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Diagonal flips in Hamiltonian triangulations on the projective plane

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Abstract

In this paper, we shall prove that any two triangulations on the projective plane with n vertices can be transformed into each other by at most $8n - 26$ diagonal flips, up to isotopy. To prove it, we focus on triangulations on the projective plane with contractible Hamilton cycles.

Keywords: Diagonal flip; Triangulation; Projective plane; Hamiltonian cycle

1. Introduction

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *k-cycle* means a cycle of length k. For two graphs H and K, let $H + K$ denote the graph obtained from H and K by joining each vertex of H to all vertices of K.

A *triangulation* on a closed surface F^2 is a simple graph on F^2 such that each face is bounded by a 3-cycle. A *diagonal flip* is an operation which replaces an edge e in the quadrilateral D formed by two faces sharing e with another diagonal of D (see Fig. 1). If the resulting graph is not simple, then we do not apply it.

Wagner proved that any two triangulations on the plane with the same number of vertices can be transformed into each other by a sequence of diagonal flips, up to isotopy [9]. This result has been extended to the torus [1], the projective plane and the Klein bottle [7]. Moreover, Negami has proved that for any closed surface F^2, there exists a positive integer

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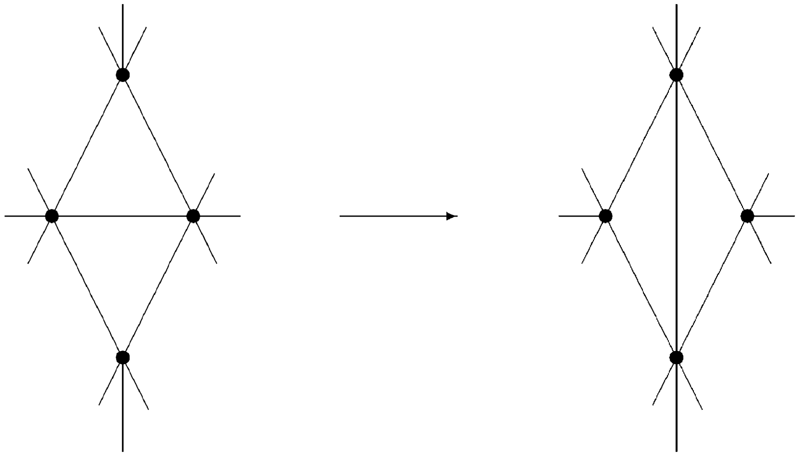

Fig. 1. Diagonal flip.

$N(F^2)$ such that any two triangulations G and G' on F^2 with $|V(G)| = |V(G')| \geqslant N(F^2)$ can be transformed into each other by a sequence of diagonal flips, up to homeomorphism [4]. There are many papers concerning with diagonal flips in triangulations and they are described in [6] for details.

In this paper, we focus on the minimum number of diagonal flips needed to transform two given triangulations on a closed surface F^2. Negami's argument in [5] shows that for the minimum number of diagonal flips needed to transform two triangulations with n vertices on a closed surface F^2, there is a quadratic bound with respect to n. However, if we restrict F^2 to the sphere, then there is a linear bound $6n - 30$ for it, as shown in [3].

In this paper, we shall prove the following theorem:

Theorem 1. *Any two triangulations on the projective plane with n vertices can be transformed into each other by at most* $8n - 26$ *diagonal flips, up to isotopy.*

This is the first result giving a linear bound for the minimum number of diagonal flips in triangulations on a closed surface other than the sphere.

For a graph G, a *Hamilton cycle* of G is a cycle passing through each vertex of G exactly once. A cycle C of G embedded in a closed surface F^2 is said to be *contractible* if C bounds a 2-cell on F^2. In order to prove Theorem 1, we show the following theorem for triangulations on the projective plane with a contractible Hamilton cycle, as in the spherical case in [3].

Theorem 2. *Let G and G′ be two triangulations on the projective plane with n vertices, each of which has a contractible Hamilton cycle. Then G and G′ can be transformed into each other by at most* $6n - 12$ *diagonal flips, preserving their Hamilton cycles.*

2. Triangulations with contractible Hamilton cycles

In this section, we deal only with triangulations which have contractible Hamilton cycles. Clearly, a contractible Hamilton cycle in a triangulation G on the projective plane separates

G into two spanning subgraphs of G. One is a maximal outer-plane graph, denoted by G_P, and the other is a triangulation of the Möbius band, denoted by G_M, in which all vertices appear on the boundary of the Möbius band. We call it a *Catalan triangulation* on the Möbius band.

