^{(\mu - a_i)h} \frac{1}{\phi_i(h)} X_i^p(n) + \sum_{j=1}^{q} |b_{ij}| L_j e^{uh} X_i^{p-1}(n) Y_j^p(n) \right\}
\]

\[
+ \sum_{j=1}^{q} \alpha_{m+j} p \left\{ e^{(\mu - c_j)h} \frac{1}{\psi_j(h)} Y_j^p(n) + \sum_{i=1}^{m} |d_{ji}| M_i e^{uh} Y_j^{p-1}(n) X_i(n) \right\}
\]

\[
= \sum_{i=1}^{m} \alpha_i p \left\{ e^{(\mu - a_i)h} \frac{1}{\phi_i(h)} X_i^p(n) + \sum_{j=1}^{q} |b_{ij}| e^{uh} \left[ L_j^f \right] X_i(n) \right\}
\]

\[
\times \cdots \times \left( |b_{ij}| ^p L_j^f X_i(n) \right) \times \left( |b_{ij}| ^p L_j^f Y_j(n) \right) \right]\}
\]

\[
+ \sum_{j=1}^{q} \alpha_{m+j} p \left\{ e^{(\mu - c_j)h} \frac{1}{\psi_j(h)} Y_j^p(n) + \sum_{i=1}^{m} e^{uh} \left[ M_i \right] Y_j^{p-1}(n) \right\}
\]

\[
\times \cdots \times \left( |d_{ji}| ^p M_i \right) Y_j(n) \times \left( |d_{ji}| ^p M_i \right) \right]\}
\]
\[ \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) |u_i(n+1)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) |v_j(n+1)|^p \leq e^{-p \mu h} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) |u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) |v_j(n)|^p \right) \] (4.15)

for \( n \geq n_0, n \neq n_k \). The impulse effects of (4.2c) with (4.3) on \( V(n) \) at \( n = n_k \) are estimated as follows:

\[ \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) |u_i(n_k^+)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) |v_j(n_k^+)|^p = \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) |u_i(n_k^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) |v_j(n_k^-)|^p \leq \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) n_{ik}^p |u_i(n_k^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) \sigma_{jk}^p |v_j(n_k^-)|^p \leq \Lambda_k^p \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h) |u_i(n_k^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h) |v_j(n_k^-)|^p \right) \] (4.16)

for \( k \in \mathbb{N} \). Upon solving (4.15) for \( n = n_0, n_0 + 1, \ldots, n_1 - 1, \)
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n+1)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n)|^p
\]
\[
\leq e^{-p\mu h(n+1-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_0)|^p \right)
\]

which we rewrite as
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n)|^p
\]
\[
\leq e^{-p\mu h(n-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_0)|^p \right)
\]

for \( n = n_0 + 1, n_0 + 2, \ldots, n_1 \). We have from (4.16) for \( k = 1 \) and the above that
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_1^+)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_1^+)|^p
\]
\[
\leq A_1^p \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_1^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_1^-)|^p \right)
\]
\[
\leq A_1^p e^{-p\mu h(n_1-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_0)|^p \right)
\]

Again we solve (4.15) for \( n = n_1, n_1 + 1, \ldots, n_2 - 1 \) to get
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n+1)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n+1)|^p
\]
\[
\leq e^{-p\mu h(n+1-n_1)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_1^+)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_1^+)|^p \right)
\]

which can be rewritten and estimated as
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n)|^p
\]
\[
\leq e^{-p\mu h(n-n_1)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_1^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_1^-)|^p \right)
\]
\[
\leq A_1^p e^{-p\mu h(n-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_0)|^p \right)
\]

