Note

Enumeration of certain finite semigroups of transformations

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Abstract

Let $\text{Sing}_n$ be the semigroup of singular self-maps of $X_n = \{1, \ldots, n\}$, let $R_n = \{\alpha \in \text{Sing}_n : (\forall y \in \text{Im} \alpha) \| yx^{-1} \| \geq \| \text{Im} \alpha \| \}$ and let $E(R_n)$ be the set of idempotents of $R_n$. Then it is shown that $R_n = (E(R_n))^2$. Moreover, expressions for the order of $R_n$ and $E(R_n)$ are obtained in terms of the $k$th-upper Stirling number of the second kind, $S(n, r, k)$; defined as the number of partitions of $X_n$ into $r$ subsets each of size not less than $k$.

1. Introduction

Let $X_n = \{1, 2, \ldots, n\}$, let $\text{Sing}_n$ be the semigroup (under composition) of all singular self-maps of $X_n$ and let $E$ be its set of idempotents. It is well known (and indeed obvious) that $\text{Sing}_n$ has order $n^n - n!$. Only slightly less obvious is the formula

$$|E| = \sum_{r=1}^{n-1} \binom{n}{r} r^{n-r}$$

due to Tainiter [10] (also to be found in [4, Ex. 2.2.2(a)].)

Various enumerative problems of an essentially combinatorial nature have been considered in certain subsemigroups of $\text{Sing}_n$ and at least three cases are worth mentioning.

In an elegant paper in 1971, Howie [5] considered the semigroup

$$|\mathcal{O}_n| = \{\alpha \in \text{Sing}_n : (\forall x, y \in X_n) x \leq y \Rightarrow x\alpha \leq y\alpha\}$$

\*This work was carried out while the author was a research student at the University of St. Andrews, Scotland, UK.
consisting of all order-preserving mappings of $X_n$. It is shown there that

$$|O_n| = \binom{2n-1}{n-1} - 1, \quad |E(O_n)| = f_{2n} - 1,$$

where $f_m$ denotes the $m$th Fibonacci number, defined recursively by

$$f_1 = f_2 = 1, \quad f_m = f_{m-1} + f_{m-2}.$$ 

More recently, Borwein et al. [2] considered the semigroup of all partial one-to-one self-maps of $X_n$ and its subsemigroup of order-decreasing mappings of $X_n$ (the latter denoted there by $b_n$). In particular they showed that

$$|b_n| = \sum_{r=1}^{n} S(n+1, n-r+1) = B_{n+1},$$

where $S(m, r)$ (see Section 3 for its definition) and $B_n$ are known as the Stirling number of the second kind and Bell's number, respectively.

Borrowing an idea from Howie [5], I (in [12]) considered the semigroup

$$S^-_n = \{ \alpha \in \text{Sing}_n : (\forall x \in X_n) \ x \alpha \leq x \}$$

consisting of all order-decreasing mappings in $\text{Sing}_n$, where I showed that

$$|E(S^-_n)| = B_n - 1$$

and once again $B_n$ is the Bell's number. It is trivially observed that the order of $S^-_n$ is $n! - 1$.

In this paper, we consider a subsemigroup $R_n$ of $\text{Sing}_n$ consisting of all those elements $\alpha$ for which $|a_i \alpha^{-1}| \geq |\text{Im } \alpha|$ (for all $a_i \in \text{Im } \alpha$). In Section 2, we show that $R_n$ is a semiband and its depth is 2. In Section 3, we introduce what we call the $k$th-upper Stirling number of the second kind and hence obtain formulae for the order of $R_n$ and its number of idempotents.

2. Preliminaries; products of two idempotents

For standard terms in semigroup theory see for example [4] or [6].

As in [8], let $\alpha \in \text{Sing}_n$ be denoted by

$$\alpha = \left( \begin{array}{ccc} A_1 & A_2 & \ldots & A_r \\ a_1 & a_2 & \ldots & a_r \end{array} \right),$$

where $A_1, A_2, \ldots, A_r$ are pairwise disjoint subsets of $X_n$ called the blocks of $\alpha$ with $A_i \alpha = a_i$. Notice that the image of $\alpha$ is

$$\text{Im } \alpha = \{a_1, a_2, \ldots, a_r\} \quad \text{and} \quad A_1 \cup A_2 \cup \ldots \cup A_r = X_n$$
If \( a_i \in A_i \) we say that \( A_i \) is stationary; otherwise it is non-stationary. If we denote by \( F(a) \) the set

\[
\{ x \in X_n : x\alpha = x \}
\]

and by \( f(\alpha) \) its cardinal, then the number of stationary blocks of \( \alpha \) is equal to \( f(\alpha) \). The element \( \alpha \) is idempotent if and only if (a) all the blocks of \( \alpha \) are stationary; or (b) \( f(\alpha) = |\text{Im} \, \alpha| \).

