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Interior Spikes of a Singularly Perturbed Neumann Problem with Potentials

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Abstract—In this paper, we prove that a singularly perturbed Neumann problem with potentials admits the existence of interior spikes concentrating in maxima and minima of an auxiliary functional depending only on the potentials.

Keywords—Interior spikes, Singularly perturbed Neumann problem.

1. INTRODUCTION

In this paper, we study the existence of interior spikes of the following problem:

$$\begin{aligned} -\varepsilon^2 \mathrm{div}\,(J(x)\nabla u) + V(x)u = u^p, &\qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, &\qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ with external normal ν, $N \geq 3$, $1 < p < (N+2)/(N-2)$, $J\colon \mathbb{R}^N \to \mathbb{R}$, and $V\colon \mathbb{R}^N \to \mathbb{R}$ are C^2 functions.

In [1], the first author, extending the classical results by Ni and Takagi in [2,3], proved that there exist solutions of (1) that concentrate at maximum and minimum points of a suitable auxiliary function defined on the boundary $\partial\Omega$ and depending only on J and V. Here we study the existence of solutions which concentrate in the interior of Ω and we will show that the concentration occurs at maximum and minimum points of the same auxiliary function introduced in [1], but now defined in Ω. We assume that the reader has familiarity with [1].

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When $J \equiv 1$ and $V \equiv 1$, interior spikes have been found by Wei (see [4]) showing that concentration occurs at local maxima of the distance function $\operatorname{dist}(\cdot, \partial\Omega)$.

On J and V, we will do the following assumptions.

(J) $J \in C^2(\Omega, \mathbb{R})$, J and D^2J are bounded; moreover, $J(x) \geq C > 0$, for all $x \in \Omega$;
(V) $V \in C^2(\Omega, \mathbb{R})$, V and D^2V are bounded; moreover, $V(x) \geq C > 0$, for all $x \in \Omega$.

Let us introduce an auxiliary function which will play a crucial role in the study of (1). Let $\Gamma: \Omega \to \mathbb{R}$ be the function defined by

$$\Gamma(Q) = V(Q)^{(p+1)/(p-1)-N/2} J(Q)^{N/2}. \tag{2}$$

Let us observe that by (J) and (V), Γ is well defined. Our main result is the following theorem.

THEOREM 1.1. *Let $Q_0 \in \Omega$. Suppose (J) and (V) hold. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1) possesses a solution u_ε concentrating at Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, provided that one of the two following conditions holds.*

(a) *Q_0 is a nondegenerate critical point of Γ.*
(b) *Q_0 is an isolated local strict minimum or maximum of Γ.*

Notation

- If $u: \mathbb{R}^N \to \mathbb{R}$ and $P \in \mathbb{R}^N$, we set $u_P \equiv u(\cdot - P)$.
- If U^Q is the function defined in (5), when there is no misunderstanding, we will often write U instead of U^Q. Moreover, if $P = Q/\varepsilon$, then $U_P \equiv U^Q(\cdot - P)$.
- If $\varepsilon > 0$, we set $\Omega_\varepsilon \equiv \Omega/\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$.
- With $o_\varepsilon(1)$, we denote a quantity which tends to zero as $\varepsilon \to 0$.

2. PRELIMINARY LEMMAS AND SOME ESTIMATES

First of all, we perform the change of variables $x \mapsto \varepsilon x$ and so problem (1) becomes

$$\begin{aligned} -\operatorname{div}(J(\varepsilon x)\nabla u) + V(\varepsilon x)u = u^p, &\qquad \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, &\qquad \text{on } \partial\Omega_\varepsilon, \end{aligned} \tag{3}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Of course, if u is a solution of (3), then $u(\cdot/\varepsilon)$ is a solution of (1).

