Characteristic polynomials of some graph coverings

Hirobumi Mizuno, Iwao Sato

* Department of Computer Science and Information Mathematics, University of Electro-Communications, 1-5-1, Chofugaoka, Chofu, Tokyo 182, Japan
b Department of Mechanical Engineering, The Tsuruoka Technical College, Tsuruoka, Yamagata 997, Japan

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Abstract

We give a formula for the characteristic polynomial of the derived graph covering of a graph with voltages in any finite group.

Graphs treated here are finite simple graphs. Let G be a graph and A(G) its adjacency matrix. Then the characteristic polynomial \( \Phi(G; \lambda) \) of G is defined by \( \Phi(G; \lambda) = \det(\lambda I - A(G)) \). The eigenvalues of A(G) are called the eigenvalues of G.

Schwenk [6] studied relations between the characteristic polynomials of some related graphs. Kitamura and Nihei [4] discussed the structure of regular double coverings of graphs by using their eigenvalues. Chae et al. [2] gave the complete computations of the characteristic polynomials of \( K_2 \) (or \( K_2 \))-bundles over graphs. Kwak and Lee [5] computed the characteristic polynomial of a graph bundle when its voltage assignment takes in an abelian group. Sohn and Lee [7] introduced weighted graph bundles and showed that the characteristic polynomial of a weighted \( K_2(K_2) \)-bundles over a weighted graph \( G_w \) can be expressed as a product of characteristic polynomials of two weighted graphs whose underlying graphs are G. Furthermore, they gave the signature of a link whose corresponding weighted graph is a double covering of that of a given link. In this paper, we establish an explicit decomposition formula for the characteristic polynomial of the derived graph covering of a graph with voltages in any finite group.

Let \( D(G) \) be the arc set of the symmetric digraph corresponding to G and \( \Gamma \) a finite group. Then a mapping \( \alpha: D(G) \to \Gamma \) is called an ordinary voltage assignment if \( \alpha(v, u) = \alpha(u, v)^{-1} \) for each \((u, v) \in D(G)\). The pair \((G, \alpha)\) is called an ordinary voltage assignment.

*Corresponding author.
The derived graph \( G^* \) of the ordinary voltage graph \((G, \alpha)\) is defined as follows:

\[
V(G^*) = V(G) \times \Gamma, \quad E(G^*) \subseteq E(G) \times \Gamma,
\]

and \((u, g, v, h)\) are adjacent in \( G^* \) if and only if \( uv \in E(G) \) and \( h = g\alpha(u, v) \). The graph \( G^* \) is called a derived graph covering of \( G \) with voltages in \( \Gamma \).

For propositions concerning the representation of groups the reader is referred to \([1]\). For a square matrix \( B \), we define \( \Phi(B; \lambda) := \det(\lambda I - B) \).

The block diagonal sum \( M_1 + \cdots + M_s \) of square matrices \( M_1, \ldots, M_s \) is defined as the square matrix

\[
\begin{pmatrix}
M_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & M_s
\end{pmatrix}
\]

If \( M_1 = \cdots = M_{a_1} = N_1, \quad M_{a_1+1} = \cdots = M_{a_1+a_2} = N_2, \ldots, M_{s-a_1+1} = \cdots = M_s = N_s \),

we write \( M_1 + M_2 + \cdots + M_s = a_1 \circ N_1 + a_2 \circ N_2 + \cdots + a_s \circ N_s \).

**Theorem 1.** Let \( G \) be a graph, \( \Gamma \) a finite group and \( \alpha : D(G) \to \Gamma \) an ordinary voltage assignment. Furthermore, let \( \rho_1 = 1, \rho_2, \ldots, \rho_s \) be the irreducible representations of \( \Gamma \), and \( f_i \) the degree of \( \rho_i \) for each \( i \), where \( f_1 = 1 \). For \( g \in \Gamma \), the matrix \( A_g \) is defined as follows:

\[
A_g = (a_{uv}^{(g)}), \quad a_{uv}^{(g)} = \begin{cases} 1 & \text{if } \alpha(u, v) = g \text{ and } (u, v) \in D(G), \\ 0 & \text{otherwise}. \end{cases}
\]

Then one has

\[
\Phi(G^*; \lambda) = \Phi(G; \lambda) \prod_{j=2}^h \left\{ \Phi \left( \sum_{g \in \Gamma} \rho_j(g) \otimes A_g; \lambda \right) \right\}^{f_j},
\]

where \( \otimes \) is the Kronecker product of matrices.

