Note
A multiplicative inequality for vertex Folkman numbers

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Abstract
Let $G$ be a graph and $a_1, \ldots, a_r$ be positive integers. The symbol $G \rightarrow (a_1, \ldots, a_r)$ denotes that in every $r$-coloring of the vertex set $V(G)$ there exists a monochromatic $a_i$-clique of color $i$ for some $i \in \{1, \ldots, r\}$. The vertex Folkman numbers $F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \rightarrow (a_1, \ldots, a_r) \text{ and } K_q \not\subseteq G\}$ are considered. Let $a_i, b_i, c_i, i \in \{1, \ldots, r\}, s, t$ be positive integers and $c_i = a_i b_i, 1 \leq a_i \leq s, 1 \leq b_i \leq t$. Then we prove that

$$F(c_1, c_2, \ldots, c_r; st + 1) \leq F(a_1, a_2, \ldots, a_r; s + 1) F(b_1, b_2, \ldots, b_r; t + 1).$$

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We consider only finite, undirected graphs without loops and multiple edges. We call a $p$-clique of the graph $G$ a set of $p$ vertices, each two of which are adjacent. The largest positive integer $p$, such that the graph $G$ contains a $p$-clique is called a clique number of $G$ and is denoted by $\text{cl}(G)$. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph $G$, respectively. The symbol $K_n$ denotes the complete graph on $n$ vertices. Let $G_1, \ldots, G_k$ be graphs with pairwise disjoint vertex sets. The Zykov sum $G_1 + \cdots + G_k$ is the graph which is obtained from the union of the graphs $G_1, \ldots, G_k$, by connecting each vertex of $G_i$ with each vertex of $G_j$, for every $i \neq j$.

Definition. Let $a_1, \ldots, a_r$ be positive integers. We say that the $r$-coloring

$$V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j$$

of the vertices of the graph $G$ is $(a_1, \ldots, a_r)$-free, if $V_i$ does not contain an $a_i$-clique for each $i \in \{1, \ldots, r\}$. The symbol $G \rightarrow (a_1, \ldots, a_r)$ means that there is no $(a_1, \ldots, a_r)$-free coloring of the vertices of $G$.

For arbitrary positive integers $a_1, \ldots, a_r$ and $q$ define

$$H(a_1, \ldots, a_r; q) = \{G : G \rightarrow (a_1, \ldots, a_r) \text{ and } \text{cl}(G) < q\}.$$
The vertex Folkman numbers are defined by the equality

\[ F(a_1, \ldots, a_r; q) = \min \{|V(G)| : G \in H(a_1, \ldots, a_r; q)\}. \]

It is clear that \( G \rightarrow (a_1, \ldots, a_r) \) implies \( \text{cl}(G) \geq \max\{a_1, \ldots, a_r\} \). Folkman [1] proved that there exists a graph \( G \) such that \( G \rightarrow (a_1, \ldots, a_r) \) and \( \text{cl}(G) = \max\{a_1, \ldots, a_r\} \). Therefore

\[ F(a_1, \ldots, a_r; q) \text{ exists if and only if } q > \max\{a_1, \ldots, a_r\}. \] (1)

Some basic important properties of Folkman numbers are obtained in [5]. Some latest results and a detailed bibliography can be found in [7]. However, in order to understand this paper the given above definition of vertex Folkman numbers is enough.

Our goal is to prove the following:

**Theorem.** Let \( a_1, b_1, c_1, i \in \{1, \ldots, r\}, s, t \) be positive integers such that \( c_i = a_i b_i, 1 \leq a_i \leq s, 1 \leq b_i \leq t \). Then

\[ F(c_1, c_2, \ldots, c_r; st + 1) \leq F(a_1, a_2, \ldots, a_r; s + 1)F(b_1, b_2, \ldots, b_r; t + 1). \]

**Remark.** The Folkman numbers in the theorem exist according to (1).

In order to prove the theorem we shall need the operation composition of graphs (see [2]) which is defined as follows. Let \( A \) and \( B \) be two graphs without common vertices and \( V(A) = \{u_1, \ldots, u_n\} \) and \( V(B) = \{v_1, \ldots, v_m\} \). Define a new graph \( A[B] \) on \( nm \) vertices in the following way. The vertex set is

\[ V(A[B]) = \{w_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}. \]

The edge set \( E(A[B]) \) is defined as follows:

If \( i \neq k \) then \( \{w_{ij}, w_{ks}\} \in E(A[B]) \iff \{u_i, u_k\} \in E(A) \). \hspace{1cm} (2)

If \( i = k \) then \( \{w_{ij}, w_{ks}\} \in E(A[B]) \iff \{v_j, v_k\} \in E(B) \). \hspace{1cm} (3)

We shall say that the graph \( A[B] \) is a composition of the graphs \( A \) and \( B \).

We consider the subgraph \( B_j \) of the graph \( A[B] \) induced by the vertices \( \{w_{j1}, w_{j2}, \ldots, w_{jm}\} \). It is straightforward from (3) that

\[ B_j \text{ is isomorphic to } B, \quad j \in \{1, \ldots, n\}. \] (4)

It is clear from (4) that the graph \( A[B] \) is obtained from the graphs \( A \) and \( B \) in the following way. We take \( n \) isomorphic copies \( B_1, \ldots, B_n \) of the graph \( B \), where \( n = |V(A)| \). According to (2) if the vertices \( u_i \) and \( u_j, i, j \in \{1, \ldots, n\} \) are adjacent in \( A \) then we connect each vertex from \( B_i \) with each vertex from \( B_j \) and if the vertices \( u_i \) and \( u_j \) are not adjacent in \( A \) then we connect no vertex from \( B_i \) with any vertex from \( B_j \).

