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On the Unbounded Behaviour for Some Non-autonomous Systems in Banach Spaces

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By modifying our previous methods (1992, *J. Nonlinear Anal. TMA* **19**, 741–751; 1993, *Proc. Amer. Math. Soc.* **117**, 951–956), and by using the notion of integral solution introduced by Ph. Bénilan (1972, "Equations d'évolution dans un espace de Banach quelconque et applications," thesis, Université Paris XI, Orsay), we study the asymptotic behaviour of unbounded trajectories for the quasi-autonomous dissipative system $du/dt + Au \ni f$, where X is a real Banach space, A an accretive (possibly multivalued) operator in $X \times X$, and $f - f_\infty \in L^p((0, +\infty); X)$ for some $f_\infty \in X$ and $1 \leqslant p < \infty$.

1. INTRODUCTION

Let X be a real Banach space; the norms of both X and X^* are denoted by $\| \ \|$. We denote weak convergence in X by $\rightharpoonup$ and strong convergence by $\to$.

For notions concerning the geometry of Banach spaces, m-accretive operators, nonlinear semi-groups generated by them, weak and integral solutions, etc., we refer to [2–7, 13, 14, 19], recalling only the basic facts.

We studied in [8, 9] the asymptotic behaviour of bounded weak solutions (if any) for the quasi-autonomous dissipative system

$$\begin{aligned} &\frac{du}{dt} + Au \ni f \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

where A is a monotone operator in a real Hilbert space H, $u_0 \in H$, and $f - f_\infty \in L^1((0, +\infty); H)$ for some $f_\infty \in H$, thus giving simple proofs for theorems containing results of Baillon, Brézis, Browder, Bruck, Pazy, and Reich (see [8, 9, 14, 15, 18, 19, 21] for appropriate references).

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In our setting, this study seems to be still an open problem when the Hilbert space H is replaced by a Banach space X. The asymptotic behaviour of unbounded orbits for non-expansive mappings in H was first studied by Pazy [20]. Extensions of his result to more general Banach spaces and to nonlinear contraction semi-groups were subsequently studied by many authors, e.g., Kobayasi [16], Kohlberg and Neyman [17], Plant and Reich [22], Hulbert and Reich [15], and Reich [23–25].

In [10] we gave a simple short proof of the generalization of Pazy's result to non-expansive sequences in a Hilbert space H, and in [12] we extended our result to the case of a Banach space X, thus containing the previous results mentioned above.

In [11] we studied the asymptotic behaviour of unbounded weak solutions (if any) for the system (1.1), where A is a monotone operator in H, $u_0 \in H$, and $f - f_\infty \in L^p((0, +\infty); H)$ for some $f_\infty \in H$ and $1 \leqslant p < \infty$.

When A is maximal monotone in H, this was studied by M. G. Crandall (unpublished result; see Brézis [4], Pazy [21], Reich [24]) using the contraction semi-group generated on $\overline{D(A)}$ by $-A$.

In this paper, by modifying our previous methods in [11, 12] and by using the notion of integral solution for (1.1) introduced by Bénilan [3], we extend our results in [11] to the case of a Banach space X, where A is an accretive (possibly multivalued) operator in $X \times X$.

2. Preliminaries

DEFINITION 2.1. A curve $(u(t))_{t \geqslant 0}$ in X is a continuous function u from $[0, +\infty[$ into X (i.e., $u \in C([0, +\infty[; X)$).

DEFINITION 2.2. (a) A sequence $(x_n)_{n \geqslant 0}$ in X is almost non-expansive if $\forall i, j \geqslant 0$, $\|x_{i+1} - x_{j+1}\| \leqslant \|x_i - x_j\| + \varepsilon(i, j)$, where $(\varepsilon(i, j))_{i,j \geqslant 0}$ is bounded and $\lim_{i,j \to +\infty} \varepsilon(i, j) = 0$.

(b) A curve $(u(t))_{t \geqslant 0}$ in X is almost non-expansive if $\exists \delta > 0$ such that $\forall s, t \geqslant 0$, $\forall h \in [0, \delta]$, $\|u(t+h) - u(s+h)\| \leqslant \|u(t) - u(s)\| + \varepsilon(s, t)$, where $(\varepsilon(s, t))_{s,t \geqslant 0}$ is bounded and $\lim_{s,t \to +\infty} \varepsilon(s, t) = 0$. (Note that in this definition $h \in [0, \delta]$ may be replaced by h belonging to any bounded interval $[0, N]$.)

