ELSEVIER
Topology and its Applications 107 (2000) 131–145
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

On some categorical properties of the functor U_R ☆

Yuri V. Sadovnichy

Chair of General Topology and Geometry, Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119899, Russia

Received 3 November 1998; received in revised form 11 February 1999

Abstract

We deal with the unit ball $U_R(X)$ of non-negative Radon measures on a Tychonoff space X. U_R is a functor in the category *Tych*. It is proved that U_R has all properties of a normal functor, with the exception of point preservation. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Radon measure; Functor; Borel selection

AMS classification: 46E27; 54E70

Introduction

The functor P_R of Radon probability measures in the category *Tych* of all Tychonoff spaces and continuous mappings was defined and studied by Banakh [2,3]. The author obtained analogous results for the functor P_β of probability measures with compact supports [9,11–14]. In [15] the functor $U_\tau : \mathit{Tych} \to \mathit{Tych}$ of τ-additive measures with norm $\leqslant 1$ was defined and studied.

In this paper we investigate the functor $U_R : \mathit{Tych} \to \mathit{Tych}$ of Radon measures with norm $\leqslant 1$. This functor is a subfunctor of the functor U_τ and an extension of the functor $U : \mathit{Comp} \to \mathit{Comp}$, where *Comp* is the category of all compacta (Hausdorff compact spaces) and continuous mappings, and for a compactum K the space $U(K)$ is the unit ball of the set $M_r(K)$ of all regular Borel measures in K equipped with $*$-weak topology.

Our main goal is to show that U_R has all properties of a normal functor, with the exception of point preservation. Namely:

(1) U_R is monomorphic (preserves embeddings);

☆ This paper was written while the author was supported by Russian Foundation for Basic Research, grant 97-01-00357.

E-mail address: vsadov@nw.math.msu.su (Yu.V. Sadovnichy).

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00090-5

(2) U_R is epimorphic (transfers surjections into mappings with dense images);
(3) U_R preserves intersections of closed subsets;
(4) U_R preserves inverse images;
(5) U_R preserves weight;
(6) U_R is continuous with respect to inverse limits.

We prove also that the functor U_R has some additional properties. For example, U_R preserves perfect mappings, Čech-complete spaces, paracompact p-spaces. Besides, U_R transfers open mappings between metric spaces, having local Borel selections, into open mappings.

In Section 1 we give necessary information concerning measures. The background is rather extensive (fills up about a half of the paper). The main reason for this is the following one. There are two approaches to measures: a measure as a function of Borel subsets, and a measure as a linear functional (integral). For compact spaces and regular τ-additive measures these two approaches coincide by Riesz' theorem. But in arbitrary Tychonoff spaces the situation is more complicated. In the main part of the article (Section 2) we use both definitions of a measure. So we have to be very punctual. For this, we map the set $M_R(X)$ of Radon measures in a Tychonoff space X into the set $M_R(\beta X)$ and study properties of this mapping and its submappings. Doing so, we can avoid certain confusions, especially when we define and study mappings of type $U_R(f)$.

All spaces in this paper are assumed to be Tychonoff. Any needed additional information from General Topology can be found, for example, in [5].

1. Preliminaries

We recall basic definitions and facts. Let $\mathcal{B}(X)$ be the σ-algebra of all Borel subsets of a space X.

Proposition 1.1 [10, Proposition 3.4]. *If $Y \subset X$, then*

$$\mathcal{B}(Y) = \mathcal{B}(X)|Y \equiv \{B \cap Y\colon\ B \in \mathcal{B}(X)\}.$$

A *Borel measure* on $\mathcal{B}(X)$ (or in X) is a countably additive non-negative function

$$\mu : \mathcal{B}(X) \to [0, +\infty).$$

The set of all Borel measures in X is denoted by $M(X)$. For $\mu \in M(X)$ we set $\|\mu\| = \mu(X)$.

Definition 1.2. A Borel measure μ in X is called
(a) a *probability measure* if $\|\mu\| = 1$;
(b) *regular* if $\mu(B) = \sup\{\mu(F)\colon\ F \subset B,\ F$ is closed$\}$;
(c) *weakly Radon* if $\mu(X) = \sup\{\mu(K)\colon\ K \subset X,\ K$ is compact$\}$;
(d) *Radon* if $\mu(B) = \sup\{\mu(K)\colon\ K \subset B,\ K$ is compact$\}$ for any $B \in \mathcal{B}(X)$;
(e) *τ-additive* if for every open set $G_0 \subset X$ we have $\mu(G_0) = \sup\{\mu(G)\colon\ G \in \mathcal{G}_0\}$, where $\mathcal{G}_0$ is an arbitrary upwards directed family of open subsets of X with $G_0 = \bigcup \mathcal{G}_0$.

The sets of all regular, τ-additive, Radon measures in X are denoted by $M_r(X)$, $M_\tau(X)$, $M_R(X)$, respectively. Evidently,

$$M_R(X) \subset M_\tau(X) \quad \text{for an arbitrary } X. \tag{1.1}$$

Corollary 6.11 from [10] implies

Proposition 1.3. *Every τ-additive measure is regular.*

In view of Proposition 1.3, the next statement is trivial.

Proposition 1.4. *For every compactum K we have*

$$M_R(K) = M_\tau(K) = M_r(K).$$

It is easy to see that one can partially strengthen this assertion.

Proposition 1.5. *For every Lindelöf space X we have*

$$M_\tau(X) = M_r(X).$$

Let

$$U_R(X) = \{\mu \in M_R(X)\colon \|\mu\| \leqslant 1\}.$$

In a similar way we define the sets $U_\tau(X)$ and $U_r(X)$. By $P_R(X)$, $P_\tau(X)$ and $P_r(X)$ we denote the subsets of $U_R(X)$, $U_\tau(X)$ and $U_r(X)$ consisting of all probability measures. If K is a compactum, then for the sake of brevity we denote by $U(K)$ the set $U_R(K) = U_\tau(K) = U_r(K)$. Similarly, by $P(K)$ we denote the set of probability measures $P_R(K) = P_\tau(K) = P_r(K)$.