for \( n = n_1 + 1, n_1 + 2, \ldots, n_2 \). We have from (4.16) for \( k = 2 \) and the above that
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_2^+)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_2^+)|^p
\]
\[
\leq A_2^p \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_2^-)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_2^-)|^p \right)
\]
\[
\leq A_1^p A_2^p e^{-p\mu h(n_2-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \phi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
Inductively,
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_{k-1}^{+})|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_{k-1}^{+})|^p \\
\leq A_0^p A_1^p \cdots A_{k-1}^p e^{-p\mu h(n_{k-1}-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for \( k \in \mathbb{N} \), and
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n)|^p \\
\leq A_0^p A_1^p \cdots A_{k-1}^p e^{-p\mu h(n-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for \( n = n_{k-1}, n_{k-1} + 1, \ldots, n_k, k \in \mathbb{N} \), where \( A_0 = 1 \). On applying the condition \( (4.8) \),
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_{k-1}^{+})|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_{k-1}^{+})|^p \\
\leq \delta^{p(k-1)} e^{-p\mu h(n_{k-1}-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for \( k \in \mathbb{N} \), and
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n)|^p \\
\leq \delta^{p(k-1)} e^{-p\mu h(n-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for \( n = n_{k-1}, n_{k-1} + 1, \ldots, n_k, k \in \mathbb{N} \). Observe that \( h \Delta n_k \geq \theta \) for all \( k \in \mathbb{N} \) and
\[
h(n_{k-1}-n_0) = h[(n_{k-1}-n_{k-2})+(n_{k-2}-n_{k-3})+\cdots+(n_2-n_1)+(n_1-n_0)] \\
\geq (k-1)\theta \\ 
\text{for } k \in \mathbb{N},
\]
\[
h(n-n_0) = h[(n-n_{k-1})+(n_{k-1}-n_0)] \geq (k-1)\theta \\ 
\text{for } n_{k-1} \leq n \leq n_k, k \in \mathbb{N}.
\]
Thus,
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_{k-1}^{+})|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_{k-1}^{+})|^p \\
\leq e^{-p(\mu-\frac{\ln k}{h})h(n_{k-1}-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for \( k \in \mathbb{N} \), and
\[
\sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n)|^p \\
\leq e^{-p(\mu-\frac{\ln k}{h})h(n-n_0)} \left( \sum_{i=1}^{m} \alpha_i \phi_i^{-1}(h)|u_i(n_0)|^p + \sum_{j=1}^{q} \alpha_{m+j} \psi_j^{-1}(h)|v_j(n_0)|^p \right)
\]
for $n_{k-1} \leq n \leq n_k$, $k \in \mathbb{N}$. These can be estimated further and summarised into

$$
\sum_{j=1}^{m} |u_i(n)|^p + \sum_{j=1}^{q} |v_j(n)|^p \leq \beta e^{-p(\mu - \frac{\ln \delta}{\theta}) h(n-n_0)} \left( \sum_{i=1}^{m} |u_i(n_0)|^p + \sum_{j=1}^{q} |v_j(n_0)|^p \right)
$$

for $n \geq n_0$, where

$$
\beta = \frac{\max \left\{ \max_{i \in I} \{ \alpha_i \phi_i^{-1}(h) \}, \max_{j \in J} \{ \alpha_{m+j} \psi_j^{-1}(h) \} \right\}}{\min \left\{ \min_{i \in I} \{ \alpha_i \phi_i^{-1}(h) \}, \min_{j \in J} \{ \alpha_{m+j} \psi_j^{-1}(h) \} \right\}} \geq 1.
$$

The result (4.9) will follow. We see that the solution $(u(n), v(n))^T$ approaches the trivial equilibrium state exponentially in the norm $\| \cdot \|_p$ as $n \to \infty$. This completes the proof. \Box

The analysis given in Theorem 4.1 can be adopted in a straightforward manner for handling linear impulses characterised by

$$
r_{ik}(x_i(n_k^-)) = -d_{ik}(x_i(n_k^-) - x_i^*), \quad s_{jk}(y_j(n_k^-)) = -e_{jk}(y_j(n_k^-) - y_j^*) \quad (4.17)
$$

for $k \in \mathbb{N}$, where $d_{ik}, e_{jk}$ denote real numbers. The equivalent form of (4.17) is

$$
R_{ik}(u_i(n_k^-)) = (1 - d_{ik})u_i(n_k^-), \quad S_{jk}(v_j(n_k^-)) = (1 - e_{jk})v_j(n_k^-) \quad (4.18)
$$

for $k \in \mathbb{N}$. These impulses (especially those of the contraction type satisfying $0 < |1 - d_{ik}| < 1$, $0 < |1 - e_{jk}| < 1$) have been considered in the literature on impulsive neural networks [14,19–25]. In this case, the Lyapunov exponent of the impulsive network is just $\mu$ which is similar to the one for a non-impulsive network. Such exponent denotes a special case (i.e. by setting $\delta = 1$) to the Lyapunov exponent $\mu - \frac{\ln \delta}{\theta}$ obtained in the following theorem. The proof of the theorem is similar like before, and is therefore omitted.