Notation 2.3. If A is a subset of X, we denote by clco A the closed convex hull of A in X and for $x \in X$, $d(x, A) = \inf_{z \in A} \|x - z\|$.

Given a sequence $(x_n)_{n \geqslant 0}$ in X, we denote $C = \bigcap_{n=1}^{\infty} \operatorname{clco}\{(x_i - x_{i-1})_{i \geqslant n}\}$. If $\|x_n/n\|$ is bounded and X is reflexive, then $C \neq \varnothing$.

Now we recall some basic facts about the geometry of Banach spaces, accretive operators, and integral solutions for (1.1).

(a) The duality map J from X into the family of non-empty closed convex subsets of X^* is defined by

$$\forall x \in X, \qquad Jx = \{x^* \in X^* / \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle x^*, x\rangle$ denotes the value of x^* at x. Note also that we have $\forall x, y \in X$, $\forall j \in Jx$, $\langle j, x-y\rangle = \|x\|^2 - \langle j, y\rangle \geqslant \frac{1}{2}(\|x\|^2 - \|y\|^2)$.

(b) If X is reflexive and strictly convex and K a non-empty closed convex subset of X, the nearest point projection map P_K of X onto K is well defined, i.e., K is a Chebyshev set (see [14]).

(c) The norm of X is Fréchet differentiable if for each $x \in S = \{z \in X / \|z\| = 1\}$, $\lim_{t \to 0}((\|x+ty\| - \|x\|)/t)$ exists uniformly for $y \in S$.

We recall that the dual space X^* has Fréchet differentiable norm if and only if X is reflexive and strictly convex and satisfies the property

$$\text{if} \quad x_n \xrightarrow[n \to +\infty]{} x$$

and

$$\|x_n\| \xrightarrow[n \to +\infty]{} \|x\| \quad \text{then} \quad \|x_n - x\| \xrightarrow[n \to +\infty]{} 0 \qquad \text{(see [13]).}$$

(d) A subset A of $X \times X$ with domain $D(A)$ and range $R(A)$ is called an accretive (possibly multivalued) operator in X if $\|x_1 - x_2\| \leqslant \|x_1 - x_2 + \lambda(y_1 - y_2)\|$ for all $(x_i, y_i) \in A$, $i = 1, 2$, and $\lambda > 0$. This is equivalent to: $\exists f \in J(x_1 - x_2)$ such that $\langle f, y_1 - y_2\rangle \geqslant 0$ for all $(x_i, y_i) \in A$, $i = 1, 2$. A is called m-accretive if A is accretive and $R(I + A) = X$; in this case $R(I + \lambda A) = X$ for every $\lambda > 0$, and $-A$ generates a nonlinear contraction semi-group $(S(t))_{t \geqslant 0}$ on $\overline{D(A)}$. In fact this holds even if A satisfies the range condition, i.e., $R(I + \lambda A) \supset \overline{D(A)}$ for all $\lambda > 0$. Denoting $(x, y)_+ = \|x\| \lim_{t \to 0^+}((\|x + ty\| - \|x\|)/t)$ we know that $(x, y)_+ = \max_{f \in Jx} \langle f, y\rangle$. Then A is accretive if and only if $(x_1 - x_2, y_1 - y_2)_+ \geqslant 0$ for all $(x_i, y_i) \in A$, $i = 1, 2$ (see [2, 5–7, 14, 19]).

(e) An integral solution for (1.1) on $[0, T]$ is a curve $u(t)$ on $[0, T]$ satisfying $u(0) = u_0$ and the inequality

$$\tfrac{1}{2}\|u(t) - x\|^2 - \tfrac{1}{2}\|u(s) - x\|^2 \leqslant \int_s^t (u(\theta) - x, f(\theta) - y)_+ \, d\theta$$

for every $(x, y) \in A$ and every $0 \leqslant s \leqslant t \leqslant T$ (see [2, 3]).