Let $X \subset \beta X$ be the identity embedding. For $B \subset \mathcal{B}(X)$ and $\mu \in M(\beta X)$, set

$$r_X(\mu)(B) = \inf\{\mu(C)\colon C \in \mathcal{B}(\beta X),\ C \cap X = B\}. \tag{1.2}$$

Definition 1.2 is correct in view of Proposition 1.1.

Proposition 1.6 [10, Construction 3.5]. *$r_X(\mu)$ is a Borel measure in X.*

So, we have the function $r_X : M(\beta X) \to M(X)$. Now let $\mu \in M(X)$ and let $B \in \mathcal{B}(\beta X)$. Set

$$e^X(\mu)(B) = \mu(B \cap X). \tag{1.3}$$

Proposition 1.7 [10, Construction 3.7]. *$e^X(\mu)$ is a Borel measure in βX with $\|e^X(\mu)\| = \|\mu\|$.*

Thus, we have the function $e^X : M(X) \to M(\beta X)$.

The next statement is trivial:

Proposition 1.8. $r_X \circ e^X = \mathrm{id}_{M(X)}$.

The next assertion is also rather simple, but very important:

Proposition 1.9 [10, Proposition 3.8]. *A measure $\mu \in M(X)$ is τ-additive iff $e^X(\mu)$ is τ-additive. In particular, e^X maps $M_\tau(X)$ into $M_\tau(\beta X)$.*

Now let us set

$$M^*(\beta X) = \{\mu \in M_r(\beta X)\colon\ \mu(K) = 0 \text{ for any compactum } K \subset \beta X \setminus X\},$$
$$M_*(\beta X) = \{\mu \in M_r(\beta X)\colon\ \mu(\beta X) = \mu_*(X)\},$$

where

$$\mu_*(X) = \sup\{\mu(K)\colon\ K \subset X \text{ is a compactum}\}.$$

It is clear that

$$M_*(\beta X) \subset M^*(\beta X). \tag{1.4}$$

If there can be no confusion, we shall denote the restrictions of e^X and r_X onto arbitrary subsets of $M(X)$ and $M(\beta X)$ by the same symbols e^X and r_X.

Proposition 1.10. *For an arbitrary space X the functions*

$$e^X : M_\tau(X) \to M^*(\beta X) \quad \textit{and} \quad r_X : M^*(\beta X) \to M_\tau(X)$$

are bijections inverse to each other.

Proof. First of all, let us check

$$e^X \circ r_X | M^*(\beta X) = \mathrm{id}. \tag{1.5}$$

Since both μ and $e^X(r_X(\mu))$ are Borel measures in βX, to prove (1.5) it is sufficient to verify

$$\mu(K) = e^X\big(r_X(\mu)\big)(K) \tag{1.6}$$

for an arbitrary compactum $K \subset \beta X$ and $\mu \in M^*(\beta X)$. We have $e^X(r_X(\mu))(K) =$ (by (1.3)) $= r_X(\mu)(K \cap X) \leqslant$ (by (1.2)) $\leqslant \mu(K)$. So,

$$\mu(K) \geqslant e^X\big(r_X(\mu)\big)(K). \tag{1.7}$$

From (1.7) and Proposition 1.8 we obtain (1.6). Therefore, (1.5) holds.

Proposition 1.9 and (1.5) imply

$$r_X(M^*\big(\beta X)\big) \subset M_\tau(X). \tag{1.8}$$

Now let $\mu \in M_\tau(X)$, and let $K \subset \beta X \setminus X$ be a compactum. Proposition 1.7 yields

$$\mu(X) = e^X(\mu)(\beta X). \tag{1.9}$$

Further, $\mu(X) =$ (by Proposition 1.8) $= r_X(e^X(\mu))(X) \leqslant$ (by (1.2)) $\leqslant e^X(\mu)(\beta X \setminus K) \leqslant e^X(\mu)(\beta X) =$ (by (1.9)) $= \mu(X)$. Hence, $e^X(\mu)(\beta X \setminus K) = e^X(\mu)(\beta X)$ or, in other words, $e^X(\mu)(K) = 0$. Therefore, $e^X(\mu) \in M^*(\beta X)$. So,

$$e^X\big(M_\tau(X)\big) \subset M^*(\beta X). \tag{1.10}$$

Then (1.10), (1.8) and (1.5) imply

$$e^X\big(M_\tau(X)\big) = M^*(\beta X). \tag{1.11}$$

But Proposition 1.8 implies that

$$e^X : M_\tau(X) \to M(X)$$

is an injection. Consequently,

$$e^X : M_\tau(X) \to M^*(\beta X)$$

is a bijection. Applying Proposition 1.8 once more we get that

$$r|M^*(\beta X) = \big(e^X|M_\tau(X)\big)^{-1}.$$

Proposition 1.10 is proved. □

Now we may identify measures $\mu \in M_\tau(X)$ with measures $e^X(\mu) \in M^*(\beta X)$.

Proposition 1.11. $e^X(M_R(X)) = M_*(\beta X)$.

Proof. Let $\mu \in M_R(X)$. Then for an arbitrary positive ε there is a compactum $K \subset X$ such that

$$\mu(X) - \mu(K) < \varepsilon.$$

But $e^X(\mu)(\beta X) = \mu(X)$ by Proposition 1.7, and $e^X(\mu)(K) = \mu(K)$ by definition. Hence

$$e^X(\mu)(\beta X) - e^X(\mu)(K) < \varepsilon.$$

Consequently, $e^X(\mu)(\beta X) = e^X(\mu)_*(X)$. This implies that $e^X(\mu) \in M_*(\beta X)$.