**Proof.** Set \( V(G) = \{v_1, \ldots, v_n\} \) and \( \Gamma = \{g_1 = 1, g_2, \ldots, g_m\} \). Arrange the vertices of \( G^* \) in \( m \) blocks;

\[
(v_1, 1), \ldots, (v_n, 1); (v_1, g_2), \ldots, (v_n, g_2); \ldots; (v_1, g_m), \ldots, (v_n, g_m).
\]

For \( g \in \Gamma \), the matrix \( P_g = (p_{ij}) \) is defined as follows:

\[
p_{ij} = \begin{cases} 1 & \text{if } g \circ g = g_j, \\ 0 & \text{otherwise}. \end{cases}
\]

Then we have

\[
A(G^*) = \sum_{g \in \Gamma} P_g \otimes A_g.
\]
Let $\rho$ be the right regular representation of $\Gamma$. Then we have $\rho(g) = P_g$ for $g \in \Gamma$. Furthermore there exists a regular matrix $P$ such that

$$P^{-1} \rho(g)P = (1 + f_2 \circ \rho_2(g) + \cdots + f_h \circ \rho_h(g)) \quad \text{for each } g \in \Gamma.$$ 

Putting

$$B = (P^{-1} \otimes I_n)A(G^*) (P \otimes I_n)$$

we have

$$B = \sum_{\mu \in \Gamma} \{ (1 + f_2 \circ \rho_2(g) + \cdots + f_h \circ \rho_h(g)) \otimes A_g \}.$$

Therefore it follows that

$$\Phi(G^*; \lambda) = \Phi(B; \lambda) = \Phi(G; \lambda) \prod_{j=2}^{h} \left\{ \Phi \left( \sum_{\mu \in \Gamma} \rho_j(g) \otimes A_g; \lambda \right) \right\}^{f_j}. \quad \square$$

**Corollary 2.** $\Phi(G; \lambda) = \Phi(G^*; \lambda)$.

Let $G$ be a graph, $\Gamma$ a finite abelian group and $\Gamma^*$ the character group of $\Gamma$. For the mapping $f: D(G) \rightarrow \Gamma^*$, a pair $G_f = (G, f)$ is called a weighted graph. Given any weighted graph $G_f$, the adjacency matrix $A(G_f) = (a_{f_{uv}})$ of $G_f$ is the square matrix of order $|V(G)|$ defined by

$$a_{f_{uv}} = a_{uv} \cdot f(u, v).$$

The characteristic polynomial of $G_f$ is that of its adjacency matrix, and is denoted $\Phi(G_f; \lambda)$ [7]

**Corollary 3.** Let $\alpha$ be an ordinary voltage assignment on a graph $G$ in a finite abelian group $\Gamma$. Then

$$\Phi(G^*; \lambda) = \prod_{\chi \in \Gamma^*} \Phi(G_{\chi}; \lambda).$$

**Proof.** Each irreducible representation of $\Gamma$ is a linear representation, and these constitute the character group $\Gamma^*$. By Theorem 1, we have

$$\Phi(G^*; \lambda) = \Phi(G; \lambda) \prod_{\chi \in \Gamma^* \setminus \{1\}} \Phi \left( \sum_{\mu \in \Gamma} \chi(g) A_g; \lambda \right).$$

Since $\sum_{\mu \chi(g)} A_g = A(G_{\chi})$, it follows that

$$\Phi(G^*; \lambda) = \prod_{\chi \in \Gamma^*} \Phi(G_{\chi}; \lambda). \quad \square$$

**Corollary 4** (Chae et al. [2, Theorem 4]; Kitamura and Nihei [4, Theorem 1]). Let $\alpha$ be an ordinary voltage assignment on a graph $G$ in the group $Z_2 = \{1, -1\}$. Then

$$\Phi(G^*; \lambda) = \Phi(G; \lambda) \Phi(G_{\alpha}; \lambda).$$
Proof. By Corollary 3, we have
\[ \Phi(G^*; \lambda) = \Phi(G; \lambda) \Phi(G_{\chi \circ \alpha}; \lambda), \]
where \( \chi(1) = 1 \) and \( \chi(-1) = -1 \). Since \( (\chi \circ \alpha)(u, v) = \alpha(u, v) \) for each \( (u, v) \in D(G) \), it follows that \( \chi \circ \alpha = \alpha \). \( \square \)

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References