**Theorem 4.2.** Let the value $0 < h < \theta$ be fixed and $p \geq 1$ be an integer. Assume that the hypotheses $H_{1,2,3}$ hold. Suppose the conditions in (4.7) are satisfied, and the impulses characterised by (4.18) satisfy

$$
|1 - d_{ik}| \leq \delta, \quad |1 - e_{jk}| \leq \delta, \quad k \in \mathbb{N} 
$$

in which the positive numbers $\delta \geq 1$ and $\mu = \mu(h)$ satisfy $\mu > \frac{\ln \delta}{\theta}$. Then the trivial equilibrium state of the impulsive analogue (4.2) is globally exponentially stable in the sense of (4.9).

Notice that the conditions in (4.7) are independent of the time-step parameter $h$ and therefore can be used to assert the exponential convergence of the impulsive BAM network (2.1). The conditions with $2 \leq p < \infty$ represent a *mixed-type dominance* which is essentially a mix between the conditions that involve *columnwise dominance* (i.e., $p = 1$) and *rowwise dominance* (i.e., $p = \infty$). One may refer to Gopalsamy [16] for the usage of these terminologies. The presence of nonnetwork parameters, namely, $q_{i,ij}$, $c_{i,ij}$, $x_{i,ij}$, $x_{i,ij}^*$, $y_{i,ij}$, $\eta_{i,ij}$, in the sufficient conditions given by (4.7) is useful for designing the desired circuit of the exponentially convergent impulsive analogue.

The design involving columnwise dominance consists of the self-regulating feedback of each processing unit dominating its own contribution to other units. The associated conditions can be obtained from (4.7) by choosing the values $p = 1$. Consequently, $q_{p,ij} = z_{p,ij} = \xi_{p,ij} = \eta_{p,ij} = 1$ and hence, the conditions reduce to

$$
a_i > \sum_{j=1}^{q} \alpha_{m+j} |d_{ji}| M_i, \quad i \in I, \quad c_j > \sum_{i=1}^{m} \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}| L_j, \quad j \in J. \quad (4.20)
$$

In this case, the convergence of the impulsive analogue is associated with the norm $\| \cdot \|_1$ and the results are stated as follows:
Corollary 4.3. Let $0 < h < \theta$ be fixed. Assume that the hypotheses $H_{1,2,3}$ hold. Suppose the conditions in (4.8) and (4.20) are satisfied. Then the impulsive analogue (4.2) is exponentially stable in the sense of
\[ \sum_{i=1}^{m} |u_i(n)| + \sum_{j=1}^{q} |v_j(n)| \leq \beta e^{-(\mu - \frac{ln\beta + \theta}{p})h(n-n_0)} \left( \sum_{i=1}^{m} |u_i(n_0)| + \sum_{j=1}^{q} |v_j(n_0)| \right) \]
for $n \geq n_0$, where $\beta \geq 1$.

The design involving rowwise dominance consists of the self-regulating feedback of each processing unit dominating the effects of other units interacting with it. To obtain the associated conditions from (4.7), we choose the values $\alpha_{l,ij} = \xi_{l,ij} = \xi_{l,ji} = \eta_{l,ji} = 1/p$ for an integer $p > 1$. Consequently,
\[ a_i > \frac{p - 1}{p} \sum_{j=1}^{q} |b_{ij}|L_j + \frac{1}{p} \sum_{j=1}^{q} \frac{\alpha_{m+j}}{\alpha_i} |d_{ji}|M_i, \quad i \in I, \]
\[ c_j > \frac{p - 1}{p} \sum_{i=1}^{m} |d_{ji}|M_i + \frac{1}{p} \sum_{i=1}^{m} \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}|L_j, \quad j \in J. \]
By allowing $p \to \infty$ in the above,
\[ a_i > \sum_{j=1}^{q} |b_{ij}|L_j, \quad i \in I; \quad c_j > \sum_{i=1}^{m} |d_{ji}|M_i, \quad j \in J. \] (4.22)

In this case, the convergence is associated with the norm $\| \cdot \|_\infty$. Take note, however, that it is somewhat inappropriate to apply the Lyapunov sequence $V(\cdot)$ defined by (4.14) for the case $p = \infty$. By appropriately redefining the Lyapunov sequence, we obtain in the following theorem sufficient conditions for the convergence of the analogue (4.2) in the norm $\| \cdot \|_\infty$; the conditions include (by choosing $\beta_k = 1$ for $k = 1, 2, \ldots, m + q$) those usual conditions given by (4.22).