Conversely, let $\mu \in M_*(\beta X)$. Then for a given $\varepsilon > 0$ there is a compactum $K \subset X$ such that

$$\mu(\beta X) - \mu(K) < \varepsilon. \tag{1.12}$$

On the other hand, in view of Proposition 1.10 and (1.4) there exists a unique measure $\nu \in M_\tau(X)$ such that $e^X(\nu) = \mu$. Then, as above, we have $\nu(X) = \mu(\beta X)$ and $\nu(K) = \mu(K)$. Hence, (1.12) yields

$$\nu(X) - \nu(K) < \varepsilon.$$

Therefore, ν is a weakly Radon measure. On the other hand, ν is regular in view of Proposition 1.3. But, clearly, every regular weakly Radon measure is a Radon measure. Consequently, $\nu \in M_R(X)$, and $\mu \in e^X(M_R(X))$. Proposition 1.11 is proved. □

Remark 1.12.

Now we may identify measures $\mu \in M_R(X)$ $(M_\tau(X))$ with measures $e^X(\mu) \in M_*(\beta X)$ $(M^*(\beta X))$, respectively. In what follows, by Radon (τ-additive) measures we shall mean, as a rule, measures from $M_*(\beta X)$ $(M^*(\beta X))$.

Let us recall some notions and facts concerning regular measures in compacta. For a compactum K by $C(K)$ we denote the Banach space of all continuous functions

$\varphi : K \to \mathbb{R}$. By Riesz' theorem, the set $M_r(K)$ is embedded into the dual space $C(K)^*$. Besides the topology of a normed space, $C(K)^*$ can be equipped with $*$-weak topology by the identity embedding

$$C(K)^* \subset \mathbb{R}^{C(K)}.$$

Riesz' embedding

$$\int : M_r(K) \hookrightarrow C(K)^*$$

induces $*$-weak topology on $M_r(K) = M_\tau(K)$. In particular, $M_\tau(K)$ is a Tychonoff space. At last, the bijection

$$e^X : M_\tau(X) \to M^*(\beta X)$$

induces $*$-weak topology on $M_\tau(X)$ and $M_R(X)$, and other subsets of $M_\tau(X)$ for an arbitrary space X. So, Propositions 1.10 and 1.11 imply

Proposition 1.13. *The mappings*

$$e^X : M_\tau(X) \to M^*(\beta X) \quad \textit{and} \quad e^X : M_R(X) \to M_*(\beta X)$$

are homeomorphisms.

In $*$-weak topology $U(K)$ for a compactum K is compact being a closed subset of the Tychonoff cube $I^{C(K)}$ (more precise, $\prod\{[-\|\varphi\|, \|\varphi\|]\colon \varphi \in C(K)\}$). The space $P(K)$ is also compact as a closed subset of $U(K)$. Set

$$U^*(\beta X) = M^*(\beta X) \cap U(\beta X);$$
$$U_*(\beta X) = M_*(\beta X) \cap U(\beta X);$$
$$P^*(\beta X) = U^*(\beta X) \cap P(\beta X);$$
$$P_*(\beta X) = U_*(\beta X) \cap P(\beta X).$$

There are important particular cases of Proposition 1.13.

Corollary 1.14. *The mappings*

$$e^X : U_\tau(X) \to U^*(\beta X) \quad \textit{and} \quad e^X : U_R(X) \to U_*(\beta X)$$

are homeomorphisms.

Corollary 1.15. *The mappings*

$$e^X : P_\tau(X) \to P^*(\beta X) \quad \textit{and} \quad e^X : P_R(X) \to P_*(\beta X)$$

are homeomorphisms.

Remark 1.16.

In slightly different terms the first part of Corollary 1.14 was proved in [15, Section 2], and Corollary 1.15 was obtained by Banakh [2, Section 0].

For a continuous mapping $f : K_1 \to K_2$ we define the mapping $M_r(f) : M_r(K_1) \to M_r(K_2)$ by

$$M_r(f)(\mu)(\varphi) = \mu(\varphi \circ f) \tag{1.13}$$

for arbitrary $\mu \in M_r(K_1)$ and $\varphi \in C(K_2)$. In this definition we identify the measure μ with the linear functional $\int_\mu : C(K_1) \to \mathbb{R}$. The next assertion is well known and easily follows from the definitions of $*$-weak topology, $M_r(f)$ and $\|\mu\|$. As for the last definition, let us notice that $\|\mu\| = \mu(K) = \mu(1_K)$, where $1_K(x) = 1$ for any $x \in K$.

Proposition 1.17. *The mapping $M_r(f)$ is continuous and, moreover, $\|\mu\| = \|M_r(f)(\mu)\|$ for every $\mu \in M_r(K_1)$.*

Let $f_1 : K_1 \to K_2$ and $f_2 : K_2 \to K_3$ be continuous mappings between compacta. Then (1.13) implies

$$M_r(f_2 \circ f_1) = M_r(f_2) \circ M_r(f_1). \tag{1.14}$$

The equality (1.14) yields

Proposition 1.18. *$M_r : Comp \to Tych$ is a covariant functor.*

Propositions 1.17 and 1.18 imply that

$$U : Comp \to Comp \quad \text{and} \quad P : Comp \to Comp$$

are subfunctors of the functor M_r (the mappings $U(f)$ and $P(f)$ are defined as in (1.13)). It is known that P is a *normal functor* (for details see, for example, [7] or [8]). In the same way as for the functor P, one can show that the functor U has all properties of a normal functor with the exception of point preservation.