3. Asymptotic Behaviour of Unbounded Almost Non-expansive Sequences

In this section we prove a general theorem for sequences $(x_n)_{n\geq 0}$ in X satisfying

$$\|x_{i+1}-x_{j+1}\|^2 \leq \|x_i-x_j\|^2+\varepsilon(i, j) \tag{3.1}$$

for all $i, j\geq 0$, where: $\forall\varepsilon>0$, $\exists k_0$ such that $\forall j\geq k_0$, $\limsup_{n\to+\infty} (\varepsilon(n, j)/n)\leq\varepsilon$ (i.e., $\lim_{j\to+\infty}\limsup_{n\to+\infty} (\varepsilon(n, j)/n)=0$).

We then show that every almost non-expansive sequence satisfies (3.1). For notations we refer to Section 2.

THEOREM 3.1. *Assume* $(x_n)_{n\geq 0}$ *satisfies* (3. 1) and x_n/n *is bounded. Then*:

(i) $\lim_{n\to+\infty}\|x_n/n\|$ *exists.*

(ii) *If* X *is reflexive, then* $C\neq\varnothing$ *and* $\lim_{n\to+\infty}\|x_n/n\|=d(0, C)$.

(iii) *If* X *is reflexive and strictly convex, then* x_n/n *converges weakly to* $P_C 0$ *and* $\|P_C 0\|=\lim_{n\to+\infty}\|x_n/n\|$.

(iv) *If* X^* *has Fréchet differentiable norm, then* x_n/n *converges strongly to* $P_C 0$.

Proof. (i) Let $k\geq 1$ fixed, and $j_n\in J(x_n-x_{k-1})$, for $n\geq k$. Then we have: $\forall n\geq k\geq 1$,

$$\begin{aligned}\langle j_n, x_k-x_{k-1}\rangle &\geq \tfrac{1}{2}\|x_n-x_{k-1}\|^2-\tfrac{1}{2}\|x_n-x_k\|^2\\ &\geq \tfrac{1}{2}\|x_n-x_{k-1}\|^2-\tfrac{1}{2}\|x_{n-1}-x_{k-1}\|^2-\tfrac{1}{2}\varepsilon(n-1, k-1).\end{aligned}$$

Hence

$$\frac{2}{n^2}\left\langle\sum_{i=k}^{n} j_i, x_k-x_{k-1}\right\rangle\geq\left\|\frac{x_n-x_{k-1}}{n}\right\|^2-\frac{1}{n^2}\sum_{i=k}^{n}\varepsilon(i-1, k-1).$$

Let $A_n=(2/n^2)\sum_{i=k}^{n} j_i$ for $n\geq k$; we have $\|A_n\|\leq(2/n^2)\sum_{i=k}^{n}\|x_i-x_{k-1}\|$; hence $\|A_n\|$ is bounded since $\|x_n/n\|$ is bounded.

Let $(n_l)_{l\geq 1}$ be a sequence so that

$$\left\|\frac{x_{n_l}}{n_l}\right\|\xrightarrow[l\to+\infty]{}\limsup_{n\to+\infty}\left\|\frac{x_n}{n}\right\|.$$

From the weak star compactness of the closed unit ball of X^*, it follows that the sequence $(A_{n_l})_{l\geq 1}$ has a weak star cluster point $q^*\in X^*$ (obviously independent of $k\geq 1$).

Now given $\varepsilon > 0$ we choose $k_0 \geqslant 1$ as in (3.1). Then for $k \geqslant k_0 + 1$, we have

$$\langle q^*, x_k - x_{k-1} \rangle \geqslant \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2 - \varepsilon$$

since $k - 1 \geqslant k_0$ and by (3.1) we have

$$\limsup_{n \to +\infty} \frac{1}{n^2} \sum_{i=k+1}^{n} \varepsilon(i-1, k-1)$$
$$= \limsup_{n \to +\infty} \frac{1}{n^2} \sum_{i=k+1}^{n} (i-1) \frac{\varepsilon(i-1, k-1)}{(i-1)} \leqslant \varepsilon/2 < \varepsilon.$$

But we have

$$\|q^*\| \leqslant \limsup_{n \to +\infty} \|A_n\| \leqslant \limsup_{n \to +\infty} \frac{2}{n^2} \sum_{i=k}^{n} \|x_i - x_{k-1}\|$$
$$= \limsup_{n \to +\infty} \frac{2}{n^2} \sum_{i=k}^{n} i \left\| \frac{x_i - x_{k-1}}{i} \right\| \leqslant \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|.$$

Therefore we get

$$\forall k \geqslant k_0 + 1, \qquad \langle q^*, x_k - x_{k-1} \rangle \geqslant \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2 - \varepsilon \geqslant \|q^*\|^2 - \varepsilon. \tag{3.2}$$

