Oscillation of second-order damped dynamic equations on time scales

Samir H. Saker\textsuperscript{a}, Ravi P. Agarwal\textsuperscript{b,\ast}, Donal O’Regan\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
\textsuperscript{b} Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA
\textsuperscript{c} Department of Mathematics, National University of Ireland, Galway, Ireland

Received 8 June 2006
Available online 15 September 2006
Submitted by Steven G. Krantz

Abstract

The study of dynamic equations on time scales has been created in order to unify the study of differential and difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. In this paper, by employing the Riccati transformation technique we will establish some oscillation criteria for second-order linear and nonlinear dynamic equations with damping terms on a time scale $\mathbb{T}$. Our results in the special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ extend and improve some well-known oscillation results for second-order linear and nonlinear differential and difference equations and are essentially new on the time scales $\mathbb{T} = h\mathbb{N}$, $h > 0$, $\mathbb{T} = q\mathbb{N}$ for $q > 1$, $\mathbb{T} = \mathbb{N}^2$, etc. Some examples are considered to illustrate our main results.

Keywords: Oscillation; Dynamic equations; Time scale

\ast Corresponding author.

E-mail addresses: shsaker@mans.edu.eg (S.H. Saker), agarwal@fitr.edu (R.P. Agarwal), donal.oregan@nuigalway.ie (D. O’Regan).
1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [23], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice—once for differential equations and once again for difference equations.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus. Dynamic equations on a time scale have enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O’Regan, and Peterson [2] and the references cited therein. A book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [8] summarizes and organizes much of time scale calculus. For advances of dynamic equations on time scales we refer the reader to the book [9].

For completeness, we recall the following concepts related to the notation of time scale. A time scale \( \mathbb{T} \) is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that \( \sup \mathbb{T} = \infty \), and define the time scale interval \([t_0, \infty)_{\mathbb{T}}\) by \([t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}\). We assume that \( \mathbb{T} \) has the topology that it inherits from the standard topology on the real numbers \( \mathbb{R} \).

The forward and the backward jump operators on any time scale \( \mathbb{T} \) are defined by

\[
\sigma(t) := \inf \{ s \in \mathbb{T} : s > t \}, \quad \rho(t) := \sup \{ s \in \mathbb{T} : s < t \}.
\]

A point \( t \in \mathbb{T}, t > \inf \mathbb{T} \), is said to be left-dense if \( \rho(t) = t \), right-dense if \( t < \sup \mathbb{T} \) and \( \sigma(t) = t \), left-scattered if \( \rho(t) < t \) and right-scattered if \( \sigma(t) > t \). The graininess function \( \mu \) for a time scale \( \mathbb{T} \) is defined by \( \mu(t) := \sigma(t) - t \).

For a function \( f : \mathbb{T} \to \mathbb{R} \) (the range \( \mathbb{R} \) of \( f \) may be actually replaced by any Banach space) the (delta) derivative is defined by

\[
f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},
\]

if \( f \) is continuous at \( t \) and \( t \) is right-scattered. If \( t \) is not right-scattered then the derivative is defined by

\[
f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},
\]

provided this limit exists. A function \( f : [a, b] \to \mathbb{R} \) is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and \( f \) is said to be differentiable if its derivative exists. A useful formula is

\[
f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).
\]
Also, we will use $x^\Delta$ to denote $x^\Delta(t) + \mu(t)x^\Delta(t)$. We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $gg^\sigma \neq 0$) of two differentiable functions $f$ and $g$

\[ (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{gg^\sigma}. \]

An integration by parts formula reads

\[ \int_a^b f(t)g^\Delta(t) \Delta t = \left[ f(t)g(t) \right]_a^b - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t. \]

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

\[ 1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}. \]

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd-continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, then we can define the exponential function by

\[ e_p(t,s) = \exp \left( \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta \tau \right), \]

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is defined by

\[ \xi_h(z) = \begin{cases} \log(1+hz) \quad &h \neq 0, \\ z &h = 0. \end{cases} \]

Alternately, for $p \in \mathcal{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP $x^\Delta = p(t)x(t)$ with $x(t_0) = 1$.

In the last few years, much interest has focused on obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [3–7,10–20,22,25–27].

In particular, much work has been done on the following dynamic equations

\[ (p(t)x^\Delta(t))^\Delta + q(t)x^\sigma = 0, \quad (1.1) \]

\[ (p(t)x^\Delta(t))^\Delta + q(t)(f \circ x^\sigma) = 0. \quad (1.2) \]

Following this trend, to develop the qualitative theory of dynamic equations on time scales, in this paper we shall study the oscillation of the following nonlinear second-order dynamic equation with damping

\[ (a(t)x^\Delta(t))^\Delta + p(t)x^\Delta(\sigma(t)) + q(t)(f \circ x^\sigma) = 0, \quad (1.3) \]

on a time scale $\mathbb{T}$.

A number of oscillation criteria for equations with damping terms can be found in the papers by Guseinov and Kaymakçalan [22], Erbe, Peterson and Saker [19], Erbe and Peterson [18] and Erbe, Peterson and Bohner [7].

In [22], the authors considered the linear dynamic equation

\[ x^{\Delta\Delta}(t) + p(t)x^\Delta(t) + q(t)x(t) = 0, \quad (1.4) \]
when $p(t)$ and $q(t)$ are positive rd-continuous functions and established some sufficient conditions for nonoscillation. They proved that if
\[ \int_{t_0}^{\infty} p(t) \Delta t < \infty, \quad \int_{t_0}^{\infty} t q(t) \Delta t < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \int t q(t) \Delta t < \infty, \]
(1.5)
then (1.4) is nonoscillatory.

In [19], the authors considered Eq. (1.4) and the nonlinear dynamic equation
\[ x^{\Delta \Delta}(t) + p(t)x^{\Delta \sigma}(t) + q(t) \left( f \circ x^{\sigma} \right) = 0, \]
(1.6)
when $p(t)$ and $q(t)$ are positive rd-continuous functions, $xf(x) > 0$ for $x \neq 0$ and $f(x) \geq Kx$ and established some sufficient conditions for oscillation by reducing the equations to the self-adjoint form and employing the generalized Riccati transformation technique.

In [18], the authors considered Eq. (1.6) and obtained an oscillation criterion when $p(t)$ is nonnegative rd-continuous function and do not require any explicit sign assumptions on $q(t)$ and
\[ f'(x) > 0 \quad \text{and} \quad f'(x) \geq \frac{f(x)}{x} \geq \lambda > 0 \quad \text{for} \quad |x| \geq K > 0. \]
(1.7)
The oscillation criterion is obtained by comparing the oscillation of (1.6) with the self-adjoint equation
\[ \left( r(t)x^{\Delta}(t) \right)^{\Delta} + \lambda p(t)x^{\sigma} = 0, \]
(1.8)
when
\[ \int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \int_{t_0}^{\infty} e_{-p(t, t_0)} = \infty, \]
(1.9)
where $r(t) = e_{r(t, t_0)}$ and $p(t_1) = r(t)q(t)$.