Lemma 3. *Let P be a maximal outer-plane graph with $n \geqslant 3$ vertices and let v be a vertex of degree $k \geqslant 2$ in P. Then P can be transformed into a maximal outer-plane graph in which the degree of v is exactly $n-1$, by exactly $n-k-1$ ($\leqslant n-3$) diagonal flips, through maximal outer-plane graphs.*

Proof. Let xy be an edge of P not in its outer cycle and let vxy and uxy be two faces sharing xy. Since $\deg_P(v) = k$, the number of vertices not adjacent to v is $n-k-1$. Since P has no subgraph isomorphic to K_4, u and v are not adjacent in P. Therefore, we can flip xy without making multiple edges. Hence, we can increase the degree of v one by one, by diagonal flips. Therefore, the lemma follows. □

In [2], the Catalan triangulations on the Möbius band with n vertices were enumerated and it was proved that any two of them can be transformed into each other by diagonal flips, but the number of diagonal flips had never been estimated yet.

Let M^2 denote the Möbius band and let ∂M^2 denote the boundary of M^2. Let K be a Catalan triangulation on M^2 with m vertices. Let $v_1, v_2, \ldots, v_m$ be the vertices of K lying on ∂M^2 in this cyclic order. An edge $v_i v_j$ is said to be *trivial* if cutting along $v_i v_j$ separates a disk D from M^2. Clearly, the subgraph of K induced by the vertices on D is a maximal outer-plane graph, which is said to be *bounded by* $v_i v_j$. Edges of K which are not trivial are said to be *essential*.

Suppose that a Catalan triangulation K on the Möbius band M^2 has no trivial edge. An essential edge e of K incident to a vertex of degree 3 is called a *spoke*. The subgraph of K induced by the essential edges which are not spokes is said to be the *zigzag frame* of K, which is uniquely taken. It is easy to see that the zigzag frame of K is a cycle of an odd length homotopic to the center line of M^2. Moreover, if K has no trivial edge and no spoke, then K is 4-regular.

Lemma 4. *Let G be a triangulation on the projective plane with $n \geqslant 7$ vertices. If G has a contractible Hamilton cycle C, then G can be transformed into $K + K_1$ by at most $n-1$ diagonal flips, where K is some Catalan triangulation on the Möbius band.*

Proof. Let G_P and G_M be the maximal outer-plane graph and the Catalan triangulation on the Möbius band, each of which is a spanning subgraph of G with boundary C.

We shall make a vertex of degree 2 in G_M by at most three diagonal flips, without breaking the simpleness of G. If G_M has a trivial edge xy, then xy bounds an outer-plane graph L. It is easy to see that L has a vertex of degree 2 other than x and y. Thus, we have nothing to do, and hence we may suppose that G_M has no trivial edge.

First, if G_M has no trivial edge and no spoke, then G_M is 4-regular. Since G_P is outer-planar, G_P has a vertex of degree 2 in G_P, say v with two neighbors p and s. Suppose that G_M has faces pqv, qrv and rsv meeting at v, and faces vrs, rts and tus meeting at s in

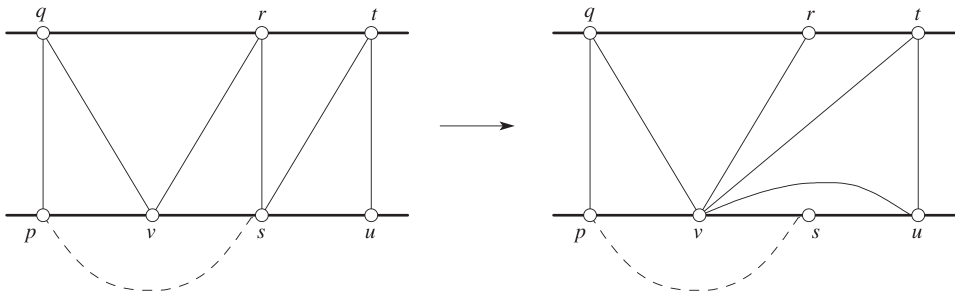

Fig. 2. Two diagonal flips making a vertex of degree 2.