If $f : X \to Y$ is a continuous mapping between Tychonoff spaces, we can define the mapping $M(f) : M(X) \to M(Y)$ by

$$M(f)(\mu)(B) = \mu\big(f^{-1}(B)\big), \tag{1.15}$$

where $B \in \mathcal{B}(Y)$. The next statement is well known:

Proposition 1.19. *If $f : K_1 \to K_2$ is a continuous mapping between compacta, then $M(f) = M_r(f)$.*

Corollary 1.20. *Let $f : K_1 \to K_2$ be a continuous mapping, and let F be a closed subset of K_1. Then for any $\mu \in M_r(K_1)$ we have*

$$\mu(F) \leqslant M_r(\mu)\big(f(F)\big).$$

If $f : X \to Y$ is a continuous mapping, then by

$$\beta f : \beta X \to \beta Y$$

we denote the natural extension of f over βX. It easily follows from the definition that

$$(\beta f)^{-1}(\beta Y \setminus Y) \subset \beta X \setminus X. \tag{1.16}$$

Proposition 1.21. *If $f : K_1 \to K_2$ is a continuous mapping, then*

$$M_r(\beta f)\big(M^*(\beta X)\big) \subset M^*(\beta Y) \quad \text{and} \quad M_r(\beta f)\big(M_*(\beta X)\big) \subset M_*(\beta Y).$$

Proof. Let $\mu \in M^*(\beta X)$, and let K be a compact subset of $\beta Y \setminus Y$. According to (1.16),

$$(\beta f)^{-1}(K) \subset \beta X \setminus X.$$

Hence, $\mu((\beta f)^{-1}(K)) = 0$. Then $M_r(\beta f)(\mu)(K) =$ (in view of Proposition 1.19) $= \mu((\beta f)^{-1}(K)) = 0$. Consequently, $M_r(\beta f)(\mu) \in M^*(\beta Y)$.

Now let $\mu \in M_*(\beta X)$, and let ε be an arbitrary positive number. There is a compact set $K \subset X$ such that

$$\mu(\beta X) - \mu(K) < \varepsilon.$$

Then $M_r(\beta f)(\mu)(\beta Y) - M_r(\beta f)(\mu)(\beta f(K)) =$ (by Proposition 1.19) $= \mu((\beta f)^{-1}(\beta Y)) - \mu((\beta f)^{-1}(\beta f(K))) \leqslant \mu(\beta X) - \mu(K) < \varepsilon$. Therefore, $M_r(\beta f)(\mu) \in M_*(\beta Y)$, since $\beta f(K) = f(K)$ is a compact subset of Y. Proposition 1.21 is proved. □

Corollary 1.22. *If $f : X \to Y$ is a continuous mapping, then*

$$U(\beta f)\big(U^*(\beta X)\big) \subset U^*(\beta Y) \quad \text{and} \quad U(\beta f)\big(U_*(\beta X)\big) \subset U_*(\beta Y).$$

The first part of this corollary was proved in [15, Section 2].

2. The functor U_R and its basic properties

We start with definitions. If $f : X \to Y$ is a continuous mapping, we set

$$U_R(f) = U(\beta f)|U_*(\beta X). \tag{2.1}$$

By virtue of Corollary 1.22, the definition (2.1) gives us the mapping

$$U_R(f) : U_R(X) \to U_R(Y). \tag{2.2}$$

Here we identify $U_R(Z)$ with $U_*(\beta Z)$ for any Z. If we prefer to consider $U_R(Z)$ as the space of Radon measures in Z, then in view of Propositions 1.8 and 1.13, the definition (2.1) can be written as

$$U_R(f) = r_Y \circ U_*(\beta f) \circ e^X. \tag{2.3}$$

The mapping $U_\tau(f)$ is defined in the same way.

Theorem 2.1 [15, Theorem 2.2]. *U_τ is a covariant functor in the category Tych which extends the functor $U : Comp \to Comp$.*

The definition (2.1), Theorem 2.1, the statement (1.4) and Corollary 1.22 yield

Theorem 2.2. *U_R is a covariant functor in the category Tych, that is an extension of the functor $U : Comp \to Comp$ and a subfunctor of the functor $U_\tau : Tych \to Tych$.*

Proposition 2.3. *The functor U_R preserves the class of injective mappings.*

Proof. Let $f : X \to Y$ be an injective mapping and $\mu_1, \mu_2 \in U_R(X)$, $\mu_1 \neq \mu_2$. Every Radon measure on X is uniquely defined by its values on compact subsets of X. Then there exists a compactum $K \subset X$ such that $\mu_1(K) \neq \mu_2(K)$. Then $f(K)$ is a compact subspace of Y. Moreover, $U_R(f)(\mu_1)(f(K)) = \mu_1(f^{-1}(f(K))) = \mu_1(K) \neq \mu_2(K) = U_R(f)(\mu_2)(f(K))$, so $U_R(f)(\mu_1) \neq U_R(f)(\mu_2)$. Proposition 2.3 is proved. □

Proposition 2.4. *The functor U_R preserves the class of all embeddings.*

Proof. In [15, Theorem 3.3] it was proved that U_τ preserves embeddings. Hence, an application of Theorem 2.2 finishes the proof. □

Proposition 2.5. *The functor U_R preserves inverse images, i.e., for any continuous mapping $f : X \to Y$ and for any subset $A \subset Y$, the equality $U_R(f)^{-1}(U_R(A)) = U_R(f^{-1}(A))$ holds.*

Proof. It is clear that $U_R(f^{-1}(A)) \subset U_R(f)^{-1}(U_R(A))$. We will show that

$$U_R(f)^{-1}\big(U_R(A)\big) \subset U_R\big(f^{-1}(A)\big).$$

Let $\mu \in U_R(X)$ be a measure such that $U_R(f)(\mu) \in U_R(A)$. Let $\varepsilon > 0$. On the one hand, there is a compactum $K_1 \subset X$ such that

$$\mu(\beta X) - \mu(K_1) < \frac{\varepsilon}{2}. \tag{2.4}$$

On the other hand, $U_R(A) \subset U_R(Y)$, because of Proposition 2.4. Hence, there is a compactum $K_2 \subset A$ such that

$$U_R(f)(\mu)(\beta Y) - U_R(f)(\mu)(K_2) < \frac{\varepsilon}{2}. \tag{2.5}$$

Proposition 1.19, the inequality (2.5) and the definition (2.1) imply

$$\mu\big((\beta f)^{-1}(\beta Y)\big) - \mu\big((\beta f)^{-1}(K_2)\big) < \frac{\varepsilon}{2},$$

or

$$\mu(\beta X) - \mu\big((\beta f)^{-1}(K_2)\big) < \frac{\varepsilon}{2}. \tag{2.6}$$