Hence, since $\varepsilon > 0$ was arbitrary, we have

$$\frac{1}{2} \|q^*\|^2 + \frac{1}{2} \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2$$
$$\geqslant \frac{1}{2} \|q^*\|^2 + \frac{1}{2} \liminf_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2 \geqslant \liminf_{n \to +\infty} \left\langle q^*, \frac{x_n}{n} \right\rangle$$
$$\geqslant \liminf_{k \to +\infty} \langle q^*, x_k - x_{k-1} \rangle \geqslant \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2$$
$$\geqslant \frac{1}{2} \|q^*\|^2 + \frac{1}{2} \limsup_{n \to +\infty} \left\| \frac{x_n}{n} \right\|^2 \geqslant \|q^*\|^2.$$

This implies that $\limsup_{n \to +\infty} \|x_n/n\| = \liminf_{n \to +\infty} \|x_n/n\|$ and completes the proof of (i).

(ii) If X is reflexive, any weak subsequential limit of x_n/n belongs to C; hence $C \neq \varnothing$ and $d(0, C) \leqslant \liminf_{n \to +\infty} \|x_n/n\| = \lim_{n \to +\infty} \|x_n/n\|$. From (3.2) in (i) it follows that $\forall z \in C$,

$$\frac{1}{2}\lim_{n\to+\infty}\left\|\frac{x_n}{n}\right\|^2+\frac{1}{2}\|z\|^2$$
$$\geqslant\frac{1}{2}\|q^*\|^2+\frac{1}{2}\|z\|^2\geqslant\langle q^*, z\rangle$$
$$\geqslant\limsup_{n\to+\infty}\left\|\frac{x_n}{n}\right\|^2=\lim_{n\to+\infty}\left\|\frac{x_n}{n}\right\|^2\geqslant\|q^*\|^2.$$

This implies that

$$\|q^*\|\leqslant\lim_{n\to+\infty}\left\|\frac{x_n}{n}\right\|\leqslant\inf_{z\in C}\|z\|=d(0, C).$$

Hence $\lim_{n\to+\infty}\|x_n/n\|=d(0, C)$, which proves (ii).

(iii) Because X is reflexive, x_n/n has a weakly convergent subsequence; let $x_{n_i}/n_i \rightharpoonup_{i\to+\infty} p\in C$; because X is also strictly convex, we have $d(0, C)=\|P_C 0\|$. Therefore $\|p\|\leqslant\liminf_{i\to+\infty}\|x_{n_i}/n_i\|=\lim_{n\to+\infty}\|x_n/n\|=\|P_C 0\|$ by (ii); hence we must have $p=P_C 0$; this shows that $x_n/n\rightharpoonup_{n\to+\infty} p_C 0$ with $\|P_C 0\|=\lim_{n\to+\infty}\|x_n/n\|$ and completes the proof of (iii).

(iv) This is an immediate consequence of (iii) and the characterization of X given in Section 2, (c).

Now we prove our results for almost non-expansive sequences in X. The first lemma is classical and its proof can be found, for example, in [11, Lemma 3.1]; therefore we omit its proof.

LEMMA 3.2. *Let $(a_n)_{n\geqslant 1}$ be a sequence of non-negative real numbers satisfying $a_{n+p}\leqslant a_n+a_p+\varepsilon(p)$ for all $n\geqslant p\geqslant 1$, where $\lim_{p\to+\infty}(\varepsilon(p)/p)=0$. Then the sequence $(a_n/n)_{n\geqslant 1}$ converges as $n\to+\infty$.*

PROPOSITION 3.3. *For an almost non-expansive sequence $(x_n)_{n\geqslant 0}$ in X, $\lim_{n\to+\infty}\|x_n/n\|$ exists.*

Proof. For $n\geqslant 1$, let $a_n=\|x_n-x_0\|$; then we have

$$\forall n\geqslant p\geqslant 1,\qquad a_{n+p}=\|x_{n+p}-x_0\|\leqslant a_p+\|x_{n+p}-x_p\|$$
$$\leqslant a_p+a_n+\sum_{i=0}^{p-1}\varepsilon(n+i, i).$$

Now let $\eta(i)=\sup_{n\geqslant 1}\varepsilon(n+i, i)$ for $i\geqslant 0$, and $\varepsilon(p)=\sum_{i=0}^{p-1}\eta(i)$ for $p\geqslant 1$. Then we have $a_{n+p}\leqslant a_n+a_p+\varepsilon(p)$, where $\lim_{p\to+\infty}(\varepsilon(p)/p)=0$; hence an application of Lemma 3.2 completes the proof of the proposition.