G_M. (See the left-hand side of Fig. 2.) Observe that since $\deg_{G_P}(v) = 2$, any diagonal flip in G_M increasing the degree of v yields no edge forming multiple edges with an edge in G_P. Moreover, since $n \geqslant 7$, we have $vt, vu \notin E(G_M)$; otherwise, we would have $u = q$ and $p = t$. Therefore, rs can be replaced with vt, and next st can be replaced with vu. Now s has degree 2 in the resulting graph on M^2, which is obtained by two diagonal flips. (See the right-hand side of Fig. 2.)

Finally suppose that G_M has spokes but no trivial edges. We first suppose that G_M has two consecutive spokes pq and pr such that q and r are adjacent on C and $\deg_{G_M}(q) = \deg_{G_M}(r) = 3$. Let pqs, pqr and prt be three faces meeting at p. It is easy to see that a diagonal flip can replace an edge pq with sr without making multiple edges in G_M, but G_P might already have an edge sr. In this case, by the planarity of G_P, G does not have an edge qt because of the obstruction of sr. Therefore, we can make r have degree 2 by one diagonal flip.

Now consider the case when the vertices of degree 3 in G_M are independent. Since $n \geqslant 7$, the zigzag frame of G_M has length at least 5. (For otherwise, i.e., if the zigzag frame has length 3 and all vertices of degree 3 are independent, then we have $n \leqslant 6$, a contradiction.) Let pq be a spoke with $\deg_{G_M}(q) = 3$ and shared by two faces pqs and pqt. Note that $4 \leqslant \deg_{G_M}(s), \deg_{G_M}(t) \leqslant 5$. Apply a diagonal flip of pq to make a vertex of degree 2 in G_M. If impossible, G_P already has an edge st. (Here, if G is assumed to be 4-connected, then this does not happen, because $G - \{p, s, t\}$ must be connected.) If G_P has an edge st, then we can make q have degree 5 or 6 and s have degree 2 by at most three diagonal flips, flipping the edges incident to s in G_M, not on ∂M^2, to make them be incident to q, similarly to the case when G_M is 4-regular. (Note that only the final case requires at most three diagonal flips to make a vertex of degree 2 and it does not happen in the 4-connected case. Hence this proves the following remark.)

We turn our attention to G_P. Let G'_M denote a Catalan triangulation with a vertex v of degree 2 obtained from G_M by at most three diagonal flips. Then we can apply any diagonal flip in G_P increasing the degree of v, without making multiple edges with an edge of G_M. Observe that $\deg_{G_P}(v) \geqslant 3$, since every vertex of a triangulation on a closed surface has degree at least 3. Therefore, at most $n - 4$ diagonal flips can make v have degree $n - 1$ in G_p, by Lemma 3. In the resulting graph, v is adjacent to all other vertices, and the graph with v removed is obviously a Catalan triangulation with $n - 1$ vertices. □

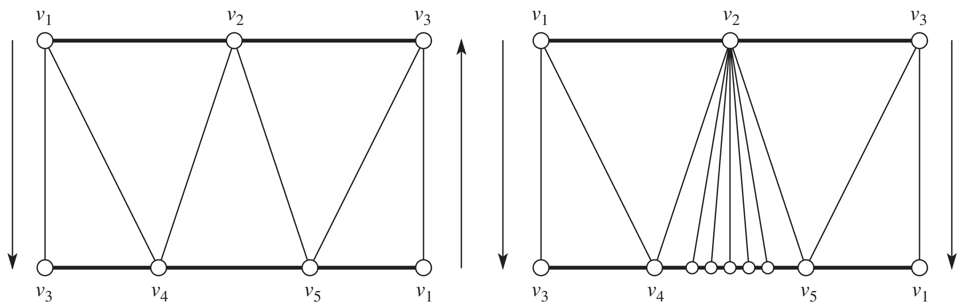

Fig. 3. K_5 and the standard form Γ_m.