Theorem 4.4. Let $0 < h < \theta$ be a fixed value, and the hypotheses $H_{1,2,3}$ hold. Suppose there exist positive numbers $\beta_k$ ($k = 1, 2, \ldots, m + q$) for which the conditions
\[ a_i > \sum_{j=1}^{q} \frac{\beta_{m+j}}{\beta_i} |b_{ij}|L_j, \quad i \in I; \quad c_j > \sum_{i=1}^{m} \frac{\beta_i}{\beta_{m+j}} |d_{ji}|M_i, \quad j \in J \] (4.23)
hold. Suppose, further, that the condition (4.8) holds. Then the impulsive analogue (4.2) is exponentially stable in the sense of
\[ |u_i(n)| \leq \rho Ke^{-\frac{ln\beta + \theta}{p}h(n-n_0)}, \quad |v_j(n)| \leq \rho Ke^{-\frac{ln\beta + \theta}{p}h(n-n_0)} \]
for all $i \in I$, $j \in J$, $n \geq n_0$, in which $\rho \geq 1$ and $K = \max\{\|u(n_0)\|_\infty, \|v(n_0)\|_\infty\}$ with $\|u(n_0)\|_\infty = \max_{i \in I}\{u_i(n_0)\}$ and $\|v(n_0)\|_\infty = \max_{j \in J}\{v_j(n_0)\}$.

Proof. We begin by substituting
\[ X_i(n) = \beta_i^{-1} |u_i(n)|, \quad i \in I; \quad Y_j(n) = \beta_{m+j}^{-1} |v_j(n)|, \quad j \in J \]
for $n \geq n_0$, into the system (4.10) to obtain
\[ X_i(n+1) \leq e^{-a_i h} X_i(n) + \phi_i(h) \sum_{j=1}^{q} \frac{\beta_{m+j}}{\beta_i} |b_{ij}|L_j Y_j(n), \quad i \in I, \quad n \geq n_0, \quad n \neq n_k, \]
\[ Y_j(n+1) \leq e^{-c_j h} Y_j(n) + \psi_j(h) \sum_{i=1}^{m} \frac{\beta_i}{\beta_{m+j}} |d_{ji}|M_i X_i(n), \quad j \in J, \quad n \geq n_0, \quad n \neq n_k. \] (4.26)
Associated with this system is a Lyapunov sequence $V(\cdot)$ defined by
\[ V(n) = \max \{ X_{i_0}(n), Y_{j_0}(n) \} \quad \text{for} \quad n \geq n_0, \] (4.27)
where $i_0 = i_0(n) \in \mathcal{I}$, $j_0 = j_0(n) \in \mathcal{J}$ and

$$X_{i_0}(n) = \max_{i \in \mathcal{I}} \{X_i(n)\}, \quad Y_{j_0}(n) = \max_{j \in \mathcal{J}} \{Y_j(n)\}.$$  

Clearly, $X_i(n) \leq V(n)$, $Y_j(n) \leq V(n)$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, $n \geq n_0$. Along the solutions of (4.26), we have either

$$V(n + 1) \leq e^{-a_i h} V(n) + \phi_{i_0}(h) \sum_{j=0}^{q} \frac{\beta_{m+j}}{\beta_{i_0}} |b_{ij}| L_j Y_j(n) \leq e^{-a_i h} + \phi_{i_0}(h) \sum_{j=1}^{q} \frac{\beta_{m+j}}{\beta_{i_0}} |b_{ij}| L_j V(n),$$

or (similarly)

$$V(n + 1) \leq \left(e^{-c_j h} + \psi_{j_0}(h) \sum_{i=1}^{m} \frac{\beta_{i}}{\beta_{m+j}} |d_{ji}| M_i \right) V(n)$$

for $n \geq n_0$, $n \neq n_k$. In either case,

$$V(n + 1) \leq e^{-\mu h} V(n) \quad \text{for} \quad n \geq n_0, \quad n \neq n_k$$