PROPOSITION 3.4. *Every almost non-expansive sequence* $(x_n)_{n\geq 0}$ *in* X *satisfies* (3.1).

Proof. We have $\|x_{i+1}-x_{j+1}\|^2 \leq [\|x_i-x_j\| + \varepsilon(i,j)]^2 = \|x_i-x_j\| + \varepsilon_1(i,j)$, where $\varepsilon_1(i,j)=\varepsilon(i,j)\ [\varepsilon(i,j)+2\,\|x_i-x_j\|]$. For $\varepsilon>0$ given, let $k_0\geq 0$ be such that $\varepsilon(i,j)\leq\varepsilon$ for all $i,j\geq k_0$; then for every $j\geq k_0$ we have

$$\limsup_{i\to+\infty}\frac{\varepsilon_1(i,j)}{i}\leq\varepsilon\limsup_{i\to+\infty}\left[\frac{\varepsilon+2\,\|x_i-x_j\|}{i}\right]$$

$$=2\varepsilon\limsup_{i\to+\infty}\left\|\frac{x_i}{i}\right\|=2\varepsilon\lim_{i\to+\infty}\left\|\frac{x_i}{i}\right\|$$

by Proposition 3.3. This shows that $(x_n)_{n\geq 0}$ satisfies (3.1) and completes the proof.

COROLLARY 3.5. *For every almost non-expansive sequence* $(x_n)_{n\geq 0}$ *in* X, *all conclusions of Theorem* 3.1 *are satisfied.*

Proof. x_n/n is bounded by Proposition 3.3, and $(x_n)_{n\geq 0}$ satisfies (3.1) by Proposition 3.4; hence the conclusions of Theorem 3.1 are satisfied.

Remark 3.6. In Kohlberg and Neyman [17] it is shown that the assumptions on X in (iii) and (iv) of Theorem 3.1 are also necessary for the respective conclusion to hold (even for non-expansive sequences).

Remark 3.7. In order to study the asymptotic behaviour of bounded sequences in a Hilbert space H, we defined in [8, 9] the notion of a "*uniformly*" almost non-expansive sequence" $(x_n)_{n\geq 0}$ as

$$\forall i,j,k\geq 0,\qquad \|x_{i+k}-x_{j+k}\|^2\leq\|x_i-x_j\|^2+\varepsilon(i,j),$$

where $\lim_{i,j\to+\infty}\varepsilon(i,j)=0$.

For bounded sequences $(x_n)_{n\geq 0}$ in H, this is equivalent to

$$\forall i,j,k\geq 0,\qquad \|x_{i+k}-x_{j+k}\|\leq\|x_i-x_j\|+\varepsilon(i,j),$$

where $\lim_{i,j\to+\infty}\varepsilon(i,j)=0$.

This study seems to be still an open problem in a Banach space X. However, for the study of unbounded behaviour, this uniformity condition is not needed and the results are valid in a Banach space, as the conclusions of Corollary 3.5 show.

4. Asymptotic Behaviour of Unbounded Almost Non-Expansive Curves

In order to apply our results to quasi-autonomous dissipative systems we prove the following theorem for almost non-expansive curves in X, by modifying the proof of [11, Theorem 5.1].

THEOREM 4.1. *Let $(u(t))_{t \geq 0}$ be an almost non-expansive curve in X and* $C_h = \bigcap_{n=0}^{\infty} \operatorname{clco}\{(u(t+h)-u(t))\}_{t \geq n}$ for $h>0$. Then:

(i) $\lim_{t \to +\infty} \|u(t)/t\|$ *exists.*

(ii) *If X is reflexive, then for every $h>0$, $C_h \neq \emptyset$ and* $\lim_{t \to +\infty} \|u(t)/t\| = d(0, C_h)/h$.

(iii) *If X is reflexive and strictly convex, then for every $h>0$, $u(t)/t$ converges weakly to $(1/h) P_{C_h} 0$ as $t \to +\infty$, and $(1/h) \|P_{C_h} 0\| =$* $\lim_{t \to +\infty} \|u(t)/t\|$.