As shown in the above proof, we have the following remark.

Remark 5. In Lemma 4, if we assume the 4-connectedness of G, then the number of diagonal flips can be improved to $n-2$.

Consider a Catalan triangulation on the Möbius band shown in the left-hand side of Fig. 3, which is a unique Catalan triangulation with five vertices isomorphic to K_5. Let $e = v_4v_5$ be an edge of the Catalan triangulation K_5 lying on the boundary of the Möbius band. Subdivide e by m vertices as shown in the right-hand side of Fig. 3, where the Möbius band is obtained by identifying the arrows indicated in the left-hand and the right-hand sides of the rectangles. The resulting graph is called the *standard form* of the Catalan triangulations and denoted by Γ_m.

The following is the most essential argument in this paper.

Lemma 6. *Every Catalan triangulation K on the Möbius band with n vertices can be transformed into the standard form Γ_{n-5} by at most $2n-3$ diagonal flips.*

Proof. Suppose that K has p trivial edges. Then it is easy to see that the unique sub-Catalan triangulation, denoted by K', of K with no trivial edges is obtained from K by successively removing a vertex of degree 2. Clearly, K' has exactly $n-p$ vertices.

Suppose that K' has q spokes and let $r = n-p-q$. Then the zigzag frame $v_1 \ldots v_r$, of K' has an odd length $r \geqslant 3$. Let q_i be the number of spokes of K' incident to v_i, for $i = 1, \ldots, r$. We may suppose that $q_1+q_3+\cdots+q_r \geqslant q_2+q_4+\cdots+q_{r-1}$. (For otherwise, we can replace v_i by v_{i-1} for each i, because the subscripts are cyclic and taken modulo r.) Let $q_2+q_4+\cdots+q_{r-1} = m$ and hence we have $2m \leqslant q$. (See Fig. 4.)

If $r = 3$, then by the simpleness of graphs, we have $q_2, q_3 \geqslant 1$. Hence, we can flip an edge v_2v_3 to make the zigzag frame have length 5. So, suppose that $r \geqslant 5$. Apply diagonal flips to all m spokes incident to $v_2, v_4, \ldots, v_{r-1}$ to make them trivial one by one. The number of diagonal flips we did is exactly

$$q_2+q_4+\cdots+q_{r-1} = m. \tag{1}$$

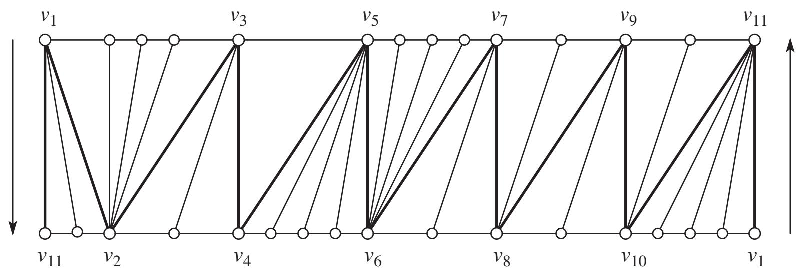

Fig. 4. K' with a zigzag frame $v_1v_2\dots v_r$.