(4.28)

in which $\mu = \mu(h)$ for a given $0 < h < \theta$ is a positive number that satisfies

$$e^{-a_i h} + \phi_{i_0}(h) \sum_{j=0}^{q} \frac{\beta_{m+j}}{\beta_{i_0}} |b_{ij}| L_j \leq e^{-\mu h} \quad \text{for all} \quad i \in \mathcal{I},$$

$$e^{-c_j h} + \psi_{j_0}(h) \sum_{i=1}^{m} \frac{\beta_{i}}{\beta_{m+j}} |d_{ji}| M_i \leq e^{-\mu h} \quad \text{for all} \quad j \in \mathcal{J}.$$

In what follows, we claim that

$$V(n_{k-1}^+) \leq V(n_0) A_0 A_1 \cdots A_{k-1} e^{-\mu h(n_{k-1}^+ - n_0)}, \quad k \in \mathbb{N},$$

$$V(n) \leq V(n_0) A_0 A_1 \cdots A_{k-1} e^{-\mu h(n_+ - n_0)}, \quad n_{k-1} \leq n \leq n_k. \tag{4.29}$$

Certainly, the claim is valid for $k = 1$ since $V(n_0^+) = V(n_0)$ and

$$V(n + 1) \leq V(n_0) e^{-\mu h(n+1 - n_0)} \quad \text{for} \quad n_0 \leq n \leq n_1 - 1,$$

the outcome upon solving the equation (4.28). This can be rewritten as

$$V(n) \leq V(n_0) e^{-\mu h(n_0 - n_0)} \quad \text{for} \quad n_0 \leq n \leq n_1.$$

Let us assume that the claim (4.29) is valid for $k = l > 1$. At the instant $n = n_l^+$, we have from (4.2c), (4.3), (4.25), (4.27) and (4.29) that

$$\text{either}$$

$$X_{i_0}(n_l^+) \leq A_1 X_{i_0}(n_l^-) = A_1 V(n_l^-) \leq V(n_0) A_0 A_1 \cdots A_{l-1} A_l e^{-\mu h(n_l^- - n_0)}$$

or

$$Y_{j_0}(n_l^+) \leq A_1 Y_{j_0}(n_l^-) \leq A_1 V(n_l^-) \leq V(n_0) A_0 A_1 \cdots A_{l-1} A_l e^{-\mu h(n_l^- - n_0)}.$$

In either case,

$$V(n_l^+) \leq V(n_0) A_0 A_1 \cdots A_{l-1} A_l e^{-\mu h(n_l^+ - n_0)}.$$

This estimate is then extended by the Eq. (4.28) to give

$$V(n + 1) \leq e^{-\mu h(n+1 - n_l)} V(n_l^+) \leq V(n_0) A_0 A_1 \cdots A_{l-1} A_l e^{-\mu h(n+1 - n_0)}, \quad n_l \leq n \leq n_l + 1.$$  

We rewrite the estimate as

$$V(n) \leq V(n_0) A_0 A_1 \cdots A_{l-1} A_l e^{-\mu h(n_0 - n_0)} \quad \text{for} \quad n_l \leq n \leq n_{l+1}.$$  

Thus, by induction, we see that the claim (4.29) is valid for any $k \in \mathbb{N}$.

On applying the condition (4.8) to (4.29),

$$V(n_{k-1}^+) \leq V(n_0) \delta^{k-1} e^{-\mu h(n_{k-1}^+ - n_0)}, \quad k \in \mathbb{N},$$

$$V(n) \leq V(n_0) \delta^{k-1} e^{-\mu h(n_0 - n_0)}, \quad n_{k-1} \leq n \leq n_k.$$
We observe that $h \Delta n_k \geq \theta$ and

\[
h(n_{k-1} - n_0) = h[(n_{k-1} - n_{k-2}) + (n_{k-2} - n_{k-3}) + \cdots + (n_2 - n_1) + (n_1 - n_0)] \geq (k - 1)\theta, \quad k \in \mathbb{N},
\]
\[
h(n - n_0) = h[(n - n_{k-1}) + (n_k - n_0)] \geq (k - 1)\theta, \quad n_{k-1} \leq n \leq n_k, \quad k \in \mathbb{N}.
\]