Solutions of (3) are critical points $u \in H^1(\Omega_\varepsilon)$ of

$$f_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla u|^2\,dx + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)u^2\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1}.$$

We look for solutions of (3) near a U^Q, the unique solution of the *limiting problem*

$$\begin{aligned} -J(Q)\,\Delta u + V(Q)\,u = u^p, &\qquad \text{in } \mathbb{R}^N, \\ u > 0, &\qquad \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u, & \end{aligned} \tag{4}$$

for an appropriate choice of $Q \in \Omega$. It is easy to see that

$$U^Q(x) = V(Q)^{1/p-1}\,\bar{U}\left(x\sqrt{\frac{V(Q)}{J(Q)}}\right), \tag{5}$$

where $\bar{U}$ is the unique solution of

$$\begin{aligned} -\Delta\bar{U} + \bar{U} &= \bar{U}^p, && \text{in } \mathbb{R}^N, \\ \bar{U} &> 0, && \text{in } \mathbb{R}^N, \\ \bar{U}(0) &= \max_{\mathbb{R}^N} \bar{U}, \end{aligned}$$

which is radially symmetric and decays exponentially at infinity together with its first derivatives.

For the sake of brevity, we will often write U instead of U^Q. If $P = \varepsilon^{-1}Q \in \Omega_\varepsilon$, we set $U_P \equiv U^Q(\cdot - P)$ and

$$Z^\varepsilon \equiv \{U_P : P \in \Omega_\varepsilon\}.$$

LEMMA 2.1. *For all $Q \in \Omega$ and for all ε sufficiently small, if $P = Q/\varepsilon \in \Omega_\varepsilon$, then*

$$\|\nabla f_\varepsilon(U_P)\| = O(\varepsilon). \tag{6}$$

PROOF. Repeating the calculations of [1], we get

$$\begin{aligned} (\nabla f_\varepsilon(U_P) \mid v) = &\int_{(\Omega-Q)/\varepsilon} [-J(Q)\,\Delta U + V(Q)\,U - U^p]\,v_{-P} + J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial\nu} v \\ &+ \int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q))\,\nabla U\cdot\nabla v_{-P} + \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q))\,U v_{-P}. \end{aligned}$$

Hence, since $U \equiv U^Q$ is a solution of (4), we get

$$\begin{aligned} (\nabla f_\varepsilon(U_P) \mid v) = J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial\nu} v &+ \int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q))\,\nabla U\cdot\nabla v_{-P} \\ &+ \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q))\,U v_{-P}. \end{aligned} \tag{7}$$

Let us estimate the first of these three terms

$$\left| J(Q)\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial\nu} v\right| \le C\|v\|_{L^2(\partial\Omega_\varepsilon)} \left(\int_{\partial\Omega_\varepsilon} \left|\frac{\partial U_P}{\partial\nu}\right|^2\right)^{1/2}.$$

First of all, we observe that there exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $v \in H^1(\Omega_\varepsilon)$, we have

$$\|v\|_{L^2(\partial\Omega_\varepsilon)} \le C\|v\|_{H^1(\Omega_\varepsilon)}.$$

Using the exponential decay of U and its derivatives, it is easy to see that

$$\int_{\partial\Omega_\varepsilon} \left|\frac{\partial U_P}{\partial\nu}\right|^2 = \int_{\partial((\Omega-Q)/\varepsilon)} \left|\frac{\partial U}{\partial\nu}\right|^2 = o(\varepsilon). \tag{8}$$

Arguing as in [1], one can prove that

$$\int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q))\,\nabla U\cdot\nabla v_{-P} + \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q))\,U v_{-P} = O(\varepsilon)\|v\|. \tag{9}$$

Now the conclusion follows immediately from (7)–(9). ∎

We here present some useful estimates that will be used in the sequel.

PROPOSITION 2.2. *Let $P = Q/\varepsilon \in \Omega_\varepsilon$. Then, we have*

$$\int_{\Omega_\varepsilon} U_P^{p+1} = \int_{\mathbb{R}^N} \left(U^Q\right)^{p+1} + o(\varepsilon),$$

$$\int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} U_P = o(\varepsilon),$$

$$J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + V(Q) \int_{\Omega_\varepsilon} U_P^2 = \int_{\mathbb{R}^N} \left(U^Q\right)^{p+1} + o(\varepsilon),$$

$$\int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla U_P|^2 = J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \varepsilon \int_{\mathbb{R}^N} J'(Q)[x] \left|\nabla U^Q\right|^2 + o(\varepsilon), \tag{10}$$

$$\int_{\Omega_\varepsilon} V(\varepsilon x) U_P^2 = V(Q) \int_{\Omega_\varepsilon} U_P^2 + \varepsilon \int_{\mathbb{R}^N} V'(Q)[x] \left(U^Q\right)^2 + o(\varepsilon). \tag{11}$$