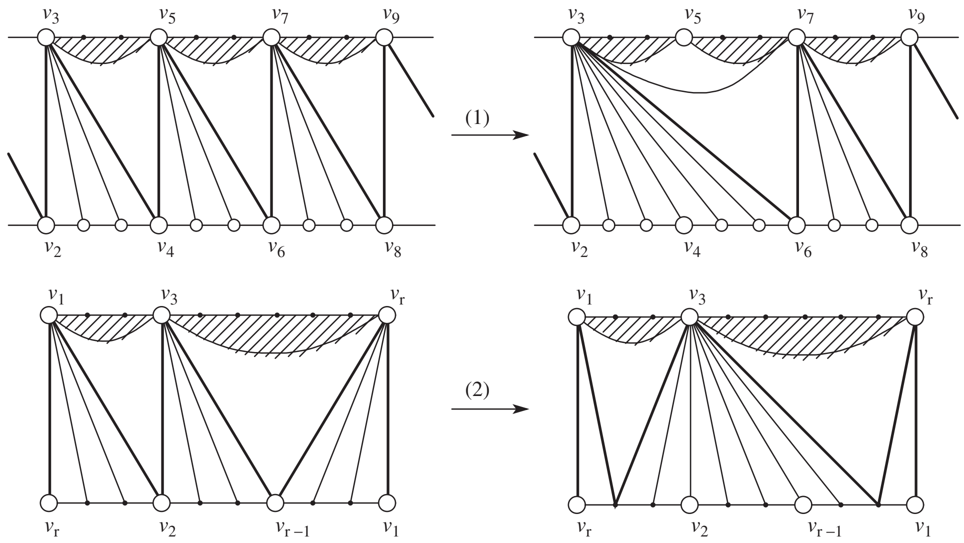

Fig. 5. Reducing the length of the zigzag frame.

Note that even if $r = 3$, the estimation (1) is true. Though we need one more diagonal flip of v_2v_3 to increase the length of the zigzag frame, this diagonal flip decreases q_2 and q_3 by one, respectively.

Next reduce the length of the zigzag frame from r to 5. In particular, we first apply a diagonal flip of v_4v_5, secondly flip q_5 spokes incident to v_5, and finally flip v_5v_6. (See Fig. 5(1).) The number of diagonal flips we did is $q_5 + 2$. In the resulting graph, the zigzag frame has length $r - 2$, and exactly one new trivial edge v_3v_7 appears. As far as that the length of the zigzag frame is at least 7, we apply these operations. If its length is exactly 5, then we apply $q_1 + q_r$ diagonal flips, as shown in Fig. 5(2). Then the total number of

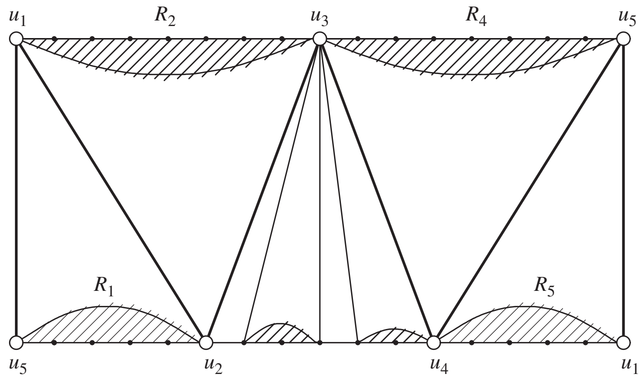

Fig. 6. The Catalan triangulation H.

diagonal flips we did is

$$(q_5 + 2) + (q_7 + 2) + \cdots + (q_{r-2} + 2) + q_1 + q_r \leqslant (q - m) + 2\left(\frac{r-5}{2}\right). \tag{2}$$

Let H' be the current Catalan triangulation obtained from K'. The zigzag frame of H' has length exactly 5, and all spokes of H' are incident to v_3. Moreover, H' has $\frac{1}{2}(r-5)+m$ trivial edges, since all m spokes incident to $v_2, v_4, \ldots, v_{r-1}$ in K' are replaced with trivial edges of H', and since decreasing the length of the zigzag frame of K' by two yields exactly one new trivial edge. Let H be the Catalan triangulation consisting of H' and all trivial edges of K. Then H has exactly $p + \frac{1}{2}(r-5) + m$ trivial edges.

Now, renaming vertices, we put H with the zigzag frame $u_1u_2u_3u_4u_5$ as shown in Fig. 6, where $u_1 = v_1$, $u_3 = v_3$ and $u_5 = v_r$. The four triangular faces $u_1u_2u_5$, $u_1u_2u_3$, $u_3u_4u_5$ and $u_4u_5u_1$ of H come from K'. Let R_i denote the outer-plane graph bounded by an edge $u_{i-1}u_{i+1}$ and containing the edge $u_{i-1}u_{i+1}$, for $i \neq 3$. (Note that R_i might be just an edge.)

The *region* F_i of the zigzag frame of H is the union of the faces bounded by the two edges $u_{i-1}u_i$, u_iu_{i+1} and the path on ∂M^2 connecting u_{i-1} and u_{i+1}, for each i, where the subscripts are taken modulo 5. Now we shall transform H into a Catalan triangulation in which all the regions of the zigzag frame, except one corresponding to F_3, consists of just one face.