Thus,

\[
V(n^+_{k-1}) \leq V(n_0)e^{-(\mu - \frac{\ln k}{\theta})h(n_{k-1} - n_0)}, \quad k \in \mathbb{N},
\]
\[
V(n) \leq V(n_0)e^{-(\mu - \frac{\ln k}{\theta})h(n - n_0)}, \quad n_{k-1} \leq n \leq n_k, \quad k \in \mathbb{N}.
\]

This can be summarised (since the Lyapunov exponent $\mu - \frac{\ln k}{\theta}$ is independent of $k \in \mathbb{N}$) as

\[
V(n) \leq V(n_0)e^{-(\mu - \frac{\ln k}{\theta})h(n - n_0)} \quad \text{for } n \geq n_0.
\]

We have from $X_i(n) \leq V(n), \ Y_j(n) \leq V(n)$ for all $i \in \mathcal{I}, \ j \in \mathcal{J}, \ k \in \mathbb{N},$ and also from (4.25) that

\[
\left|u_i(n)\right| \leq \rho K e^{-(\mu - \frac{\ln k}{\theta})h(n - n_0)}, \quad \left|v_j(n)\right| \leq \rho K e^{-(\mu - \frac{\ln k}{\theta})h(n - n_0)}
\]

for all $i \in \mathcal{I}, \ j \in \mathcal{J}, \ k \in \mathbb{N},$ where $\rho = \frac{\max e^{1, \ldots, m+q}\{\tilde{\rho}_k\}}{\min e^{1, \ldots, m+q}\{\tilde{\rho}_k\}} \geq 1.$ The proof is complete. \(\square\)

5. Discussion and computer simulations

We demonstrate several capabilities of the impulsive analogue (4.2) in providing computer simulations of its continuous-time counterpart – the counterpart is a transformed version of the impulsive BAM network (2.1) – in resisting the destabilising effects of nonlinear impulses with large magnitude.

It is not hard to see that the convergence estimate in accordance with Theorem 4.1 for the continuous-time impulsive network is given by $e^{-(\mu - \frac{\ln k}{\theta})(t-t_0)}$ for $t > t_0$ in which the value $\mu^*$ is determined from the conditions given by (4.7), namely,

\[
\begin{align*}
\mu^* - a_i + \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}| \right)^{p_\xi} L_j^{p_\xi} &+ \cdots + |b_{ij}|^{p_\xi} L_j^{p_\xi} \\
+ \frac{1}{p} \sum_{j=1}^{q} \frac{\alpha_{m+j}}{\alpha_i} |d_{ji}|^{p_\xi} M_i^{p_\xi} &\leq 0 \quad \text{for all } i \in \mathcal{I},
\end{align*}
\]

\[
\begin{align*}
\mu^* - c_j + \frac{1}{p} \sum_{i=1}^{m} \left( |d_{ji}| \right)^{p_\xi} M_i^{p_\xi} &+ \cdots + |d_{ji}|^{p_\xi} M_i^{p_\xi} \\
+ \frac{1}{p} \sum_{i=1}^{m} \frac{\alpha_i}{\alpha_{m+j}} |b_{ij}|^{p_\xi} L_j^{p_\xi} &\leq 0 \quad \text{for all } j \in \mathcal{J},
\end{align*}
\]

and that $\mu^* > \frac{\ln \delta}{\theta}$ with $\delta > 1$ appearing in the condition (4.8) as a means for controlling the size of the impulses. In fact, it can be immediately seen that the results of Theorem 4.1 (and also the other theorems) reduce to the corresponding results for the exponential stability of the continuous-time counterpart if the value $h$ approaches zero.

For our impulsive analogue, we observe that the constant $\mu = \mu(h)$ for a given fixed value $0 < h < \theta$ can only attain a value within the range $0 < \mu < \mu^*.$ If one assumes $h$ large—and so is $\theta,$ then the value $\mu$ needs to be close to zero in order to satisfy the inequalities given by (4.11). In the limit $h \to 0,$ the function $G_i(\mu)$ – and so is $F_j(\cdot)$ – satisfying (4.11) exhibits the following:

\[
\lim_{h \to 0} \ G_i(\mu) = \lim_{h \to 0} \left\{ e^{(\mu - a_i)h} - 1 \right\} \frac{1}{(1 - e^{-\alpha_i h})/\alpha_i} + \frac{1}{p} \sum_{j=1}^{q} \left( |b_{ij}|^{p_\xi} L_j^{p_\xi} + \cdots + |b_{ij}|^{p_\xi} L_j^{p_\xi} \right)
\]
\[+ \frac{1}{p} \sum_{j=1}^{q} \frac{\alpha_{m+j}}{\alpha_i} |d_{ji}|^{p_\xi} M_i^{p_\xi} \]
we provide several computer simulations produced by the analogue that Fig. 1 leads us to allowing preserves network. \( h(d) \) corresponding to

\[
\ln \mu > \mu t \quad \text{in which } \gamma \text{ is subject to impulses characterised by the functions } \\
\mu > \text{magnitude so long as } \text{the exponential convergence of the continuous-time network especially when subjected to impulses with large.}
\]