(iv) *If X^* has Fréchet differentiable norm, then for every $h>0$, $u(t)/t$ converges strongly as $t \to +\infty$ to $(1/h) P_{C_h} 0$.*

Proof. For $n \geq 0$ let $x_n = u(t_n)$, where for $t_0 \geq 0$ and $0<h<\delta$ fixed, t_n is defined by $t_n = nh + t_0$; without loss of generality we may assume that $t_0 = 0$. We have $\|x_{i+1} - x_{j+1}\| = \|u((i+1)h) - u((j+1)h)\| \leq \|u(ih) - u(jh)\| + \varepsilon(ih, jh) = \|x_i - x_j\| + \varepsilon(ih, jh)$. Therefore $(x_n)_{n \geq 0}$ is an almost non-expansive sequence in X, and hence the conclusions of Corollary 3.5 hold for $(x_n)_{n \geq 0}$ with C replaced by $K_h = \bigcap_{n=0}^{\infty} \operatorname{clco}\{(u(i+1)h) - u(ih))_{i \geq n}\}$ for $h>0$. Since we have $u(t_n)/t_n = x_n/nh$ it follows that the conclusions of Theorem 4.1 hold for the sequence $u(t_n)/t_n$ with C_h replaced by K_h.

If X is reflexive and strictly convex, we have $(k/h) P_{K_{h/k}} 0 = (1/h) P_{K_h} 0$ for every $k \geq 1$; this follows by considering the sequence $t'_n = n(h/k)$ and noting that t_n is a subsequence of t'_n. Now to show that $(1/h) P_{K_h} 0$ is independent of h and also to complete the proof, it suffices to show that

$$\forall 0<h<\delta, \qquad \lim_{n \to +\infty} \sup_{t_n \leq t \leq t_{n+1}} \left\| \frac{u(t)}{t} - \frac{u(t_n)}{t_n} \right\| = 0.$$

We have

$$\forall n \geq 1, \forall t_n \leq t \leq t_{n+1},$$

$$\begin{aligned}
\left\| \frac{u(t)}{t} - \frac{u(t_n)}{t_n} \right\| &= \frac{\|(t_n - t)\, u(t_n) + t_n(u(t) - u(t_n))\|}{t_n t} \\
&\leq \frac{t_{n+1} - t_n}{t_n} \left\| \frac{u(t_n)}{t_n} \right\| + \frac{\|u(t) - u(t_n)\|}{t_n} \\
&\leq \frac{1}{n} \left\| \frac{u(t_n)}{t_n} \right\| + \frac{\|u(0) - u(t-t_n)\| + \sum_{i=0}^{n-1} \varepsilon(t_i, t_i + (t - t_n))}{t_n}.
\end{aligned}$$

Hence

$$\forall n \geqslant 1, \qquad \sup_{t_n \leqslant t \leqslant t_{n+1}} \left\| \frac{u(t)}{t} - \frac{u(t_n)}{t_n} \right\| \leqslant \frac{1}{n} \left\| \frac{u(t_n)}{t_n} \right\| + \frac{2}{nh} \sup_{0 \leqslant t \leqslant h} \|u(t)\|$$
$$+ \frac{1}{nh} \sum_{i=0}^{n-1} \sup_{0 \leqslant s \leqslant h} \varepsilon(t_i, t_i + s) \xrightarrow[n \to +\infty]{} 0$$

since $\|u(t_n)/t_n\|$ converges as $n \to +\infty$, hence is bounded, and $\sup_{0 \leqslant s \leqslant h} \varepsilon(t_i, t_i + s) \xrightarrow[i \to +\infty]{} 0$ since $(u(t))_{t \geqslant 0}$ is an almost non-expansive curve. This completes the proof of the theorem.

Remark 4.2. Observations similar to Remarks 3.6 and 3.7 apply here for almost non-expansive curves in X.

5. ASYMPTOTIC BEHAVIOUR OF UNBOUNDED TRAJECTORIES FOR $du/dt + Au \ni f$

In this section wse consider the quasi-autonomous dissipative system (1.1), where A is an accretive (possibly multivalued) operator in X, $u_0 \in X$, and $f \in L^1_{\text{loc}}((0, +\infty); X)$. We study the asymptotic behaviour of the unbounded (or, better stated, "not necessarily bounded") integral solution $(u(t))_{t \geqslant 0}$ of this system (if any) under suitable conditions on f. The bounded case in a Hilbert space was treated in [8, 9]. First we recall the following theorem on the existence and uniqueness of integral solutions for (1.1) and refer to Barbu [2, Theorem 3.2.1., p. 124] for its proof.