PROOF. Let us prove the first formula. We have

$$\int_{\Omega_\varepsilon} U_P^{p+1} = \int_{(\Omega - Q)/\varepsilon} \left(U^Q\right)^{p+1} = \int_{\mathbb{R}^N} \left(U^Q\right)^{p+1} - \int_{\mathbb{R}^N \setminus (\Omega - Q)/\varepsilon} \left(U^Q\right)^{p+1}.$$

Using the exponential decay of U^Q, it is easy to see that

$$\int_{\mathbb{R}^N \setminus (\Omega - Q)/\varepsilon} \left(U^Q\right)^{p+1} \le \int_{\mathbb{R}^N \setminus B_{1/\sqrt{\varepsilon}}} \left(U^Q\right)^{p+1} = C \int_{1/\sqrt{\varepsilon}}^{\infty} r^{N-1} \left(U^Q(r)\right)^{p+1} dr = o(\varepsilon).$$

Using the exponential decay of U^Q, also the second formula can be proved in a similar way. The other equations can be proved as in [1]. ∎

3. THE FINITE-DIMENSIONAL REDUCTION

In this section, we perform a finite-dimensional reduction, following some ideas introduced in [5]. The symbol $T_{U_P} Z^\varepsilon$ denotes the tangent space to Z^ε at U_P. Let $L_{\varepsilon,Q} : (T_{U_P} Z^\varepsilon)^\perp \to (T_{U_P} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon,Q} v \mid w) = D^2 f_\varepsilon(U_P)[v, w]$.

LEMMA 3.1. *There exists $C > 0$ such that for ε small enough, one has that*

$$\|L_{\varepsilon,Q} v\| \ge C \|v\|, \qquad \forall\, v \in (T_{U_P} Z^\varepsilon)^\perp. \tag{12}$$

PROOF. The proof of (12) is completely analogous to that of equation (21) in [1], so we omit the details. ∎

LEMMA 3.2. *For $\varepsilon > 0$ small enough, there exists a unique $w = w(\varepsilon, Q) \in (T_{U_P} Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(U_P + w) \in T_{U_P} Z$. Such a $w(\varepsilon, Q)$ is of class C^2, respectively, $C^{1,p-1}$, with respect to Q, provided that $p \ge 2$, respectively, $1 < p < 2$. Moreover, the functional $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_{Q/\varepsilon} + w(\varepsilon, Q))$ has the same regularity of w and satisfies*

$$\nabla \mathcal{A}_\varepsilon(Q_0) = 0 \iff \nabla f_\varepsilon\left(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0)\right) = 0.$$

PROOF. Let $\mathcal{P} = \mathcal{P}_{\varepsilon,Q}$ denote the projection onto $(T_{U_P} Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{U_P} Z^\varepsilon)^\perp$ of the equation $\mathcal{P} \nabla f_\varepsilon(U_P + w) = 0$. One has that $\nabla f_\varepsilon(U_P + w) = \nabla f_\varepsilon(U_P) + D^2 f_\varepsilon(U_P)[w] + R(U_P, w)$ with $\|R(U_P, w)\| = o(\|w\|)$, uniformly with respect to U_P. Therefore, our equation is

$$L_{\varepsilon,Q} w + \mathcal{P} \nabla f_\varepsilon(U_P) + \mathcal{P} R(U_P, w) = 0.$$

According to Lemma 3.1, this is equivalent to

$$w = N_{\varepsilon,Q}(w), \qquad \text{where } N_{\varepsilon,Q}(w) = -\left(L_{\varepsilon,Q}\right)^{-1} \left(\mathcal{P} \nabla f_\varepsilon(U_P) + \mathcal{P} R(U_P, w)\right).$$