We know from [2] that

\[ \text{cl}(A[B]) = \text{cl}(A)\text{cl}(B). \] (5)

**Proof of Theorem.** Consider the graphs

\[ A \in H(a_1, a_2, \ldots, a_r; s + 1) \quad \text{and} \quad B \in H(b_1, b_2, \ldots, b_r; t + 1), \] (6)

where \( |V(A)| = F(a_1, a_2, \ldots, a_r; s + 1) \) and \( |V(B)| = F(b_1, b_2, \ldots, b_r; t + 1) \). Let us denote \( V(A) = \{u_1, \ldots, u_n\}, V(B) = \{v_1, \ldots, v_m\}, C = A[B], V(C) = V(A[B]) = \{w_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\} \). We shall prove that \( C \rightarrow (c_1, \ldots, c_r) \). Again we shall denote by \( B_j \) the subgraph of the graph \( A[B] \) induced by the vertices \( \{w_{j1}, w_{j2}, \ldots, w_{jm}\} \). Let us take an arbitrary \( r \)-coloring \( W_1 \cup \cdots \cup W_r \) of \( V(C) \). Consider the sets

\[ V_i^{(j)} = W_i \cap V(B_j), \quad i \in \{1, \ldots, r\}, \quad j \in \{1, \ldots, n\}. \]
Since $V(B_j) = V_1^{(j)} \cup \cdots \cup V_r^{(j)}$ we obtain from (4) and $B \in H(b_1, b_2, \ldots, b_r; t + 1)$ that

$$V_1^{(j)} \cup \cdots \cup V_r^{(j)} \text{ is not a } (b_1, b_2, \ldots, b_r)\text{-free coloring of } V(B_j).$$  \hfill (7)

We define an $r$-coloring of $V(A)$ in the following way. We color the vertex $u_j \in V(A)$, $j \in \{1, \ldots, n\}$ in color $k$, if $V_k^{(j)}$ contains a $b_k$-clique of the graph $B_j$ (and hence of $C$). We know from (7) that such $k$ exists. If there are several indices $k$ such that $V_k^{(j)}$ contains a $b_k$-clique, we choose just one of them (for example, we may chose the smallest $k$ among them). According to (6), $A \to (a_1, \ldots, a_r)$. Thus for some $i \in \{1, \ldots, r\}$ there is a monochromatic $a_i$-clique $Q$ of $A$ in the $i$th color of the just now defined $r$-coloring of $V(A)$. Without loss of generality we can assume that the clique $Q$ consists of the vertices $\{u_1, u_2, \ldots, u_{a_i}\}$. Now it is straightforward from (2) that

the graph $C = A[B]$ contains $B_1 + \cdots + B_{a_i}$. \hfill (8)

According to the definition of the given above $r$-coloring of $V(A)$ we have

$$V_i^{(j)} \text{ contains a monochromatic } b_i\text{-clique } Q_j \text{ for } j \in \{1, \ldots, a_i\}.$$ \hfill (9)

Since $V_1^{(j)} \subseteq W_i$, (8) and (9) give that $Q_1 \cup \cdots \cup Q_{a_i}$ is an $a_i b_i$-clique of $C$ in $W_i$. As $a_i b_i = c_i$ this gives that $W_1 \cup \cdots \cup W_r$ is not a $(c_1, \ldots, c_r)$-free coloring of $V(C)$ and thus $C \to (c_1, \ldots, c_r)$ is proved. On the other hand, it follows from (5) and (6) that $cl(C) = cl(A) cl(B) \leq s t$. Thus we have $C \in H(c_1, c_2, \ldots, c_r; s t + 1)$ and $F(c_1, c_2, \ldots, c_r; s t + 1) \leq |V(C)|$. As $|V(C)| = |V(A)||V(B)|$ and $|V(A)| = F(a_1, a_2, \ldots, a_r; s + 1)$, $|V(B)| = F(b_1, b_2, \ldots, b_r; t + 1)$ the Theorem is proved. \hfill \Box

**Corollary 1.** We have

$$F(kl, kl, \ldots, kl; kl + 1) \leq F(k, k, \ldots, k; k + 1) F(l, l, \ldots, l; l + 1).$$

Proof. We put $a_1 = \cdots a_r = k, b_1 = \cdots b_r = l, s = k, t = l$ in the Theorem. \hfill \Box

Let $C_5$ denote the simple cycle on five vertices. From $C_5 \to (2, 2)$ we have $F(2, 2; 3) \leq 5$. It is easy to see that in fact $F(2, 2; 3) = 5$. By putting $k = l = 2, r = 2$ in Corollary 1 and taking into consideration the equality $F(2, 2; 3) = 5$ we obtain:

**Corollary 2.** $F(4, 4; 5) \leq 25$, [3].

**Remark.** The proof of Corollary 2 given in [3] is based on the fact that $C_5 \cdot C_5 \to (4, 4)$.

This upper bound improves the bound 26 on this number from [4]. The best known lower bound for this number is $F(4, 4; 5) \geq 16$, [6].

In [6] (see also [4]) Nenov proved the following recurrent inequality:

$$F(p + 1, p + 1; p + 2) \leq (p + 1) F(p, p; p + 1).$$

From this inequality and Corollary 2 we trivially obtain

$$F(p, p; p + 1) \leq \frac{25}{24} p!, \quad p \geq 4.$$

This improves the bound $F(p, p; p + 1) \leq \lfloor 2p!(e - 1) \rfloor - 1$ given in [5].

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References