Now define a subset of \( \text{Sing}_n \) by

\[
R_n = \{ \alpha \in \text{Sing}_n : |A_i| \geq |\text{Im} \, \alpha| (\forall i) \}.
\]

(2.1)

In general, if \( S \) is a semigroup with set \( E(S) \) of idempotents then we say that \( S \) is idempotent-generated or a semiband if \( \langle E(S) \rangle = S \). If there exists a least natural number \( n \) for which \( (E(S))^n = S \), we call \( n \) the depth of \( S \) and write \( \Delta(S) = n \). The main algebraic result about \( R_n \) is

**Theorem 2.1.** Let \( R_n \) be as defined in (2.1). Then \( R_n \) is a regular subsemiband of \( \text{Sing}_n \) and \( \Delta(R_n) = 2 \) if and only if \( n > 3 \).

**Proof.** First notice that it is easy to show that \( R_n \) is a subsemigroup of \( \text{Sing}_n \) and if \( n \leq 3, R_n \) is a right zero semigroup, and so is of depth one. To show that \( R_n \) is indeed regular, consider

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix} \in R_n.
\]

First we partition \( X_n \setminus \text{Im} \, \alpha \) into subsets \( D_i \) such that \( |D_i| = |A_i| - 1 \). (This is always possible though not uniquely.) Now define \( \alpha' \) by

\[
z \alpha' = y_i \in a_i \alpha^{-1} (\forall z \in D_i \cup \{a_i\})
\]

for all \( i \). It is then clear that \( \alpha' \in R_n \) and \( \alpha \alpha' \alpha = \alpha \) since

\[
A_i \alpha \alpha' \alpha = a_i \alpha' \alpha = y_i \alpha = a_i = A_i \alpha \quad (i \in \{1, \ldots, r\}).
\]

To show finally that \( R_n \) is a semiband, notice that the condition \( |A_i| \geq |\text{Im} \, \alpha| \) implies that every non-stationary block contains an element of \( X_n \setminus \text{Im} \, \alpha \). For \( i = 1, \ldots, r \) define \( c_i \) to be equal to \( a_i \) if the block \( A_i \) is stationary, and otherwise to be an element of \( A_i \cap (X_n \setminus \text{Im} \, \alpha) \). Now partition \( X_n \) into subsets \( B_1, B_2, \ldots, B_r \), such that \( a_i, c_i \in B_i \) and \( |B_i| \geq |\text{Im} \, \alpha| \) for \( i = 1, 2, \ldots, r \). (This is possible since no \( c_i \) can coincide with \( a_j \) with \( j \neq i \).) Then

\[
\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} B_1 & B_2 & \ldots & B_r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix}
\]
a product of two idempotents in $R_a$. However, since clearly $R_a$ contains some non-idempotents we conclude that

$$\Delta(R_a) = 2,$$

as required.

3. Combinatorial results

Our aim in this section is to find a formula for the orders of $R_a$ and $E(R_a)$. But first we need the following definitions and results.

The Stirling number of the second kind, denoted by $S(n, r)$, is usually defined as the number of partitions of an $n$-element set $X_n$ into $r$ (non-empty) subsets (see [1] or [3] for example). It satisfies the recurrence relation

$$S(n, 1) = 1 = S(n, n), \quad S(n, r) = S(n - 1, r - 1) + rS(n - 1, r).$$

Let $S(n, r, k)$ be the number of partitions of an $n$-element set $X_n$ into $r$ subsets each of size not less than $k > 0$. Evidently,

$$S(n, r, 1) = S(n, r),$$

where $S(n, r)$ is the usual (unrestricted) Stirling number of the second kind. It is also fairly obvious that,

$$S(n, r, k) = 0 \quad \text{if} \quad n < kr \quad \text{and} \quad S(n, 1, k) = 1 \quad \text{if} \quad n \geq k.$$

We shall refer to $S(n, r, k)$ as the $k$th upper Stirling number of the second kind. As with $S(n, r)$, it is difficult to obtain an explicit expression (in terms of $n$ and $r$) for $S(n, r, k)$; however, they do satisfy the following recurrence relation which reduces to that of $S(n, r)$ when $k = 1$.