Set $K_3 = (\beta f)^{-1}(K_2)$. Then

$$\mu(K_3) = \mu(K_1 \cap K_3) + \mu(K_3 \setminus K_1).$$

Consequently, $\frac{1}{2}\varepsilon >$ (by (2.6)) $> \mu(\beta X) - \mu(K_3) = \mu(\beta X) - \mu(K_1 \cap K_3) - \mu(K_3 \setminus K_1) \geqslant \mu(\beta X) - \mu(K_1 \cap K_3) - \mu(\beta X \setminus K_1) >$ (by (2.4)) $> \mu(\beta X) - \mu(K_1 \cap K_3) - \frac{1}{2}\varepsilon$. Therefore,

$$\mu(\beta X) - \mu(K_1 \cap K_3) < \varepsilon. \tag{2.7}$$

But $X \cap (\beta f)^{-1}(Z) = f^{-1}(Z)$ for any $Z \subset Y$. Hence, $K_1 \cap K_3$ is a compact subset of $f^{-1}(A)$. So, the inequality (2.7) shows that $\mu \in U_R(f^{-1}(A))$. Proposition 2.5 is proved. □

Theorem 2.6. *The functor U_R preserves the class of perfect mappings.*

Proof. Let $f : X \to Y$ be a perfect mapping of Tychonoff spaces. Then $\beta f : \beta X \to \beta Y$ is a perfect mapping, being a continuous mapping between compacta. By the same reason, $U(\beta f)$ is perfect. Hence, to prove the perfectness of $U_R(f)$, we have to check, in accordance with (2.1), that

$$U(\beta f)^{-1}\bigl(U_R(Y)\bigr) = U_R(X).$$

To check this equality, it suffices, in view of Proposition 2.5, to show that

$$(\beta f)^{-1}(Y) = X. \tag{2.8}$$

Since f is perfect, $\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$ [5, Theorem 3.7.15]. Adding (1.16) to this inclusion we obtain (2.8). Theorem 2.6 is proved. □

From Proposition 2.4 and Theorem 2.6 we have

Corollary 2.7. *The functor U_R preserves the class of all closed embeddings.*

Definition 2.8. Let $F_i : \mathcal{C} \to \mathcal{C}'$, $i = 1, 2$, be covariant functors from a category $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ into a category $\mathcal{C}' = (\mathcal{O}', \mathcal{M}')$. A family of morphisms $\Phi = \{\varphi_X : F_1(X) \to F_2(X),\ X \in \mathcal{O}\} \subset \mathcal{M}'$ is said to be a *natural transformation* of the functor F_1 into the functor F_2 if for any morphism $f : X \to Y$ from $\mathcal{M}$ the diagram

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\varphi_X} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\varphi_Y} & F_2(Y) \end{array}$$

is commutative.

For any Tychonoff space X, let $\delta_X : X \to U_R(X)$ be the mapping which maps every point $x \in X$ into its Dirac measure $\delta(x)$.

From (1.4) and [15, Theorem 3.6] it follows:

Theorem 2.9. *The family $\delta = \{\delta_X\}$ defines a natural transformation of the identity functor* Id : *Tych* $\to$ *Tych into the functor U_R : Tych $\to$ Tych. Moreover, every component $\delta_X : X \to U_R(X)$ is a closed embedding.*

By analogy with [15, Proposition 3.1] we can prove

Proposition 2.10. *Let $f : X \to Y$ be a mapping such that $f(X)$ is everywhere dense in Y. Then $U_R(f)(U_R(X))$ is everywhere dense in $U_R(Y)$.*

Lemma 2.11. *Let X be a Tychonoff space, and let $B \subset X$ be its Borel subset. Then $U_R(B) = U_\tau(B) \cap U_R(X) \subset U(\beta X)$.*

Proof. It is clear that $U_R(B) \subset U_\tau(B) \cap U_R(X)$. Let $\mu \in U_\tau(B) \cap U_R(X)$. Since B is a Borel subset in X, there exists a Borel subset $B_1 \subset \beta X$ such that $r_X(\mu)(B) = \mu(B_1)$ and $B_1 \cap X = B$. Further, $\mu_*(X) = \mu(\beta X)$. Since the measure μ is regular, for every $\varepsilon > 0$ there exists a compactum $K_1 \subset B_1$ such that $\mu(B_1 \setminus K_1) < \frac{1}{2}\varepsilon$. From the definition of $\mu_*(X)$ it follows that there exists a compactum $K_2 \subset X$ such that $\mu(K_2) > \mu(\beta X) - \frac{1}{2}\varepsilon$. Then $K = K_1 \cap K_2 \subset B_1 \cap X = B$ is a compactum in B. Since $B_1 \setminus K = B_1 \setminus (K_1 \cap K_2) = (B_1 \setminus K_1) \cup (B_1 \setminus K_2) \subset (B_1 \setminus K_1) \cup (\beta X \setminus K_2)$, we have $\mu(B_1 \setminus K) \leqslant \mu(B_1 \setminus K_1) + \mu(\beta X \setminus K_2) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. But $K \subset B$ and ε is arbitrary. Thus, $\mu_*(B) = \mu(B_1)$. Therefore, $\mu \in U_R(B)$. Lemma 2.11 is proved. □

From [15, Theorem 3.4] and Lemma 2.11 we obtain

Theorem 2.12. *The functor U_R preserves the intersection of closed subsets, i.e, for any Tychonoff space X and for any its closed subsets X_α, $\alpha \in A$, we have $U_R(\bigcap_{\alpha\in A} X_\alpha) = \bigcap_{\alpha\in A} U_R(X_\alpha)$.*

Now we will examine the continuity of the functor U_R. Let A be an upwards directed partially ordered set, and let $\{X_\alpha, p_\alpha^\gamma\}$ be a spectrum consisting of Tychonoff spaces. By $\varprojlim X_\alpha$ we denote the limit of this spectrum, by $p_\alpha : \varprojlim X_\alpha \to X_\alpha$, $\alpha \in A$, we denote the limit projections. The spectrum $\{X_\alpha, p_\alpha^\gamma\}$ generates the spectrum $\{U_R(X_\alpha), U_R(p_\alpha^\gamma)\}$. We denote its limit by $\varprojlim U_R(X_\alpha)$ and its limit projections by $\mathrm{pr}_\alpha : \varprojlim U_R(X_\alpha) \to U_R(X_\alpha)$. The mappings $U_R(p_\alpha) : U_R(\varprojlim X_\alpha) \to U_R(X_\alpha)$ generate a mapping $T : U_R(\varprojlim X_\alpha) \to \varprojlim U_R(X_\alpha)$. Since the functor U is continuous in the category *Comp*, the mapping T is an homeomorphism if X_α is compact for any α.