Also, in [7] the authors considered Eq. (1.6) when $f'(x) > 0$ and established some new oscillation criteria when no explicit sign assumptions on $p(t)$ and $q(t)$ are required. The results are obtained by reducing the equation to the nonlinear equation
\[ \left( r(t)x^{\Delta}(t) \right)^{\Delta} + p(t)f(x^{\sigma}) = 0, \]
(1.10)
where $r(t)$ and $p(t)$ are as defined above. However the common restriction (1.7) is required and $f'(x) \geq k > 0$ is also required. This does not hold in the general case, for example, for the function
\[ f(x) = x \left( \frac{1}{9} + \frac{1}{1 + x^2} \right) \]
we see that
\[ f'(x) = \frac{(x^2 - 2)(x^2 - 5)}{9(1 + x^2)^2} \]
changes sign four times. Hence, none of the aforementioned results established in [7,18] can be applied in this case.

The presence of the damping term in (1.3) calls for a modified approach to the study of the oscillatory properties of solutions. Since, we cannot rewrite Eq. (1.3) in the self-adjoint form (1.8), because the damping term contains $x^{\Delta}(t)$, we study (1.3) in its general form.

We will use some of the following assumptions:
(H1) \( a, p \) and \( q \) are positive real-valued rd-continuous functions;
(H2) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is such that \( uf(u) > 0 \);
(H3) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is such that \( f(u) \geq \kappa u \) for \( u \neq 0 \) and some \( \kappa > 0 \);
(H4) \( f : \mathbb{R} \rightarrow \mathbb{R} \) is such that \( f'(u) \geq k \) for \( u \neq 0 \) and some \( k > 0 \).

As will be seen later, in order to discuss the oscillatory properties of (1.3), it is necessary to distinguish two cases:

\[(H_5) \int_{t_0}^{\infty} \left( \frac{1}{a(t)} e^{-\frac{p(t)}{a(t)}} (t, t_0) \right) \Delta t = \infty, \]

\[(H_6) \int_{t_0}^{\infty} \left( \frac{1}{a(t)} e^{-\frac{p(t)}{a(t)}} (t, t_0) \right) \Delta t < \infty. \]

Our attention is restricted to those solutions of (1.3) which exist on some half-line \([t_x, \infty)\) and satisfy \( \sup\{|x(t)|: t > T\} > 0 \) for any \( T \geq t_x \). We assume the standing hypothesis that (1.3) does possess such solutions. A solution \( x(t) \) of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Note that, assumption (H4) allows \( f \) to be of superlinear or sublinear growth, say

\[ f(x) = x^\gamma, \quad \gamma > 0 \] (quotient of odd positive integers). \hspace{1cm} (1.11)

We note that in the linear case, i.e., when \( f(u) = u \), (1.6) becomes

\[ x^{\Delta \Delta}(t) + p(t)x^{\Delta \sigma}(t) + q(t)x^\sigma \]
\[ = x^{\Delta \Delta}(t) + p(t)[x^{\Delta}(t) + \mu(t)x^{\Delta \Delta}(t)] + q(t)[x(t) + \mu(t)x^{\Delta}(t)] \]
\[ = (1 + \mu p)x^{\Delta \Delta}(t) + (p + \mu q)x^{\Delta}(t) + qx(t) \]
\[ = \frac{1}{1 + \mu p} \left[ x^{\Delta \Delta}(t) + \frac{(p + \mu q)}{1 + \mu p} x^{\Delta}(t) + \frac{q}{1 + \mu p} x(t) \right] = 0. \]

Hence, the oscillation of (1.6), when \( f(u) = u \), is equivalent to the oscillation of the equation

\[ y^{\Delta \Delta}(t) + p_1(t)y^{\Delta \Delta}(t) + p_2(t)y(t) = 0, \] \hspace{1cm} (1.12)

where

\[ p_1(t) := \frac{(p + \mu q)}{(1 + \mu p)}, \quad p_2(t) := \frac{q}{(1 + \mu p)} \quad \text{and} \quad 1 + \mu p \neq 0, \]

and the oscillation of (1.12) is equivalent to the oscillation of (1.6) when \( f(u) = u \), where

\[ p(t) := \frac{(p_1 - \mu p_2)}{(1 - \mu p_1 - \mu^2 p_2)}, \]
\[ q(t) := p_2 \left[ 1 + \frac{\mu p_1 - \mu^2 p_2}{1 - \mu p_1 - \mu^2 p_2} \right] \quad \text{and} \quad 1 - \mu p_1 - \mu^2 p_2 \neq 0. \]

In this paper, by employing the Riccati transformation technique we will establish some sufficient conditions for the oscillation of (1.3). The paper is organized as follows: In Section 2, we develop
the Riccati transformation technique to give some sufficient conditions for the oscillation of all solutions of (1.3) when (H6) holds. In Section 3, we consider the case when (H7) holds and establish some conditions that ensure that all solutions are either oscillatory or tends to zero.

Our results in this paper, are essentially new and complement the nonoscillation conditions for Eq. (1.4) for the linear case that has been established in [22].

Note that, in the special case when $T = \mathbb{R}$, $\sigma(t) = 0$, $\mu(t) = 0$, $x^\Delta(t) = x'(t)$. In this case, (1.3) involves the second-order differential equations
\begin{align}
  x''(t) + q(t)x(t) &= 0, \quad (1.13) \\
  x''(t) + p(t)x'(t) + q(t)x(t) &= 0, \quad (1.14)
\end{align}

and
\begin{equation}
  x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0. \quad (1.15)
\end{equation}

In this case, our results extend and improve some well-known oscillation results for second-order linear and nonlinear differential equations that have been established by Atkinson [1], Fite [21], Leighton [24], Waltman [33], Wintner [34], Yan [35] and Yeh [36].

When $T = \mathbb{N}$, $\sigma(t) = t + 1$, $\mu(t) = 1$, $x^\Delta(t) = \Delta x(t) = x(t + 1) - x(t)$. In this case, Eq. (1.3) involves the second-order difference equations
\begin{align}
  \Delta^2 x_n + q_n x_{n+1} &= 0, \quad (1.16) \\
  \Delta^2 x_n + p_n \Delta x_{n+1} + q_n f(x_{n+1}) &= 0, \quad (1.17)
\end{align}

and
\begin{equation}
  \Delta(a_n \Delta x_n) + p_n \Delta x_{n+1} + q_n f(x_{n+1}) = 0. \quad (1.18)
\end{equation}

In this case our results extend and improve some well-known oscillation results for Eq. (1.16) which have been established by Györi and Lalli [29] and Zhang and Chen [37], and for Eq. (1.17) results which have been established by Thandapani and Lalli [30], Thandapani, Pandian and Lalli [31], Thandapani, Ravi and Graef [32].