Here we focus on the outer-plane graph R_2 and first suppose that $|V(R_2)| \geqslant 3$. By Lemma 3, we can make u_1 have degree $|V(R_2)| - 1$ by at most $|V(R_2)| - 3$ diagonal flips. Let $u_1, x_1, \ldots, x_l, u_3$ be the vertices of R_2 lying on ∂M^2 in this order. Apply five diagonal flips of u_1u_3, u_1u_2, u_1x_l, u_1u_5 and u_4u_5 in this order, if $l \geqslant 2$. (See Fig. 7.) If $l = 1$, then apply three diagonal flips of u_1u_3, u_1u_2, u_4u_5 in this order. In the resulting graph, each of two regions of the zigzag frame corresponding to F_2 and F_5 is just a face. The number of diagonal flips we did is at most $|V(R_2)| - 3 + 5$.

Secondly we suppose that $|V(R_2)| = 2$. If we also have $|V(R_5)| = 2$, then we have nothing to do for F_2 and F_5. So, suppose that $|V(R_5)| \geqslant 3$. Similarly to the above case, at most $|V(R_5)| - 3$ diagonal flips make u_4 have degree $|V(R_5)| - 1$ in R_5 and we apply two

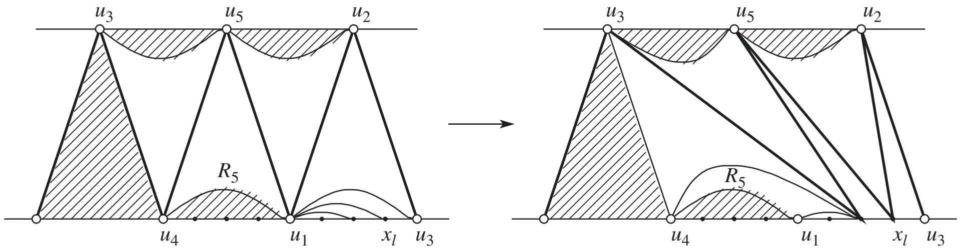

Fig. 7. Moving vertices of R_4 and R_5.

diagonal flips of u_1u_4 and u_4u_5. In the resulting graph, the two regions corresponding to F_2 and F_5 are just faces. Then the number of diagonal flips we did is at most $|V(R_5)|-3+2$. Note that by the above operations, the number of trivial edges decreases by one, if $|V(R_2)|\geqslant 3$ or $|V(R_5)|\geqslant 3$.

We can do the same procedures for the regions R_1 and R_4. Let L denote the resulting graph in which exactly four regions are just faces. Hence, the number of diagonal flips transforming H into L is at most

$$\max\{|V(R_2)|+2, |V(R_5)|-1\}+\max\{|V(R_4)|+2, |V(R_1)|-1\} \leqslant p+\tfrac{1}{2}(r-5)+m+8, \tag{3}$$

since

$$(|V(R_1)|-2)+(|V(R_2)|-2)+(|V(R_4)|-2)+(|V(R_5)|-2) \leqslant p+\tfrac{1}{2}(r-5)+m.$$

Note that we can assume that the number of trivial edges of L is at most $p+\frac{1}{2}(r-5)+m-1$, since we may suppose that at least one of R_1, R_2, R_4 and R_5 has at least three vertices. (For otherwise, we do not need to add (3) to the estimation of the maximum number of diagonal flips, and this case requires a few number of diagonal flips.)

Finally we flip all trivial edges of L, all of which are incident to u_3. Since the number of trivial edges of L is at most $p+\frac{1}{2}(r-5)+m-1$, the number of diagonal flips transforming L into the standard form is at most

$$p+\frac{1}{2}(r-5)+m-1=p+\frac{r}{2}+m-\frac{7}{2}. \tag{4}$$

Therefore, by (1)–(4), the total number of diagonal flips is at most

$$m+(q-m+r-5)+\left(p+\frac{r}{2}+m+\frac{11}{2}\right)+\left(p+\frac{r}{2}+m-\frac{7}{2}\right) = 2p+q+2m+2r-3\leqslant 2(p+q+r)-3=2n-3,$$

since $q\geqslant 2m$. Therefore, the lemma follows. □

3. Triangulations with contractible Hamilton cycles

In the previous section, we described only the result on triangulations with contractible Hamilton cycles. In this section, we shall mention how we can obtain triangulations with contractible Hamilton cycles from any triangulations.