A comparison with (5.1) leads us to allowing \( \mu \to \mu^* \) as \( h \to 0 \).

The discussion suggests that the value \( h \) needs to be appropriately chosen in order for the analogue (4.2) preserves the exponential convergence of the continuous-time network especially when subjected to impulses with large magnitude so long as \( \mu > \frac{\ln \delta}{\sigma} \). In Fig. 1, we provide several computer simulations produced by the analogue that is subject to impulses characterised by the functions

\[
R_{ik}(u_i(n^-_k)) = \gamma \tanh(u_i(n^-_k)), \quad S_{jk}(v_j(n^-_k)) = \gamma \tanh(v_j(n^-_k)),
\]

in which \( \gamma = 1.9 \times 10^6 \) denotes the impulse magnitude and the positive integers \( n_k \) correspond to fixed moments \( t_k = 5, 10, 15, \ldots \). Note that \( \sigma = 5 \). The network parameters have been assigned with values so that \( \mu^* = 4.1 \) satisfies \( \mu^* > \frac{\ln \gamma}{\sigma} \). The values \( \mu = \mu(h) \) corresponding to step-sizes \( h = 0.1, 0.2, 0.5, 1 \) are calculated as \( \mu = 3.9845, 3.8405, 3.2384, 2.072, \) respectively. One can verify that the first three values satisfy the condition \( \mu > \frac{\ln \gamma}{\sigma} \) for containing the impulse magnitude \( \gamma = 1.9 \times 10^6 \) so that exponential stability is achieved by the analogue.
$$\mu = 4, \text{ impulse } = (2.9 \times 10^8) \tanh(u), h = 0.01.$$  

$$\mu = 4.4, \text{ impulse } = (2.9 \times 10^8) \tanh(u), h = 0.01.$$  

$$\mu = 4.6, \text{ impulse } = (2.9 \times 10^8) \tanh(u), h = 0.01.$$  

$$\mu = 5, \text{ impulse } = (2.9 \times 10^8) \tanh(u), h = 0.01.$$  

Fig. 2. The strong convergence behaviour of the impulsive BAM network is preserved by the impulsive analogue as the values of the self-regulating feedback parameters $a_i, c_j$ are increased—that is, $\mu = 4, 4.4, 4.6, 5$.

(see subplots (a)–(c) of Fig. 1). For $\mu = 2.072$ corresponding to $h = 1$, the condition $\mu > \frac{\ln \gamma}{\theta}$ is violated. Thus erroneous dynamics in the form of divergence which occur at the impulse moments are produced by the impulsive analogue (see subplot (d) of Fig. 1).

The other interesting feature possessed by the impulsive analogue is the stability conditions given by (4.7) which do not contain the time-step parameter. This means that once the impulse magnitude has already been contained according to $\mu > \frac{\ln \gamma}{\theta}$ for a given $0 < h < \theta$, enlarging the value $\mu$ by increasing the values of the self-regulating feedback parameters $a_i, c_j$ will indicate strong convergence in the impulsive analogue (see Fig. 2). This convergence certainly reflects the strong convergence behaviour of the impulsive BAM network.

6. Conclusion

We have provided a unified treatment in analysing the exponential convergence of impulsive discrete-time BAM networks. The results for the discrete-time networks can be used for ascertaining the exponential stability of the corresponding continuous-time networks by allowing $h \to 0$. Secondly, a relation between the Lyapunov exponent $\mu - \frac{\ln \delta}{\theta}$ of an impulsive BAM network and the exponent $\mu$ of a non-impulsive BAM network (i.e., when $\theta \to \infty$) or a BAM network with linear impulses of the contraction type (i.e., when $\delta = 1$) has been explicitly obtained.

References