If $T = h\mathbb{Z}$, $h > 0$, $\sigma(t) = t + h$, $\mu(t) = h$, $x^\Delta(t) = \Delta_h x(t) = \frac{x(t+h) - x(t)}{h}$, our results are essentially new for the second-order difference equation
\begin{equation}
  \Delta_h (a(t)\Delta_h x(t)) + p(t)\Delta_h x(t+h) + q(t)x(t+h) = 0. \quad (1.19)
\end{equation}

If $T = q\mathbb{N}^+ = \{t: t = q^k, k \in \mathbb{N}, q > 1\}$, $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^\Delta(t) = \Delta_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}$, our results are essentially new for the $q$-difference equation
\begin{equation}
  \Delta_q (a(t)\Delta_q x(t)) + p(t)\Delta_q x(qt) + q(t)x(qt) = 0. \quad (1.20)
\end{equation}

If $T = \mathbb{N}_0^2 = \{t^2: t \in \mathbb{N}_0\}$, $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$, $\Delta_N x(t) = \frac{x((\sqrt{t}+1)^2) - x(t)}{1+2\sqrt{t}}$, our results are essentially new for the difference equation
\begin{equation}
  \Delta_N (a(t)\Delta_N x(t)) + p(t)\Delta_N x((\sqrt{t}+1)^2) + q(t)x((\sqrt{t}+1)^2) = 0. \quad (1.21)
\end{equation}

If $T = \mathbb{T}_n = \{t_n: n \in \mathbb{N}_0\}$ where $\{t_n\}$ is the set of the harmonic numbers defined by
\begin{align*}
  t_0 &= 0, \quad t_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N}_0,
\end{align*}
we have \( \sigma(t_n) = t_n + 1 \), \( \mu(t_n) = \frac{1}{n+1} \), \( x^\Delta(t_n) = \Delta_{t_n} x(t_n) = (n+1)x(t_n) \), and our results are essentially new for the difference equation

\[
\Delta_{t_n} \left( a(t_n) \Delta_{t_n} x(t_n) \right) + p(t_n) \Delta_{t_n} x(t_n+1) + q(t_n) x(t_n+1) = 0.
\]

When \( T_2 = \{ \sqrt{n}: n \in \mathbb{N}_0 \} \), we have \( \sigma(t) = \sqrt{t^2 + 1} \) and \( \mu(t) = \sqrt{t^2 + 1} - t \), \( x^\Delta(t) = \Delta_2 x(t) = (x(\sqrt{t^2 + 1}) - x(t))/\sqrt{t^2 + 1} - t \), and Eq. (1.3) becomes the second-order difference equation

\[
\Delta_2 \left( a(t)(\Delta_2 x(t)) \right) + p(t) x(\sqrt{t^2 + 1}) + q(t) f \left( x(\sqrt{t^2 + 1}) \right) = 0.
\]

When \( T_3 = \{ 3\sqrt{n}: n \in \mathbb{N}_0 \} \), we have \( \sigma(t) = 3\sqrt{t^3 + 1} \) and \( \mu(t) = 3\sqrt{t^3 + 1} - t \), \( x^\Delta(t) = \Delta_3 x(t) = (x(3\sqrt{t^3 + 1}) - x(t))/3\sqrt{t^3 + 1} - t \), and Eq. (1.3) becomes the second-order perturbed delay difference equation

\[
\Delta_3 \left( a(t)(\Delta_3 x(t)) \right) + p(t) x(3\sqrt{t^3 + 1}) + q(t) f \left( x(3\sqrt{t^3 + 1}) \right) = 0.
\]

### 2. The case where (H5) holds

Throughout this section, we will assume (H5) holds. In the following, we establish some oscillation criteria for (1.3) when \( f \) is not required to be differentiable.

**Theorem 2.1.** Assume that (H1)–(H3) and (H5) hold. Furthermore, assume that there exists a positive real rd-continuous differentiable function \( \rho(t) \) such that

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left[ \kappa \rho(s) q(s) - \frac{a(s) \psi^2(s)}{4\rho(s)} \right] \psi = \infty, \tag{2.1}
\]

where

\[
\psi(t) = \frac{a^{\sigma} \rho^\Delta(t) - \rho(t) p(t)}{a^{\sigma}}. \tag{2.2}
\]

Then every solution of (1.3) is oscillatory.

**Proof.** Suppose to the contrary that \( x(t) \) is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that \( x(t) > 0 \) for \( t \geq t_1 > t_0 \). We shall consider only this case, since in view of (H2), the proof of the case when \( x(t) \) is eventually negative is similar. Now, we claim that \( x^\Delta(t) \) has a fixed sign on the interval \([t_2, \infty)\) for some \( t_2 \geq t_1 \). From (1.3), since \( q(t) > 0 \) and \( f(x(t)) > 0 \), we have

\[
(a(t)x^\Delta(t))^\Delta + p(t)x^\Delta = -q(t_n) f \left( x^{\sigma} \right) < 0,
\]

i.e.,

\[
(a(t)x^\Delta(t))^\Delta + p(t)x^\Delta < 0.
\]

By setting \( y(t) = a(t)x^\Delta(t) \), we immediately see that \( y^\Delta(t) + \frac{p(t)}{a^{\sigma}(t)} y^{\sigma} < 0 \), which implies that \( (y(t)e^\frac{p(t)}{a^{\sigma}(t)})^\Delta < 0 \). Then \( y(t)e^\frac{p(t)}{a^{\sigma}(t)} \) is decreasing and thus \( y(t) \) is eventually of one sign. Then \( x^\Delta(t) \) has a fixed sign for all sufficiently large \( t \) and we have one of the following:
Case (i): $x^\Delta(t)$ is eventually positive.

Case (ii): $x^\Delta(t)$ is eventually negative.