THEOREM 5.1. *Let C be a closed convex cone in X and A a closed accretive operator in $X \times X$ such that $R(I + \lambda A) \supset C \supset \overline{D(A)}$ for every $\lambda > 0$. Let $u_0 \in \overline{D(A)}$ and $f \in L^1((0, T); X)$ be such that $f(t) \in C$ a.e. on $(0, T)$. Then the system (1.1) has a unique integral solution $u(t)$ such that $u(t) \in \overline{D(A)}$ for every $t \in [0, T]$. Moreover, if u and v are respectively integral solutions of $du/dt + Au \ni f$ and $dv/dt + Av \ni g$ on $[0, T]$, then*

$$\tfrac{1}{2} \|u(t) - v(t)\|^2 \leqslant \tfrac{1}{2} \|u(s) - v(s)\|^2 + \int_s^t (u(\theta) - v(\theta), f(\theta) - g(\theta))_+ \, d\theta$$

for $0 \leqslant s \leqslant t \leqslant T$.

COROLLARY 5.2. *If $f, g \in L^1((0, T); X)$, u and v are respectively integral solutions of $du/dt + Au \ni f$ and $dv/dt + Av \ni g$ on $[0, T]$, then*

$$\|u(t) - v(t)\| \leqslant \|u(s) - v(s)\| + \int_s^t \|f(\theta) - g(\theta)\| \, d\theta$$

for $0 < s \leqslant t \leqslant T$.

Proof. Since $(x, y)_+ \leqslant \|x\| \, \|y\|$, the inequality in Theorem 5.1 implies that

$$\forall 0 \leqslant s \leqslant t \leqslant T,$$

$$\frac{1}{2} \|u(t) - v(t)\|^2 \leqslant \frac{1}{2} \|u(s) - v(s)\|^2 + \int_s^t \|u(\theta) - v(\theta)\| \, \|f(\theta) - g(\theta)\| \, d\theta.$$

Now the result follows by using Brézis [4, Lemma A.5, p. 157].

COROLLARY 5.3. *If* $f \in L^1_{\text{loc}}((0, +\infty); X)$ *and* u *is an integral solution for* (1.1), *then we have*

$$\forall h \geqslant 0, \forall t \geqslant s \geqslant 0,$$

$$\|u(t+h) - u(s+h)\| \leqslant \|u(t) - u(s)\| + \int_s^{s+h} \|f(\theta + (t-s)) - f(\theta)\| \, d\theta.$$

Proof. Applying Corollary 5.2 with $g(\theta) = f(\theta + (t-s))$ and $v(\theta) = u(\theta + (t-s))$ gives the result.

PROPOSITION 5.4. *If for every* $T > 0$, u *is an integral solution for the system* (1.1) *on* $[0, T]$, *and if* f *satisfies*

$$\exists f_\infty \in X, \exists \delta > 0 \qquad \text{such that} \quad \lim_{s \to +\infty} \int_s^{s+\delta} \|f(\theta) - f_\infty\| \, d\theta = 0 \quad (5.1)$$

then $(u(t))_{t \geqslant 0}$ *is an almost non-expansive curve in* X.

Proof. This follows from Corollary 5.3 by taking

$$\varepsilon(t, s) = \begin{cases} \int_s^{s+\delta} \|f(\theta + (t-s)) - f(\theta)\| \, d\theta & \text{if} \quad t \geqslant s \\ \int_t^{t+\delta} \|f(\theta + (s-t)) - f(\theta)\| \, d\theta & \text{if} \quad s \geqslant t. \end{cases}$$

In fact we have $\forall s, t \geqslant 0$,

$$\varepsilon(t, s) \leqslant \int_s^{s+\delta} \|f(\theta) - f_\infty\| \, d\theta + \int_t^{t+\delta} \|f(\theta) - f_\infty\| \, d\theta.$$

(Note also that (5.1) implies the same conclusion for *every* $\delta > 0$.)