The following gives an important sufficient condition for a graph on the projective plane to have a contractible Hamilton cycle.

Lemma 7 (*Thomas and Yu [8]*). *Every* 4*-connected graph on the projective plane has a contractible Hamilton cycle*.

The following lemma is essential.

Lemma 8. *Let G be a triangulation on the projective plane with n vertices. Then G can be transformed into a* 4*-connected triangulation by at most $n-6$ diagonal flips*.

Proof. Observe that a triangulation on the projective plane has no separating 3-cycle if and only if it is 4-connected. We first show that G has at most $n-6$ separating 3-cycles, by induction on n. It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of K_6, which has no separating 3-cycle. Therefore, the lemma holds when $n=6$.

When $n \geqslant 7$, we may assume that G has a separating 3-cycle $C = xyz$, and it is *innermost*, that is, there is no separating 3-cycle in the 2-cell bounded by C. Cutting along C, we can decompose G into a plane triangulation G_1 with no separating 3-cycle and a triangulation G_2 on the projective plane. By induction hypothesis, G_2 has at most $|V(G_2)|-6$ separating 3-cycles. Let M denote the number of separating 3-cycles in G. Then we have

$$M \leqslant |V(G_2)| - 6 + 1 = (n - |V(G_1)| + 3) - 5 \leqslant n - 6,$$

since $|V(G_1)| \geqslant 4$.

Now we shall show that there is a diagonal flip decreasing the number of separating 3-cycles by at least one. Let $C = xyz$ be a separating 3-cycle in G and $e = xy$. Let $xayb$ be the quadrilateral formed by two triangular faces sharing e, where a lies in the 2-cell region bounded by C. Consider the diagonal flip of e replacing xy with ab. In the resulting graph G', the separating cycle C in G has disappeared.

We shall show that no new separating 3-cycle arises in G', by possibly re-choosing e. Suppose that G' has a new separating 3-cycle C'. Then C' contains both a and b; otherwise, C' would be contained in G. We must have $C' = abz$, where we assume that x is contained in the 2-cell region bounded by C' in G'. This means that $V(G_1) = \{x, y, z, a\}$ since C is innermost in G. In this case, the edge yz can be flipped to destroy a 3-cycle byz and make no new separating 3-cycle, because byz separates a and other vertices outside byz. Therefore, at most $n-6$ diagonal flips can make the graph be 4-connected. □

4. Proof of theorems

It is well-known that the smallest triangulation on the projective plane is the unique triangular embedding of K_6. Let xy be one of its edges, and suppose that two faces xyz

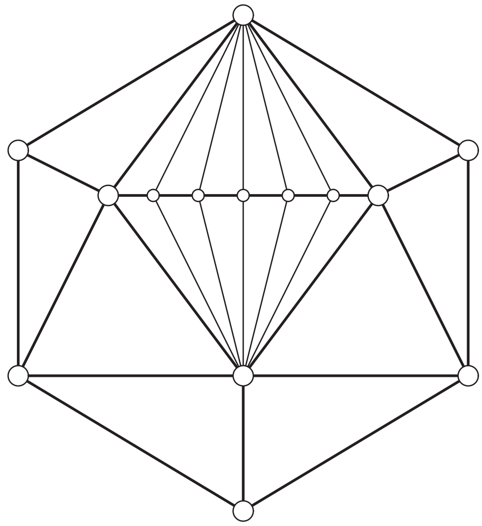

Fig. 8. The standard form Ψ_m of triangulations on the projective plane.

and xyw share xy. Subdivide xy by m vertices $v_1, \ldots, v_m$ and add $2m$ edges $v_i z$, $v_i w$ for $i = 1, \ldots, m$. The resulting graph with $m + 6$ vertices is called the *standard form* of triangulations on the projective plane and denoted by Ψ_m. (See Fig. 8.) Clearly, we obtain the standard form Ψ_m from the standard form Σ_{m-1} of Catalan triangulations of the Möbius band M^2 by pasting a disk along ∂M^2, placing a vertex v at its center and joining v to all vertices of Σ_m.