First, we consider Case (i). $x^\Delta(t) \geq 0$ on $[t_2, \infty)$ for some $t_2 \geq t_1$. Then in view of (1.3) we have

$$x(t) > 0, \quad x^\Delta(t) \geq 0, \quad (a(t)x^\Delta(t))^\Delta \leq 0, \quad t \geq t_2.$$  \hspace{1cm} (2.3)

Define the function $w(t)$ by the Riccati substitution

$$w(t) := \rho(t) a(t)x^\Delta(t) x(t), \quad t \geq t_2.$$  \hspace{1cm} (2.4)

Then $w(t) > 0$, and satisfies

$$w^\Delta(t) = -\frac{\rho(t)q(t)f(x^\sigma)}{x(t)} - \rho(t)p(t)x^\Delta x(t) + \rho^\Delta(t) + \rho\sigma w\sigma - \rho(t)\left(\frac{a(t)x^\Delta(x^\Delta(t))^\Delta}{x^\sigma x(t)}\right).$$  \hspace{1cm} (2.5)

However from (2.3), we see that for $t \geq t_3 = \sigma(t_2)$

$$a(t)x^\Delta(t) \geq (ax^\Delta)^\sigma, \quad x^\sigma \geq x(t).$$  \hspace{1cm} (2.6)

Using (2.6) in (2.5), we have

$$w^\Delta(t) \leq -\frac{\rho(t)q(t)f(x^\sigma)}{x(t)} - \rho(t)p(t)x^\Delta x(t) + \frac{\rho^\Delta(t)}{\rho\sigma} - \rho(t)\frac{a(t)x^\Delta(x^\Delta(t))^\Delta}{x^\sigma x(t)},$$  \hspace{1cm} (2.7)

and hence by (H3), we have

$$w^\Delta(t) \leq -\kappa \rho(t) q(t) + \frac{\psi(t)}{\rho\sigma} w\sigma - \frac{\rho(t)(\rho\sigma)^2 a(t)^2}{\rho(t)(x^\sigma)^2},$$  \hspace{1cm} (2.8)

Then

$$w^\Delta(t) \leq -\kappa \rho(t) q(t) + \frac{a(t)\psi^2(t)}{4\rho(t)} - \left[\frac{\sqrt{\rho(t)}}{\rho\sigma} w\sigma - \frac{\psi(t)\sqrt{a(t)}}{2\sqrt{\rho(t)}}\right]^2$$

$$\leq -\left[\kappa \rho(t) q(t) - \frac{a(t)\psi^2(t)}{4\rho(t)}\right].$$  \hspace{1cm} (2.9)

Integrating from $t_3$ to $t$, we obtain

$$w(t) - w(t_3) \leq -\int_{t_3}^{t} \left[\kappa \rho(s) q(s) - \frac{a(s)\psi^2(s)}{4\rho(s)}\right] \Delta s$$  \hspace{1cm} (2.10)

which yields

$$\int_{t_3}^{t} \left[\kappa \rho(s) q(s) - \frac{a(s)\psi^2(s)}{4\rho(s)}\right] \Delta s \leq w(t_3) - w(t) < w(t_3).$$
for all large $t$. This is contrary to (2.1). Next, we consider Case (ii). Then there exists $t_2 \geq t_1$ such that $x^{\Delta}(t) < 0$ for $t \geq t_2$. Define the function $u(t) = -a(t)x^{\Delta}(t)$. Then from (1.3), we have

$$u^{\Delta}(t) + \frac{p(t)}{a(t)}u(t) \geq 0.$$ 

Thus

$$u(t) \geq u(t_2)e^{-\frac{p(t)}{a(t)}(t,t_2)},$$

so that

$$x^{\Delta}(t) \leq -u(t_2)\left(\frac{1}{a(t)}e^{-\frac{p(t)}{a(t)}(t,t_2)}\right).$$

Integrating last inequality from $t_2$ to $t$, we have

$$x(t) - x(t_2) \leq a(t_2)x^{\Delta}(t_2)\int_{t_2}^{t} \left(\frac{1}{a(s)}e^{-\frac{p(s)}{a(s)}(s,t_2)}\right) \Delta s,$$

for $t \geq t_2$. Condition (H$_5$) implies that $x(t)$ is eventually negative, which is a contradiction. The proof is complete. □

From Theorem 2.1, we can obtain different conditions for the oscillation of (1.3) using different choices of $\rho(t)$. For instance, we may let $\rho(t) = 1$, $\rho(t) = t^\lambda$, $t \geq t_0$, and $\lambda \geq 1$, or, we may let $\rho(t) = R(t,t_0) = \int_{t_0}^{t}(1/a(s))\Delta s$. From Theorem 2.1 we obtain the following corollaries.

**Corollary 2.1.** Assume that (H1)–(H3) and (H5) hold. If

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\kappa q(s) - \frac{a(s)p^2(s)}{4(a^\sigma)^2}\right] \Delta s = \infty,$$

then every solution of (1.3) is oscillatory.

**Corollary 2.2.** Assume that (H1)–(H3) and (H5) hold. If there is $\lambda \geq 1$ such that

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\kappa s^\lambda q(s) - \frac{a(s)[a^\sigma(s^\lambda)^\Delta - s^\lambda p(s)]^2}{4(a^\sigma)^2s^\lambda}\right] \Delta s = \infty,$$

then every solution of (1.3) is oscillatory.

**Corollary 2.3.** Assume that (H1)–(H3) and (H5) hold. If

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\kappa R(s,t_0)q(s) - \frac{a(s)[a^\sigma(R(s,t_0))^\Delta - R(s,t_0)p(s)]^2}{4(a^\sigma)^2R(s,t_0)}\right] \Delta s = \infty,$$

then every solution of (1.3) is oscillatory.
Corollaries 2.1–2.3 extend as well as improve the old result of Fite [21] for the linear second-order differential equation (1.13) which says that all solutions oscillate if
\[ \int_{t_0}^{\infty} q(t) \, dt = \infty. \] (2.14)

In the case \( T = \mathbb{R} \), this result was subsequently extended and improved by a number of authors, see Waltman [33], Yan [35] and Yeh [36].

In the case when \( T = \mathbb{N} \), Thandapani, Györi and Lalli [29] and Zhang and Chen [37] obtained the discrete analogy of (2.14) and proved that every solution of (1.16) oscillates if
\[ \sum_{n=n_0}^{\infty} q_n = \infty. \] (2.15)

The condition (2.15) was improved and extended by Thandapani [28], Thandapani and Lalli [30], Thandapani, Pandian and Lalli [31], Thandapani, Ravi and Graef [32].

In the time scales case, this result (the Leighton–Wintner result [24,34]) may be found in [5] (for the case \( p(t) > 0 \)) and in [17] and [18] with no explicit sign assumption on \( p(t) \).

If \( p(t) = 0 \), then (1.3) becomes
\[ x^{\Delta \Delta}(t) + q(t) f(x^\sigma) = 0. \] (2.16)

In this case from Corollary 2.2, we have the following:

**Corollary 2.4.** Assume that \((H_1)\)–\((H_3)\) hold. If
\[ \limsup_{t \to \infty} \int_{t_0}^{t} \left[ \kappa s q(s) - \frac{1}{4s} \right] \Delta s = \infty, \]
then every solution of (2.15) is oscillatory.

Corollary 2.4 extends as well as improves the oscillation result for the second-order differential equation of Atkinson [1], which says that: if
\[ \int_{t_0}^{\infty} s q(s) \, ds = \infty, \] (2.17)
then all solutions of the equation
\[ x''(t) + q(t)x^{2n+1}(t) = 0 \] (2.18)
are oscillatory, where \( q(t) > 0 \) and is continuous on \([t_0, \infty)\).