Theorem 2.13. *The mapping $T : U_R(\varprojlim X_\alpha) \to \varprojlim U_R(X_\alpha)$ is an embedding. If*

$$p_\alpha : \varprojlim X_\alpha \to X_\alpha$$

are dense (that is, $p_\alpha(\varprojlim X_\alpha)$ is everywhere dense in X_α), then $T(U_R(\varprojlim X_\alpha))$ is everywhere dense in $\varprojlim U_R(X_\alpha)$. If A is countable, then T is an homeomorphism.

Proof. Let $\{\beta X_\alpha, \beta(p_\alpha^\gamma)\}$ be the Stone–Čech compactification of the spectrum $\{X_\alpha, p_\alpha^\gamma\}$. It is clear that the limit mapping $\varprojlim X_\alpha \to \varprojlim \beta X_\alpha$ is an embedding. Moreover, $\varprojlim X_\alpha$ is everywhere dense in $\varprojlim \beta X_\alpha$ if p_α are dense. Then the mapping $\overline{T} : U(\varprojlim \beta X_\alpha) \to \varprojlim U(\beta X_\alpha)$ is an homeomorphism. From Propositions 2.4 and 2.10 we obtain the first and the second statement of this theorem.

Let A be countable. We will show that the mapping $T : U_R(\varprojlim X_\alpha) \to \varprojlim U_R(X_\alpha)$ is an homeomorphism. It is enough to prove that the mapping T is surjective. Let $\{\mu_\alpha\}_{\alpha\in A} \in \varprojlim U_R(\beta X_\alpha)$. We will show that

$$\mu = \overline{T}^{-1}(\{\mu_\alpha\}_{\alpha\in A}) \in U_R(\varprojlim X_\alpha) \subset U_R(\varprojlim \beta X_\alpha).$$

Let $\varepsilon > 0$. Let $\xi : A \to \mathbb{N}$ be a bijection. For every $\alpha \in A$ there exists a compactum $K_\alpha \subset X_\alpha$ such that $\mu_\alpha(K_\alpha) > \mu(\beta X) - \varepsilon \cdot 2^{-\xi(\alpha)}$. It is clear that the set $K = \{(x_\alpha)_{\alpha\in A} \in \varprojlim X_\alpha :\ p_\alpha(x_\alpha) \in K_\alpha, \alpha \in A\}$ is compact. Moreover,

$$\mu\big((\varprojlim X_\alpha)\setminus K\big) \leqslant \sum_{\alpha\in A}\mu\big(p_\alpha^{-1}(X_\alpha\setminus K_\alpha)\big) = \sum_{\alpha\in A}\mu_\alpha(X_\alpha\setminus K_\alpha)$$
$$\leqslant \sum_{\alpha\in A}\varepsilon\cdot 2^{-\xi(\alpha)} = \varepsilon.$$

Hence, the mapping T is surjective. Theorem 2.13 is proved. □

Definition 2.14. A mapping $f : X \to Y$ *has a Borel selection* if there exists a function $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$ and for every open set $V \subset X$ the set $s^{-1}(V)$ is a Borel subset of Y.

Proposition 2.15. *Let $f : X \to Y$ be a mapping between separable metric spaces which has a Borel selection. Then the mapping $U_R(f) : U_R(X) \to U_R(Y)$ is surjective.*

Proof. Here we consider measures as functions of Borel sets and implicitly use the definition (2.3) of the mapping $U_R(f)$. Let $s : Y \to X$ be a Borel selection of the mapping f. For an arbitrary measure $\mu \in U_R(Y)$ we take a measure $\nu \in U(X)$ such that $\nu(A) = \mu(f(A \cap s(Y)))$ for any Borel set $A \subset X$. We will show that the measure ν is Radon. It is enough to prove that for an arbitrary $\varepsilon > 0$ there is a compactum $K \subset X$ such that $\nu(K) > \nu(\beta X) - \varepsilon$. Since the mapping $s : Y \to X$ is Borel measurable, there exists a closed subset $C \subset Y$ such that $\mu(C) > \mu(\beta Y) - \frac{1}{2}\varepsilon$ and the mapping $s|C : C \to X$ is continuous [6, 2.3.5]. Since the measure μ on Y is Radon, there is a compactum $K \subset C$ such that $\mu(C \setminus K) < \frac{1}{2}\varepsilon$. Then $s(K) \subset X$ is a compactum. Moreover, $\nu(s(K)) = \mu(f(s(K) \cap s(Y))) = \mu(f(s(K))) = \mu(K) > \mu(\beta Y) - \varepsilon$. So, the measure ν on X is Radon and $U_R(f)(\nu) = \mu$. Proposition 2.15 is proved. □

Definition 2.16. A mapping $f : X \to Y$ *has local Borel selections* if for any open set $V \subset X$ there exists a Borel selection $s : Y \to X$ of the mapping f such that $s(f(V)) \subset V$.