We first prove the following theorem.

Theorem 9. *Let G be a triangulation on the projective plane with n vertices which has a contractible Hamilton cycle. Then G can be transformed into* Ψ_{n-6}, *preserving the Hamilton cycle*, *by at most* $3n - 6$ *diagonal flips. If G is* 4-*connected*, *then the number of diagonal flips is improved to* $3n - 7$.

Proof. We may suppose that $n \geqslant 7$. By Lemma 4, G can be transformed into $K + K_1$ by at most $n - 1$ diagonal flips, preserving the Hamilton cycle, where K is some Catalan triangulation on the Möbius band with $n - 1$ vertices. (By Remark 5, if G is 4-connected, the number "$n - 1$" of diagonal flips can be replaced with "$n - 2$".)

Note that no two vertices of K are joined by an edge outside K. Therefore, it suffices to prove that K can be transformed into Σ_{n-6}. By Lemma 6, it can be done by at most $2(n - 1) - 3$ diagonal flips. Therefore, G can be transformed into Ψ_{n-6} by at most $3n - 6$ ($3n - 7$ when G is 4-connected) diagonal flips, preserving the Hamilton cycle. □

Theorem 10. *Every triangulation on the projective plane with n vertices can be transformed into the standard form* Ψ_{n-6} *by at most* $4n - 13$ *diagonal flips*, *up to isotopy*.

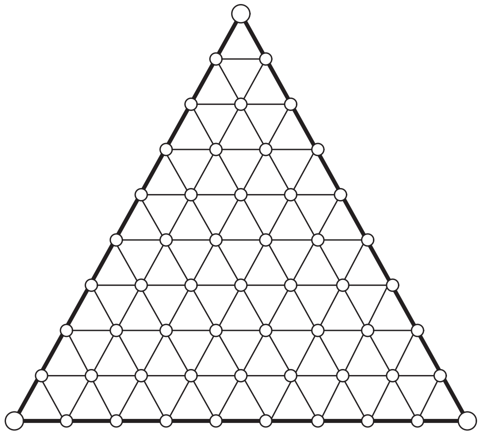

Fig. 9. A triangular mesh.

Proof. Let G be a triangulation on the projective plane with n vertices. By Lemma 8, at most $n-6$ diagonal flips transform G into a 4-connected triangulation, denoted by H. By Lemma 7, H has a contractible Hamilton cycle. Then apply Theorem 9. □

Now we shall prove Theorems 1 and 2.

Proof of Theorems 1 and 2. Theorems 1 and 2 follow from Theorems 10 and 9, respectively, via the standard form Ψ_{n-6}. □

Finally, we consider two triangulations G_1 and G_2 on the projective plane with n vertices which need many diagonal flips to transform them into each other. Let $G_1 = \Psi_{n-6}$, and let G_2 be a triangulation with maximum degree 6. For example, it is constructed from K_6 by putting a *triangular mesh* shown in Fig. 9 to each face.

The maximum degree of G_1 is $n-1$ and it is attained by two vertices, say x and y. To transform G_1 into G_2, we have to decrease the degree of x and y to six or five. Since each diagonal flip decreases the degree of a fixed vertex at most by one, each of x and y requires at least $(n-1)-6$ diagonal flips. Observe that the degree of x and y decrease simultaneously by one diagonal flip, only if this diagonal flip is applied to the edge xy. If such diagonal flips are applied at least twice in the process from G_1 to G_2, then there must be a diagonal flip joining x and y, which increases $\deg(x)+\deg(y)$. Therefore, if the number $\deg(x)+\deg(y)$ is non-increasing in the process from G_1 to G_2, then the edge xy is flipped at most once, and the number of diagonal flips transforming G_1 to G_2 is at least

$$(n-1)-6+(n-1)-6-1 = 2n-13.$$

Therefore, the order of our estimation in Theorems 2 and 1 cannot be improved.

Proposition 11. *For any natural number N, there exists a pair of triangulations G_1 and G_2 on the projective plane with $n \geqslant N$ vertices such that at least $2n-13$ diagonal flips are needed to transform them into each other.*

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