Also, Corollary 2.4 extends as well as improves the results for the difference equation (1.16) by Thandapani, Györi and Lalli [29, Theorem 1] and Zhang and Chen [37, Corollary 2.2] and for Eqs. (1.17) and (1.18) by Thandapani [28, Theorem 2], Thandapani, Pandian and Lalli [31, Theorem 2.4], Thandapani, Ravi and Graef [32, Theorem 2]. Most of the results in these papers proved that every solution oscillates if (2.15) holds and \( p_n \) is nonincreasing. However, if \( q_n = \gamma/n^2 \) the results in [28,31,32] cannot be applied and also our results do not require the condition
\[ f(u) - f(v) = g(u, v)(u - v), \quad g(u, v) \geq M > 0. \]
Example 2.1. Consider the second-order dynamic equation

\[ x^{\Delta\Delta}(t) + \frac{1}{t} x^{\Delta}(t) + \frac{1}{t^2} x^\sigma = 0, \quad t \geq 1. \]  

(2.19)

Here \( a(t) = 1 \), \( p(t) = 1/t \), and \( q(t) = 1/t^2 \). It is clear that conditions (H1)–(H3) and (H5) are satisfied. To apply Corollary 2.2, it remains to show condition (2.12). By choosing \( \lambda = 1 \), we have

\[
\lim sup_{t \to \infty} \int_{t_0}^{t} \left[ \kappa s^\lambda q(s) - \frac{a(s)a(s)s^{\Delta} - s^{\lambda} p(s)}{4(a^\sigma)^2 s^\lambda} \right] \Delta s = \lim sup_{t \to \infty} \int_{t_0}^{t} \left[ s^\frac{1}{s^2} \right] \Delta s = \infty.
\]

Then, by Corollary 2.2, every solution of (2.19) oscillates. Note that in the case when \( T = \mathbb{R} \), it is easy to see that one such solution of (2.19) is \( x(t) = \sin(\ln t) \).

The following theorem gives the extension of the Kamenev-type oscillation criterion for Eq. (1.3).

First, let us introduce the class of functions \( \mathfrak{R} \) which will be extensively used in the sequel. Let \( D_0 \equiv \{(t,s) \in T^2: t > s \geq t_0\} \) and \( \mathbb{D} \equiv \{(t,s) \in T^2: t \geq s \geq t_0\} \). The function \( H \in C_{rd}(\mathbb{D}, \mathbb{R}) \) is said belongs to the class \( \mathfrak{R} \) if

(i) \( H(t,t) = 0, \ t \geq t_0, \ H(t,s) > 0, \) on \( D_0 \),

(ii) \( H \) has a continuous \( \Delta \)-partial derivative \( H/\Delta s(t,s) \) on \( D_0 \) with respect to the second variable.

(\( H \) is rd-continuous function if \( H \) is rd-continuous function in \( t \) and \( s \).)

Theorem 2.2. Assume that (H1)–(H3) and (H5) hold. Let \( \rho(t) \) be as defined in Theorem 2.1 and let \( h, H: \mathbb{D} \to \mathbb{R} \) be rd-continuous functions such that \( H \) belongs to the class \( \mathfrak{R} \) and where

\[
\lim sup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ \kappa H(t,s)\rho(s)q(s) - \frac{(\rho^\sigma)^2 a(s)A^2(t,s)}{4\rho(s)H(t,s)} \right] \Delta s = \infty, \tag{2.20}
\]

\[
A(t,s) = H(t,s)\psi(s)\rho^\sigma + H/\Delta s(t,s) \tag{2.21}
\]

Then every solution of (1.3) is oscillatory.

Proof. Suppose to the contrary that \( x(t) \) is a nonoscillatory solution of (1.3) and let \( t_1 \geq t_0 \) be such that \( x(t) \neq 0 \) for all \( t \geq t_1 \), so without loss of generality, we may assume that \( x(t) \) is an eventually positive solution of (1.3) with \( x(t) > 0 \) for all \( t \geq t_1 \) sufficiently large. In view of Theorem 2.1 we see that \( x^{\Delta}(t) \) is eventually negative or eventually positive. If \( x^{\Delta}(t) \) is eventually negative, we are then back to Case (ii) of Theorem 2.1 and we obtain a contradiction.

If \( x^{\Delta}(t) \) is eventually positive, we assume that there exists \( t_2 \geq t_1 \) such that \( x^{\Delta}(t) \geq 0 \) for \( t_2 \geq t_1 \) and proceed as in the proof of Case (i) of Theorem 2.1 and get (2.8). From (2.8), it follows that

\[
\int_{t_2}^{t} \kappa H(t,s)\rho(s)q(s) \Delta s \leq - \int_{t_2}^{t} H(t,s)w^{\Delta}(s) \Delta s + \int_{t_2}^{t} H(t,s)\psi(s)\rho^\sigma w^\sigma \Delta s
\]
\[- \int_{t_2}^{t} H(t, s) \frac{\rho(s)}{(\rho^\sigma)^2 a(s)} (w^\sigma)^2 \Delta s. \tag{2.22}\]

Using the integration by parts formula, we have

\[
\int_{t_2}^{t} H(t, s) w^\Delta(s) \Delta s = H(t, t_2) w(t_2) - \int_{t_2}^{t} H^\Delta (t, s) w^\sigma \Delta s
\]

\[- H(t, t_2) w(t_2) - \int_{t_2}^{t} H^\Delta (t, s) w^\sigma \Delta s, \tag{2.23}\]

where $H(t, t) = 0$. Substituting (2.23) into (2.22), we obtain

\[
\int_{t_2}^{t} \kappa H(t, s) \rho(s) q(s) \Delta s \leq H(t, t_2) w(t_2) + \int_{t_2}^{t} \frac{H(t, s)}{\rho^\sigma} w^\sigma \Delta s
\]

\[- \int_{t_2}^{t} \frac{\rho(s)}{(\rho^\sigma)^2 a(s)} (w^\sigma)^2 \Delta s. \tag{2.24}\]

Hence,

\[
\int_{t_2}^{t} \kappa H(t, s) \rho(s) q(s) \Delta s \leq H(t, t_2) w(t_2) + \int_{t_2}^{t} \left[ H(t, s) \frac{\psi(s)}{\rho^\sigma} + H^\Delta (t, s) \right] w^\sigma \Delta s
\]

\[- \int_{t_2}^{t} \frac{\rho(s)}{(\rho^\sigma)^2 a(s)} (w^\sigma)^2 \Delta s. \tag{2.24}\]

Therefore, by completing the square as in Theorem 2.1, we obtain

\[
\int_{t_2}^{t} \kappa H(t, s) \rho(s) q(s) \Delta s \leq H(t, t_2) w(t_2) + \int_{t_2}^{t} \frac{(\rho^\sigma)^2 a(s) [A(t, s)]^2}{4 \rho(s) H(t, s)} \Delta s. \tag{2.25}\]