Theorem 2.17. *Let $f : X \to Y$ be an open mapping between separable metric spaces which has local Borel selections. Then the mapping $U_R(f) : U_R(X) \to U_R(Y)$ is open.*

Proof. We start with the same remark as at the beginning of the proof of Proposition 2.15. The system of sets

$$\mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) = \big\{\mu \in U_R(X) : \ \mu(U_i) - \mu_0(U_i) > -\varepsilon, \\ 1 \leqslant i \leqslant n, \ |\mu(X) - \mu_0(X)| < \varepsilon\big\},$$

where $\varepsilon > 0$, $\mu_0 \in U_R(X)$ and $U_1, \ldots, U_n$ are open sets in X, is a base of a topology on $U_R(X)$ [16, II, § 1]. One has to note that Varadarajan considered spaces of Baire measures. But in every perfectly normal, in particular, in every metric space, each Baire measure is a Borel one. Let $\mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon)$ be a basic set. We will show that $U_R(f)(\mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon))$ is a neighborhood of the measure ν_0, where $\nu_0 = U_R(f)(\mu_0) \in U_R(Y)$. First we will find a basic neighborhood $\mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon') \subset$

$\mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon)$ such that V_i, $1 \leqslant i \leqslant m$, are open subsets of X and $V_i \cap V_j = \emptyset$ if $i \neq j$.

By N we denote the set $\{1, \ldots, n\}$. Let $\exp N$ be the set of all non-empty subsets of N. It is easy to introduce a linear order on this set satisfying the property: if $B \subset A$, then $A \leqslant B$. It is clear that $|\exp N| < 2^n$. Let $\varepsilon' = \varepsilon/2^{n+1}$. For every $A \subset N$ we denote $U_A = \bigcap_{i \in A} U_i$. By induction, for every $A \subset N$ we find an open set $V_A \subset X$ such that $\overline{V_A} \subset U_A \setminus \bigcup_{B<A} \overline{V_B}$ and $\mu_0(V_A) > \mu_0(U_A \setminus \bigcup_{B<A} \overline{V_B}) - \varepsilon'$. Here $\overline{V}$ is the closure of a set V. It is clear that for any $A \subset N$ we have $\mu_0(\overline{V_A} \setminus V_A) < \varepsilon'$. Hence, $\mu_0(U_A) < \mu_0(V_A) + \sum_{B<A} \mu_0(\overline{V_B}) + \varepsilon' < \mu_0(V_A) + \sum_{B<A} \mu_0(V_B) + 2^n\varepsilon'$. Moreover, it is obvious that $\overline{V_A} \cap \overline{V_B} = \emptyset$ if $A \neq B$. We will show that $\mathcal{N}^*(\mu_0, \{V_A\colon A \subset N\}, \varepsilon') \subset \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon)$. If $\mu \in \mathcal{N}^*(\mu_0, \{V_A\colon A \subset N\}, \varepsilon')$, then

$$
\begin{aligned}
\mu(U_i) &= \sum_{i \in A} \mu(\overline{V_A}) + \mu\left(U_i \setminus \bigcup_{i \in A} \overline{V_A}\right) \\
&\geqslant \sum_{i \in A} \mu(\overline{V_A}) > \sum_{i \in A} \left(\mu_0(V_A) - \varepsilon'\right) \\
&> \sum_{i \in A} \mu_0(V_A) - 2^n\varepsilon' > \mu_0(U_i) - 2^{n+1}\varepsilon' \\
&= \mu_0(U_i) - \varepsilon, \quad 1 \leqslant i \leqslant n.
\end{aligned}
$$

Hence, $\mu \in \mathcal{N}^*(\mu, U_1, \ldots, U_n, \varepsilon)$.

We will re-denote the set $\mathcal{N}^*(\mu_0, \{V_A\colon A \subset N\}, \varepsilon')$ as $\mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon')$, where $m = |\exp N|$. By M we denote the set $\{1, \ldots, m\}$. We introduce a linear order on the set $\exp M$ with the same property as the linear order on $\exp N$. For every $A \subset M$ we put

$$
W'_A = \bigcap_{i \in A} f(V_i).
$$

Since f is an open mapping, the sets $W'_A \subset Y$ are open. Let $\delta = \varepsilon'/2^{m+1}$. By induction, for every $A \subset M$ we find an open set $W_A \subset Y$ such that

$$
\overline{W_A} \subset W'_A \setminus \bigcup_{B<A} \overline{W_B} \quad \text{and} \quad \nu_0(W_A) > \nu_0(W'_A \setminus \bigcup_{B<A} \overline{W_B}) - \delta.
$$

It is clear that $\overline{W_A} \cap \overline{W_B} = \emptyset$ if $A \neq B$, $A, B \subset M$. Moreover, $\nu_0((W'_A \setminus \bigcup_{B<A} W'_B) \setminus W_A) < \delta$ for every $A \subset M$. We will show that $\mathcal{N}^*(\nu_0, \{W_A\colon A \subset M\}, \delta) \subset U_R(f)(\mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon'))$. Let $\nu \in \mathcal{N}^*(\nu_0, \{W_A\colon A \subset M\}, \delta)$. For every $A \subset M$ and for each $i \in A$ we denote by $s_{A,i}\colon Y \to X$ a Borel selection of the mapping f such that $s_{A,i}(W_A) \subset V_i$. Let α_i^A, $i \in A$, be non-negative numbers such that $\sum_{i \in A} \alpha_i^A = 1$ and $\alpha_i^A \nu_0(W_A) \geqslant \mu_0(f^{-1}(W_A) \cap V_i)$ for every $A \subset M$. Let $s_0\colon Y \to X$ be a Borel selection of the mapping f. Let μ be a measure on X such that for any Borel set $C \subset X$ we have

$$
\mu(C) = \nu\left(f\left(s_0\left(Y \setminus \bigcup_{A \subset M} W_A\right)\right) \cap C\right) + \sum_{A \subset M} \sum_{i \in A} \alpha_i^A \nu\big(f\big(s_{A,i}(W_A) \cap C\big)\big).
$$