By (6), it follows that

$$\|N_{\varepsilon,Q}(w)\| = O(\varepsilon) + o(\|w\|). \tag{13}$$

Then, one readily checks that $N_{\varepsilon,Q}$ is a contraction on some ball in $(T_{U_P} Z^\varepsilon)^\perp$ provided that $\varepsilon > 0$ is small enough. Then, there exists a unique w such that $w = N_{\varepsilon,Q}(w)$. Given $\varepsilon > 0$ small, we can apply the implicit function theorem to the map $(Q, w) \mapsto \mathcal{P}\nabla f_\varepsilon(U_P + w)$. Then, in particular, the function $w(\varepsilon, Q)$ turns out to be of class C^1 with respect to Q. Finally, it is a standard argument, see [5], to check that the critical points of $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w)$ give rise to critical points of f_ε. ∎

REMARK 3.3. From (13), it follows immediately that

$$\|w\| = O(\varepsilon). \tag{14}$$

Moreover, repeating the arguments of [1], if $\gamma = \min\{1, p-1\}$, then, for $i = 1, \dots, N$, we infer that

$$\|\partial_{P_i} w\| = O(\varepsilon^\gamma).$$

4. THE FINITE-DIMENSIONAL FUNCTIONAL

THEOREM 4.1. *Let $Q \in \Omega$ and $P = Q/\varepsilon \in \Omega_\varepsilon$. Suppose (J) and (V). Then, for ε sufficiently small, we get*

$$\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w(\varepsilon, Q)) = c_0 \Gamma(Q) + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} J'(Q)[x]|\nabla U|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V'(Q)[x] U^2 + o(\varepsilon),$$

where Γ is the auxiliary function introduced in (2) and

$$c_0 \equiv \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} \bar{U}^{p+1}.$$

Moreover, for all $i = 1, \dots, N$, we get

$$\partial_{P_i} \mathcal{A}_\varepsilon(Q) = \varepsilon c_0 \partial_{Q_i} \Gamma(Q) + o(\varepsilon), \tag{15}$$

and hence,

$$\|\mathcal{A}_\varepsilon - c_0\Gamma\|_{C^1(\Omega)} = O(\varepsilon). \tag{16}$$

PROOF. In the sequel, to be brief, we will often write w instead of $w(\varepsilon, Q)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

First of all, reasoning as in the proofs of (10) and (11) and by (14), we can observe that

$$\int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w = J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + o(\varepsilon), \tag{17}$$

$$\int_{\Omega_\varepsilon} V(\varepsilon x) U_P\, w = V(Q)\int_{\Omega_\varepsilon} U_P\, w + o(\varepsilon). \tag{18}$$

Recalling (14), we have

$$\mathcal{A}_\varepsilon(Q) = \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)\,|\nabla(U_P + w)|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x)\,(U_P + w)^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} (U_P + w)^{p+1}$$

$$= \frac{1}{2}\int_{\Omega_\varepsilon} J(\varepsilon x)|\nabla U_P|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} V(\varepsilon x) U_P^2 - \frac{1}{2}\int_{\Omega_\varepsilon} U_P^{p+1} + \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla w + \int_{\Omega_\varepsilon} V(\varepsilon x) U_P\, w$$

$$-\int_{\Omega_\varepsilon} U_P^p w + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} U_P^{p+1} - \frac{1}{p+1}\int_{\Omega_\varepsilon}\left[(U_P + w)^{p+1} - U_P^{p+1} - (p+1)U_P^p\, w\right] + o(\varepsilon)$$

(by Proposition 2.2, (17), and (18), and with our notations)

$$= \frac{1}{2}\int_{\mathbb{R}^N} U^{p+1} + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} J'(Q)[x]|\nabla U|^2 + \frac{\varepsilon}{2}\int_{\mathbb{R}^N} V'(Q)[x] U^2 - \frac{1}{2}\int_{\mathbb{R}^N} U^{p+1}$$

$$+ J(Q)\int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q)\int_{\Omega_\varepsilon} U_P\, w - \int_{\Omega_\varepsilon} U_P^p\, w + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^N} U^{p+1} + o(\varepsilon).$$