Then for all $t \geq t_2$, we have

\[
\int_{t_2}^{t} \left[ \kappa H(t, s) \rho(s) q(s) - \frac{(\rho^\sigma)^2 a(s) [A(t, s)]^2}{4 \rho(s) H(t, s)} \right] \Delta s \leq H(t, t_2) w(t_2), \tag{2.26}\]

and this implies that

\[
\frac{1}{H(t, t_2)} \int_{t_2}^{t} \left[ \kappa H(t, s) \rho(s) q(s) - \frac{(\rho^\sigma)^2 a(s) [A(t, s)]^2}{4 \rho(s) H(t, s)} \right] \Delta s \leq w(t_2),
\]

for all large $t$, which contradicts (2.20). The proof is complete. \square

As an immediate consequence of Theorem 2.2 we get the following.
Corollary 2.5. Let the assumption (2.20) in Theorem 2.2 be replaced by

\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s) q(s) \Delta s = \infty,
\]

\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ \frac{(\rho^\sigma)^2 a(s) [H(t, s) \frac{\psi(s)}{\rho^\sigma} + H(t, s)]^2}{\rho(s) H(t, s)} \right] \Delta s < \infty,
\]

then every solution of (1.3) is oscillatory.

Remark 2.3. With an appropriate choice of the functions \( H \) and \( h \) one can derive a number of oscillation criteria for Eq. (1.3) on different types of time scales. Consider, for example, if we take the function \( H(t, s) = (t - s)^m, (t, s) \in D \), with \( m > 1 \), we see that \( H \) belongs to the class \( \mathcal{R} \). We claim that

\[
((t - s)^m)^\Delta \leq -m(t - \sigma(s))^{m-1}. \quad (2.27)
\]

We consider the following two cases:

Case 1. If \( \mu(t) = 0 \), then

\[
((t - s)^m)^\Delta = -m(t - s)^{m-1}.
\]

Case 2. If \( \mu(t) \neq 0 \), then we have

\[
((t - s)^m)^\Delta = \frac{1}{\mu(s)} [(t - \sigma(s))^m - (t - s)^m]
\]

\[
= -\frac{1}{\sigma(s) - s} [(t - s)^m - (t - \sigma(s))^m]. \quad (2.28)
\]

Using Hardy, Littlewood and Polya inequality

\[
x^m - y^m \geq \gamma y^{m-1} (x - y) \quad \text{for all } x \geq y > 0 \text{ and } m \geq 1,
\]

we have

\[
[(t - s)^m - (t - \sigma(s))^m] \geq m(t - \sigma(s))^{m-1} (\sigma(s) - s). \quad (2.29)
\]

Then, from (2.28) and (2.29), we have

\[
((t - s)^m)^\Delta \leq -m(t - \sigma(s))^{m-1},
\]

and this proves (2.27).

From Theorem 2.2, we obtain the following classical Kamenev-type oscillation criteria.

Corollary 2.6. Assume that \((H_1)-(H_3)\) and \((H_5)\) hold. Let \( \rho(t) \) be as in Theorem 2.1. If for \( m > 1 \)

\[
\lim_{t \to \infty} \frac{1}{t^m} \int_{t_0}^{t} \left[ \frac{k(t - s)^m \rho(s) q(s) - a(s)(\rho^\sigma)^2 B^2(t, s)}{4\rho(s)(t - s)^m} \right] \Delta s = \infty, \quad (2.30)
\]
where
\[ B(t, s) = (t - s)^m \frac{\psi(s)}{\rho^\sigma} + m(t - \sigma(s))^{m-1}, \quad t \geq s \geq t_0, \]
then every solution of (1.3) is oscillatory on \([t_0, \infty)\).

For another choice of the function \(H(t, s)\), we can use the factorial function
\[ H(t, s) = (t - s)^{(m)} \]
where \(t^{(m)} = t(t-1) \cdots (t-m+1), t^{(0)} = 1\) and establish new oscillation criteria for
Eq. (1.3). In this case
\[ H^{\Delta}(t - s)^{(m)} = \frac{(t - \sigma(s))^{(m)} - (t - s)^{(m)}}{\mu(s)} \geq -m(t - s)^{(m-1)}. \]
Note that in the special case when \(\mathbb{T} = \mathbb{R}\), we have \(\rho^\sigma = \rho(s)\). By choosing \(\rho(s) = 1\) and by
putting \(a(s) = 1, m = n - 1\), we have from Corollary 2.6 the following oscillation criterion for
the second-order differential equation (1.15) established by Yeh [36].

**Corollary 2.7.** [36, Theorem 3] Assume that \(p\) and \(q\) are positive functions and \(f(u) \geq ku\) for \(u \neq 0\). If for \(n > 2\)
\[ \limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t - s)^{n-1} q(s) \, ds = \infty \]
and
\[ \limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} \frac{a(s)C^2(t, s)}{(t - s)^{n-1}} \, ds < \infty, \]
where
\[ C(t, s) = (t - s)^{n-1} \frac{p(s)}{a(s)} + (n - 1)(t - s)^{n-2}, \quad t \geq s \geq t_0, \]
then every solution of (1.15) is oscillatory.

In the following theorem, we establish some sufficient conditions for oscillation of Eq. (1.3)
without including the coefficient \(p(t)\).

**Theorem 2.3.** Assume that \((H_1)-(H_3)\) and \((H_5)\) hold. Furthermore, assume that there exists a
positive real rd-continuous function \(\rho(t)\) such that
\[ \limsup_{t \to \infty} \int_{t_0}^{t} \left[ \kappa \rho(s)q(s) - \frac{(\rho^\Delta(s))^2 a(s)}{4\rho(s)} \right] \Delta s = \infty, \tag{2.31} \]
then every solution of Eq. (1.3) is oscillatory.
Proof. Suppose to the contrary that \( x(t) \) is a nonoscillatory solution of (1.3) and let \( t_1 \geq t_0 \) be such that \( x(t) \neq 0 \) for all \( t \geq t_1 \), so without loss of generality, we may assume that \( x(t) \) is an eventually positive solution of (1.3) with \( x(t) > 0 \) for all \( t \geq t_1 \) sufficiently large. In view of Theorem 2.1 we see that \( x(t) \) is eventually negative or eventually positive. If \( x(t) \) is eventually negative, we are then back to Case (ii) of Theorem 2.1 and we obtain a contradiction.

If \( x(t) \) is eventually positive, we assume that there exists \( t_2 \geq t_1 \) such that \( x(t) \geq 0 \) for all \( t \geq t_2 \) and proceed as in the proof of Case (i) of Theorem 2.1 and get (2.7). From (2.7), we have

\[
\omega(t) = -\kappa \rho(t) q(t) + \frac{\rho(t)}{\rho^\sigma} w^{\sigma} - \frac{\rho(t)}{(\rho^\sigma)^2 a(t)} (w^{\sigma})^2.
\]

(2.32)

The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted. \( \square \)

We remark that, as in the previous case, different choices of \( \rho(t) \) lead to different corollaries of Theorem 2.3. The details are left to the reader.