COROLLARY 5.5. *If for every* $T > 0$, u *is an integral solution for the system* (1.1) *on* $[0, T]$, *and if* $f - f_\infty \in L^p((0, +\infty); X)$ *for some* $f_\infty \in X$ *and* $1 \leqslant p < \infty$, *then* $(u(t))_{t \geqslant 0}$ *is an almost non-expansive curve in* X.

Proof. This follows from Proposition 5.4 since by Hölder's inequality we have

$$\forall \delta > 0, \quad \int_s^{s+\delta} \|f(\theta) - f_\infty\| \, d\theta$$

$$\leqslant \delta^{(p-1)/p} \left(\int_s^{s+\delta} \|f(\theta) - f_\infty\|^p \, d\theta \right)^{1/p} \xrightarrow[s \to +\infty]{} 0.$$

Now we state the main result of this section.

THEOREM 5.6. *Assume u is an integral solution for the system (1.1) on every interval $[0, T]$, and $f - f_\infty \in L^p((0, +\infty); X)$ for some $f_\infty \in X$ and $1 \leqslant p < \infty$ (or more generally f satisfies (5.1)), and let $C_h = \bigcap_{n=0}^{\infty} \text{clco}\{(u(t+h) - u(t))_{t \geqslant n}\}$ for $h > 0$. Then:*

(i) $\lim_{t \to +\infty} \|u(t)/t\|$ *exists.*

(ii) *If X is reflexive, then for every $h > 0$, $C_h \neq \varnothing$ and $\lim_{t \to +\infty} \|u(t)/t\| = d(0, C_h)/h$.*

(iii) *If X is reflexive and strictly convex, then for every $h > 0$, $u(t)/t$ converges weakly to $(1/h) P_{C_h} 0$ as $t \to +\infty$ and $(1/h) \|P_{C_h} 0\| = \lim_{t \to +\infty} \|u(t)/t\|$.*

(iv) *If X^* has Fréchet differentiable norm, then for every $h > 0$, $u(t)/t$ converges strongly as $t \to +\infty$ to $(1/h) P_{C_h} 0$.*

Proof. In fact by Corollary 5.5 (or Proposition 5.4) the curve $(u(t))_{t \geqslant 0}$ is an almost non-expansive curve in X. The result now follows by applying Theorem 4.1.

Remark 5.7. In Theorem 5.6 the existence and uniqueness of the integral solution u for the system (1.1) on every interval $[0, T]$ is guaranteed (by Theorem 5.1) when A is m-accretive and $u_0 \in \overline{D(A)}$. Moreover in this case, letting $D = \overline{R(A)} - f_\infty$, if in addition X^* is strictly convex, then $\lim_{t \to +\infty} \|u(t)/t\| = d(0, D)$, and in Theorem 5.6(iii), (iv) the limit of $u(t)/t$ as $t \to +\infty$ can be identified with $-v$, where v is the unique point of least norm in D. This is due to M.G. Crandall (unpublished result; cf. Brézis [4], Pazy [21], Reich [24]) when $X = H$ is a Hilbert space, and to Plant and Reich [22] and Reich [25] (for the autonomous case $f \equiv 0$) in the Banach space.

Remark 5.8. Assume f satisfies (5.1); if u is an integral solution of (1.1) and v an integral solution of the autonomous system

$$\frac{dv}{dt} + (A - f_\infty)\, v \ni 0$$

$$v(0) = u_0$$

then by Corollary 5.2 we have

$$\|u(t)-v(t)\| \leqslant \int_0^t \|f(\theta)-f_\infty\| \, d\theta.$$

Hence by (5.1) we get $\lim_{t\to+\infty} \|u(t)/t - v(t)/t\| = 0$. Therefore to prove Theorem 5.6 we may first prove it for the autonomous case (i.e., $f \equiv 0$ in (1.1)) by using [12, Theorem 3.1, Corollary 3.2] adapted for non-expansive curves in X and then use Corollary 5.2 to get the result. However, here we proved a general theorem for almost non-expansive curves in X.

Remark 5.9. Theorem 5.6 extends also, for the non-autonomous case, previous results by Reich [26, Proposition 1.2], where $\overline{R(A)}$ is assumed to have the minimum property, by Reich [25, Remarks, p. 124], where A is assumed to satisfy the range condition and the norm of X to be uniformly Gâteaux differentiable, and by Aizicovici, Londen, and Reich [1, Theorem 3.4] (for the case $b \equiv 1$ and $g \equiv 0$), where A is assumed to be m-accretive.

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