By analogy with the proof of Proposition 2.15 we can show that $\mu \in U_R(X)$ and $U_R(f)(\mu) = \nu$. We will prove that $\mu \in \mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon')$. Indeed,

$$\begin{aligned}
\mu(V_i) &\geqslant \sum_{i\in A}\alpha_i^A \nu\big(f\big(s_{A,i}(W_A)\cap V_i\big)\big) = \sum_{i\in A}\alpha_i^A \nu(W_A) \\
&> \sum_{i\in A}\alpha_i^A\big(\nu_0(W_A)-\delta\big) > \sum_{i\in A}\alpha_i^A \nu_0(W_A) - 2^m\delta \\
&\geqslant \sum_{i\in A}\mu_0\big(f^{-1}(W_A)\cap V_i\big) - 2^m\delta = \mu_0\bigg(f^{-1}\bigg(\bigcup_{i\in A} W_A\bigg)\cap V_i\bigg) - 2^m\delta \\
&= \mu_0\bigg(f^{-1}\bigg(\bigcup_{i\in A} W'_A\bigg)\cap V_i\bigg) - \mu_0\bigg(f^{-1}\bigg(\bigcup_{i\in A} W'_A \Big\backslash \bigcup_{i\in A} W_A\bigg)\bigg) - 2^m\delta \\
&= \mu_0(V_i) - 2^m\delta - \nu_0\bigg(\bigcup_{i\in A} W'_A \Big\backslash \bigcup_{i\in A} W_A\bigg).
\end{aligned}$$

By definition of the sets W'_A we have $\bigcup_{i\in A} W'_A = W'_i$. Then

$$\begin{aligned}
\nu_0\bigg(\bigcup_{i\in A} W'_A \backslash \bigcup_{i\in A} W_A\bigg) &= \nu_0\bigg(W'_i \Big\backslash \bigcup_{i\in A} W_A\bigg) \\
&= \nu_0\bigg(\bigcup_{i\in A}\bigg(W'_A \Big\backslash \bigcup_{B<A} W'_B\bigg)\Big\backslash \bigcup_{i\in A} W_A\bigg) \\
&\leqslant \sum_{i\in A}\nu_0\bigg(\bigg(W'_A \Big\backslash \bigcup_{B<A} W'_B\bigg)\Big\backslash W_A\bigg) \\
&< 2^m\delta.
\end{aligned}$$

Hence, $\mu(V_i) > \mu_0(V_i) - 2^{m+1}\delta = \mu_0(V_i) - \varepsilon'$. Theorem 2.17 is proved. □

By analogy with [4, Proposition 4.1] we can prove

Proposition 2.18. *Let $f : X \to Y$ be a continuous mapping. If $U_R(f) : U_R(X) \to U_R(Y)$ is an open mapping, then the mapping f is open too.*

By analogy with [15, Theorem 3.7] we get

Proposition 2.19. *The functor U_R preserves density, i.e., $d(U_R(X)) \leqslant d(X)$ for any infinite space X.*

From [15, Theorems 3.8 and 3.11], [16, II, § 4, Theorem 13] and Lemma 2.11 we obtain

Theorem 2.20. *The functor U_R preserves weight, i.e., $w(U_R(X)) = w(X)$ for any infinite space X.*

Theorem 2.21. *The functor U_R preserves the class of metrizable spaces.*

Theorem 2.22. *The functor U_R preserves the class of Čech-complete spaces.*

We recall that a space X is said to be a p-space if there exists a countable family $\mathcal{P}$ of open covers of the space X by sets which are open in βX such that $\bigcap\{\gamma(x)\colon \gamma \in \mathcal{P}\} \subset X$ for any point $x \in X$, where $\gamma(x) = \bigcup\{V \in \gamma\colon x \in V\}$. Arhangel'skiĭ proved in [1] that paracompact p-spaces and only them are perfectly mapped onto metric spaces. Theorems 2.6 and 2.21 yield

Theorem 2.23. *The functor U_R preserves the class of paracompact p-spaces.*

References

[1] A.V. Arhangel'skiĭ, On a class of spaces containing all metric and all locally bicompact spaces, Mat. Sb. 67 (1) (1965) 55–85 (in Russian).
[2] T.O. Banakh, Topology of probability measures spaces I, Mat. Studiï, Proc. Lwow Math. Soc. 5 (1995) 65–87 (in Russian).
[3] T.O. Banakh, Topology of probability measures spaces II, Mat. Studiï, Proc. Lwow Math. Soc. 5 (1995) 88–106 (in Russian).
[4] S. Ditor, L. Eifler, Some open mapping theorems for measures, Trans. Amer. Math. Soc. 164 (1972) 287–293.
[5] R. Engelking, General Topology, PWN, Warsaw, 1977.
[6] G. Federer, Geometric Measure Theory, Springer, Berlin, 1969.
[7] V.V. Fedorchuk, Probability measures in topology, Russian Math. Surveys 46 (1) (1991) 45–93.
[8] V.V. Fedorchuk, V.V. Filippov, General Topology. Basic Constructions, Moscow Univ. Press, Moscow, 1988 (in Russian).
[9] V.V. Fedorchuk, Yu.V. Sadovnichy, On some categorical properties of uniform spaces of probability measures, Topology Appl. 82 (1998) 131–151.
[10] R.G. Gardner, W.F. Pfeffer, Borel measures, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 961–1043.
[11] Yu.V. Sadovnichy, On a metric on spaces of probability measures, Moscow Univ. Math. Bull. 49 (4) (1994) 26–30.
[12] Yu.V. Sadovnichy, On the completion of metric spaces of probability measures, Moscow Univ. Math. Bull. 49 (5) (1994) 26–29.
[13] Yu.V. Sadovnichy, On uniformity on probability measures spaces, in: V.V. Fedorchuk et al. (Eds.), General Topology. Mappings, Products and Dimension of Spaces, Moscow Univ. Press, Moscow, 1994, pp. 119–131 (in Russian).
[14] Yu.V. Sadovnichy, Uniform spaces of probability measures, Moscow Univ. Math. Bull. 50 (2) (1995) 55–56.
[15] Yu.V. Sadovnichy, On some categorical properties of the functor U_τ, Vestnik Moskov. Univ. Ser. 1, to appear.
[16] V.S. Varadarajan, Measures on topological spaces, Mat. Sb. 55 (1) (1961) 35–100 (in Russian).