Example 2.2. Consider the dynamic equation

\[
\left( \frac{1}{t} x(t) \right)^{\Delta} + \frac{1}{\mu(t) t^2} x(t) + \frac{\gamma}{t^2} w^{\sigma} = 0, \quad t \geq 1.
\]

(2.33)

Here \( a(t) = 1/t, \, p(t) = 1/\mu(t) t^2 \) and \( q(t) = \gamma/t^2 \) with \( \kappa = 1 \) and \( \mu(t) \neq 0 \). It is clear that conditions (H1)–(H3) and (H5) are satisfied. To apply Theorem 2.3, it remains to show condition (2.31). By choosing \( \rho(t) = t \), we have

\[
\limsup_{t \to \infty} \int_{t_0}^t \left[ \kappa \rho(s) q(s) - \frac{a(s)((\rho(s))^{\Delta})^2}{4 \rho(s)} \right] \Delta s = \limsup_{t \to \infty} \int_{t_0}^t \left[ \frac{\gamma}{s^2} - \frac{1}{4s^2} \right] \Delta s = \infty, \quad \text{for} \, \gamma > 1.
\]

Thus every solution of (2.33) oscillates.

Note that in the case when \( \mathbb{T} = \mathbb{N}, \, \mu(t) = 1 \), the results by Thandapani [28, Theorem 2], Thandapani, Pandian and Lalli [31, Theorem 2.4], Thandapani, Ravi and Graef [32, Theorem 2] cannot be applied for Eq. (2.33) since

\[
\sum_{n=n_0}^{\infty} q_n < \infty,
\]

and thus our results already improve the results in [28,31,32].

In the following, we establish some oscillation criteria for (1.3) when \( f \) is differentiable.

Theorem 2.4. Assume (H1), (H2), (H4) and (H5). Furthermore, assume that there exists a positive real rd-continuous function \( \rho(t) \) such that

\[
\limsup_{t \to \infty} \int_{t_0}^t \left[ \rho(s) q(s) - \frac{a(s) \psi^2(s)}{4k \rho(s)} \right] \Delta s = \infty,
\]

(2.34)

where \( \psi \) is as defined in Theorem 2.1. Then every solution of (1.3) is oscillatory.
Proof. Suppose to the contrary that \( x(t) \) is a nonoscillatory solution of (1.3) and let \( t_1 \geq t_0 \) be such that \( x(t) \neq 0 \) for all \( t \geq t_1 \), so without loss of generality, we may assume that \( x(t) \) is an eventually positive solution of (1.3) with \( x(t) > 0 \) for all \( t \geq t_1 \) sufficiently large. In view of Theorem 2.1 we see that \( x^\Delta(t) \) is eventually negative or eventually positive. If \( x^\Delta(t) \) is eventually negative, we are then back to Case (ii) of Theorem 2.1 and obtain a contradiction.

If \( x^\Delta(t) \) is eventually positive, we assume that there exists \( t_2 \geq t_1 \) such that \( x^\Delta(t) \geq 0 \) for \( t \geq t_2 \) sufficiently large. Define the function \( w(t) \) by

\[
w(t) := \rho(t) \frac{a(t)x^\Delta(t)}{f(x(t))}, \quad t \geq t_2.
\]

Then \( w(t) > 0 \), and satisfies

\[
w^\Delta(t) = \left( ax^\Delta \right)^{\sigma} \left[ \frac{\rho(t)}{f(x(t))} \right]^\Delta + \frac{\rho(t)(a(t)x^\Delta(t))^{\Delta}}{f(x(t))}.
\]

In view of (1.3) and (2.36), we have

\[
w^\Delta(t) = -\rho(t)q(t)f(x^{\sigma}) - \rho(t)\frac{p(t)x^\Delta}{f(x(t))} + \frac{\rho^\Delta(t)}{\rho^{\sigma}}w^{\sigma} - \frac{\rho(t)(ax^{\Delta})^{\sigma}\Delta f^\Delta(x(t))}{f(x(t))f(x^{\sigma})}.
\]

But from (2.6), since \( f \) is nondecreasing, we have \( f(x^{\sigma}) \geq f(x) \) and this implies that

\[
w^\Delta(t) \leq -\rho(t)q(t) - \rho(t)\frac{p(t)x^\Delta}{f(x(t))} + \frac{\rho^\Delta(t)}{\rho^{\sigma}}w^{\sigma} - \frac{\rho(t)(ax^{\Delta})^{\sigma}\Delta f^\Delta(x(t))}{(f(x^{\sigma}))^2}.
\]

Now, by using the chain rule [8], we obtain

\[
f^\Delta(x(t)) = f'(x(\zeta))x^\Delta(t) \geq kx^\Delta(t), \quad \zeta \in [t, \sigma(t)].
\]

Then from (2.38), we have

\[
w^\Delta(t) \leq -\rho(t)q(t) - \rho(t)\frac{p(t)x^\Delta}{f(x^{\sigma})} + \frac{\rho^\Delta(t)}{\rho^{\sigma}}w^{\sigma} - \frac{k\rho(t)(ax^{\Delta})^{\sigma}x^\Delta(t)}{(f(x^{\sigma}))^2}.
\]

Again by using (2.6), we obtain

\[
w^\Delta(t) \leq -\rho(t)q(t) - \frac{p(t)\rho(t)}{a^{\sigma}\rho^{\sigma}}w^{\sigma} + \frac{\rho^\Delta(t)}{\rho^{\sigma}}w^{\sigma} - \frac{k\rho(t)}{a(t)(\rho^{\sigma})^2}(w^{\sigma})^2.
\]

The remainder of the proof is similar to that of Theorem 2.1 and is thus omitted. The proof is complete. \( \square \)

We remark that, as in the previous case, different choices of \( \rho(t) \) lead to different corollaries of the above theorem. The details are left to the reader.

Theorem 2.5. Assume that \( (H_1), (H_2), (H_4) \) and \( (H_5) \) hold. Let \( \rho(t) \) be as in Theorem 2.1 and let \( h, H : \mathbb{D} \to \mathbb{R} \) be rd-continuous functions such that \( H \) belongs to the class \( \mathfrak{R} \) and

\[
\lim_{t \to \infty} \sup_{t_0} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s)\rho(s)q(s) - \frac{(\rho^{\sigma})^2 a(s)A^2(t, s)}{4k\rho(s)H(t, s)} \right] ds = \infty.
\]

Then every solution of (1.3) is oscillatory.
Proof. The proof is similar to that of Theorem 2.3 using (2.40) and hence is omitted.