Since that U is solution of (4), we infer

$$J(Q)\int_{\Omega_\varepsilon}\nabla U_P\cdot\nabla w+V(Q)\int_{\Omega_\varepsilon}U_P\,w-\int_{\Omega_\varepsilon}U_P^p\,w$$

$$=\int_{\Omega_\varepsilon}\left[-J(Q)\,\Delta U_P+V(Q)\,U_P-U_P^p\right]w+J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}w=J(Q)\int_{\partial\Omega_\varepsilon}\frac{\partial U_P}{\partial\nu}w=o(\varepsilon).$$

By these considerations, we can say that

$$\mathcal{A}_\varepsilon(Q)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^N}U^{p+1}+\frac{\varepsilon}{2}\int_{\mathbb{R}^N}J'(Q)\,[x]|\nabla U|^2+\frac{\varepsilon}{2}\int_{\mathbb{R}^N}V'(Q)\,[x]U^2+o(\varepsilon).$$

Now the conclusion of the first part of the theorem follows observing that, since by (5),

$$U^Q(x)=V(Q)^{1/(p-1)}\,\bar{U}\left(x\sqrt{\frac{V(Q)}{J(Q)}}\right),$$

then

$$\int_{\mathbb{R}^N}U^{p+1}=V(Q)^{(p+1)/(p-1)-N/2}\,J(Q)^{N/2}\int_{\mathbb{R}^N}\bar{U}^{p+1}.$$

Estimate (15) on the derivatives of $\mathcal{A}_\varepsilon$ follows easily by repeating the arguments of [1]. ∎

5. PROOF OF THEOREM 1.1

In this section, we will state and prove two multiplicity results for (1). Theorem 1.1 will follow as a particular case.

THEOREM 5.1. *Let (J) and (V) hold and suppose Γ has a nondegenerate smooth manifold of critical points $M\subset\Omega$. There exists $\varepsilon_0>0$ such that if $0<\varepsilon<\varepsilon_0$, then (1) has at least $l(M)$ solutions that concentrate near points of M. Here $l(M)$ denotes the cup long of M (for a precise definition, see [6]).*

PROOF. Fix a δ-neighborhood M_δ of M such that the only critical points of Γ in M_δ are those in M. We will take $U=M_\delta$. For ε sufficiently small, by (16) and [6, Chapter II, Theorem 6.4], $\mathcal{A}_\varepsilon$ possesses at least $l(M)$ critical points, which are solutions of (3) by Lemma 3.2. Let $Q_\varepsilon\in M$ be one of these critical points, then $u_\varepsilon^{Q_\varepsilon}=U_{Q_\varepsilon/\varepsilon}+w(\varepsilon,Q_\varepsilon)$ is a solution of (3). Therefore,

$$u_\varepsilon^{Q_\varepsilon}\left(\frac{x}{\varepsilon}\right)\simeq U_{Q_\varepsilon/\varepsilon}\left(\frac{x}{\varepsilon}\right)=U^{Q_\varepsilon}\left(\frac{x-Q_\varepsilon}{\varepsilon}\right)$$

is a solution of (1). ∎

Moreover, when we deal with local minima (respectively, maxima) of Γ, the preceding results can be improved because the number of positive solutions of (1) can be estimated by means of the category and M does not need to be a manifold. We will give only the statement of the theorem; for the proof, see [1].

THEOREM 5.2. *Let (J) and (V) hold and suppose Γ has a compact set $X\subset\Omega$ where Γ achieves a strict local minimum (respectively, maximum), in the sense that there exist $\delta>0$ and a δ-neighborhood $X_\delta\subset\Omega$ of X such that*

$$b\equiv\inf\{\Gamma(Q):Q\in\partial X_\delta\}>a\equiv\Gamma_{|X},\qquad(\text{respectively, }\sup\{\Gamma(Q):Q\in\partial X_\delta\}<\Gamma_{|X}).$$

Then, there exists $\varepsilon_0>0$ such that (1) has at least cat (X,X_δ) *solutions that concentrate near points of X_δ, provided $\varepsilon\in(0,\varepsilon_0)$. Here* cat (X,X_δ) *denotes the Lusternik-Schnirelman category of X with respect to X_δ.*

REMARK 5.3. Let us observe that Part (a) of Theorem 1.1 is a particular case of Theorem 5.1, while Part (b) of Theorem 1.1 is a particular case of Theorem 5.2.

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