As an immediate consequence of Theorem 2.5 using \( \rho(t) = t, \ H(t, s) = (t - s)^m \) and \( m = n - 1 \), and \( \rho(t) = 1, \ H(t, s) = (t - s)^m \) and \( m = n - 1 \), we get the following results respectively.

**Corollary 2.8.** Assume that (H1), (H2), (H4) and (H5) hold. If for \( n > 2 \)

\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t - s)^{n-1} q(s) \Delta s = \infty
\]

and

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{a(s) C^2(t, s)}{(t - s)^{n-1}} \Delta s < \infty,
\]

where

\[
C(t, s) = (t - s)^{n-1} s \left( \frac{p(s)}{a(s)} - 1 \right) + (n - 1) (t - \sigma(s))^{n-2}, \quad t \geq s \geq t_0,
\]

then every solution of (1.3) is oscillatory on \([t_0, \infty)\).

**Corollary 2.9.** Assume that (H1), (H2), (H4) and (H5) hold. If for \( n > 2 \)

\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t - s)^{n-1} q(s) \Delta s = \infty
\]

and

\[
\limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{a(s) C^2(t, s)}{(t - s)^{n-1}} \Delta s < \infty,
\]

where

\[
C(t, s) = (t - s)^{n-1} \frac{p(s)}{a(s)} + (n - 1) (t - \sigma(s))^{n-2}, \quad t \geq s \geq t_0,
\]

then every solution of (1.3) is oscillatory on \([t_0, \infty)\).

Note that in the special case when \( T = \mathbb{R} \), we have \( \rho^\sigma = \rho(s) \). By choosing \( \rho(s) = 1 \) and by putting \( a(s) = 1 \), we find that Corollaries 2.8 and 2.9 reduce to the following oscillation results for the second-order differential equation (1.15) established by Yan [35] and Yeh [36].

**Corollary 2.10.** [35, Corollary 2] Assume that \( p \) and \( q \) are positive functions and \( f'(u) \geq k > 0 \). If for \( n > 2 \)

\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t - s)^{n-1} q(s) ds = \infty
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_0^t \left[ (t-s)p(s) + (n-1) \right]^2 (t-s)^{n-3} ds < \infty,
\]
then every solution of (1.15) oscillates.

**Corollary 2.11.** [36, Theorem 1] Assume that \( p \) and \( q \) are positive functions and \( f'(u) \geq k > 0 \).
If for \( n > 2 \)
\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_0^t (t-s)^{n-1} s q(s) ds = \infty
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t^{n-1}} \int_0^t s \left[ (t-s) \left( p(s) - \frac{1}{s} \right) + (n-1) \right]^2 (t-s)^{n-3} ds < \infty,
\]
then every solution of (1.15) is oscillatory.

### 3. The case where \((H_6)\) holds

Next, we consider the case when \((H_6)\) holds. We will need an additional condition:

\((H_7)\) \( (p(t))^{\Delta} \leq 0 \) for \( t \geq t_0 \), as well as
\[
\int_0^\infty q(t) \Delta t = \infty, \quad \int_0^\infty \frac{1}{a(t)} \int_0^t q(s) \Delta s \Delta t = \infty.
\]  

**Theorem 3.1.** Assume that \((H_1)-(H_3), (H_6)\) and \((H_7)\) hold. Assume further that there exists a positive function \( \rho(t) \) such that (2.1) holds. Then every solution of (1.3) oscillates or converges to zero.

**Proof.** Suppose that \( x(t) \) is a nonoscillatory solution of (1.3). Without loss of generality we may assume that \( x(t) \) is eventually positive. In view of Theorem 2.1 we see that \( x^{\Delta}(t) \) is eventually negative or eventually positive. If \( x^{\Delta}(t) \) is eventually negative, we are then back to the proof of Case (i) of Theorem 2.1 and we obtain a contradiction with (2.1). If \( x^{\Delta}(t) \) is eventually negative, then \( \lim_{n \to \infty} x(t) = b > 0 \). We prove that \( b = 0 \). If not, then \( x(t) \geq b > 0 \), and since \( f(u) \) is nondecreasing then there exists \( t_2 \geq t_1 \) such that \( f(x(t)) \geq f(b) \). Therefore, from (1.3), we have
\[
\left( a(t)x^{\Delta}(t) \right)^{\Delta} \leq -p(t)x^{\Delta}(t) - q(t)f(b).
\]  

Let \( u(t) = a(t)x^{\Delta}(t) \) for \( t \geq t_2 \). Then from (3.2), we have
\[
u^{\Delta}(t) \leq -f(b)q(t) - p(t)x^{\Delta}(t).
\]

Integrating the last inequality from \( t_2 \) to \( t \), we get
\[
u(t) \leq \nu(t_2) - f(b) \int_{t_2}^t q(s) \Delta s - \int_{t_2}^t p(s)x^{\Delta}(s) \Delta s.
\]
Now integrating by parts in the last term, we have
\[ u(t) \leq u(t_2) - f(b) \int_{t_2}^{t} q(s) \Delta s - \left[ p(s)x(s) \bigg|_{t_2}^{t} - \int_{t_2}^{t} p^\Delta(s)x^\sigma \Delta s \right]. \]

Thus by (H7), we have
\[ u(t) \leq M - f(b) \int_{t_2}^{t} q(s) \Delta s, \]

where \( M = u(t_2) + p(t_2)x^\Delta(t_2) \). In view of (H7), it is possible to choose \( t_3 \) sufficiently large such that for all \( t \geq t_3 \),
\[ u(t) \leq -\frac{f(b)}{2} \int_{t_2}^{t} q(s) \Delta s. \]

Integrating the last inequality from \( t_3 \) to \( t \), we obtain
\[ x(t) \leq x(t) - \left( \frac{f(b)}{2} \right) \int_{t_3}^{t} \left( \frac{1}{a(s)} \int_{t_2}^{s} q(u) \, du \right) \Delta s. \]

Condition (H7) implies that \( x(t) \) is eventually negative, which is a contradiction. The proof is complete. \( \square \)

As we have seen before, we can obtain different corollaries from Theorem 3.1 by choosing different \( \rho(t) \).

We remark further that since the crucial step in obtaining Theorem 3.1 is to show that eventually positive and eventually increasing solutions of (1.3) do not exist, then we have analogues of Theorems 2.2–2.5. For example the analogue of Theorem 2.2 is the following.

**Theorem 3.2.** Suppose (H1)–(H3), (H6) and (H7) hold. Let \( \rho(t) \) be as defined in Theorem 2.1 and let \( h, H : \mathbb{D} \rightarrow \mathbb{R} \) be rd-continuous functions such that \( H \) belongs to the class \( \mathcal{R} \) and (2.20) holds. Then every solution of (1.3) oscillates or converges to zero.

The other results can similarly be stated.

**References**