**Proposition 3.1** (Tomescu [11, Problem IV.21]). (a) $S(n, r, k) = \binom{n-1}{r-1}S(n - k, r - 1, k) + rS(n - 1, r, k)$;

(b) $S(n, r, k) = \frac{1}{r!} \sum_{j_1, j_2, \ldots, j_r} j_1! j_2! \ldots j_r! \frac{n!}{j_1! j_2! \ldots j_r!}$, where the sum is over all solutions $j_1, j_2, \ldots, j_r$ of the equation $j_1 + j_2 + \cdots + j_r = n$ and $j_s \geq k$ for $s = 1, 2, \ldots, r$.

Now coming back to our semigroup $R_a$, we note that since $|A_t| \geq |1m z|$ for all $t \in \{1, \ldots, r\}$, then $n \geq r^2$ and so we let $p$ be the greatest integer not exceeding $\sqrt{n}$.

On a (regular) semigroup $S$, the Green's relation $l$, $r$ and $j$ are defined, respectively, by equality of principal left, right and two-sided ideals of $S$. The intersection of $l$ and $r$ is denoted by $h$ while their join is denoted by $d$. It is clear that $h \subseteq l \subseteq d \subseteq j$ and $h \subseteq r \subseteq d \subseteq j$. 

We observe that from [6, Proposition II.4.5 and Ex.II.10]

\[(\alpha, \beta) \in l \iff \text{Im} \alpha = \text{Im} \beta\]
\[(\alpha, \beta) \in r \iff \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}\]
\[(\alpha, \beta) \in h \iff \text{Im} \alpha = \text{Im} \beta \text{ and } \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}.\]

Moreover, observe that if \((\alpha, \beta) \in r(T_n) \text{ with } \alpha \in R_n, \beta \in T_n, \) then \(\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1},\)
whence \(\beta \in R_n\) also. So suppose that \((\alpha, \beta) \in d(T_n) \cap (R_n \times R_n).\) Then there exist \(\gamma \in T_n\) such that \(\alpha \circ (T_n) \gamma \cap T_n \beta,\)
whence \(\gamma \in R_n\) by the previous observation. Thus \((\alpha, \beta) \in d(R_n).\) In conclusion we obtain:
\[
d(R_n) \subseteq \{(R_n) \subseteq \{T_n) \cap (R_n \times R_n) = d(T_n) \cap (R_n \times R_n) \subseteq d(R_n).\]
yielding equality throughout. Hence
\[(\alpha, \beta) \in d = j \iff |\text{Im} \alpha| = |\text{Im} \beta|.

Hence, from the definitions we deduce that \(R_n\) has \(S(n, r, r)\) \(r\)-classes and \((\beta)\) \(l\)-classes in each \(j\)-class \(J_r (r = 1, \ldots, p)\) where of course
\[J_r = \{z \in R_n : |\text{Im} z| = r\}.\]
Also as in Sing, each non-empty \(h\)-class (in \(J_r\)) contains \(r!\) elements, thus we have

**Lemma 3.2.** \(|J_r| = (\beta)S(n, r, r). r!, \ r = 1, \ldots, p.\)

**Theorem 3.3.** Let \(R_n\) be as defined in (2.1). Then
\[|R_n| = \sum_{r=1}^{p} \left(\begin{array}{c} n \\ r \end{array}\right) S(n, r, r). r!.

**Proof.** It follows directly from Lemma 3.2. \(\square\)

**Theorem 3.4.** Let \(R_n\) be as defined in (2.1) and \(E\) be its set of idempotents. Then
\[|E| = \sum_{r=1}^{p} \left(\begin{array}{c} n \\ r \end{array}\right) S(n - r, r, r - 1). r!.

**Proof.** Let \(|\text{Im} \alpha| = f(\alpha) = r\) and \(F(\alpha) = \{x_1, x_2, \ldots, x_r\}.\) Then there are \((\beta)\) ways of choosing the \(r\) elements of \(F(\alpha).\) The remaining \(n - r\) elements of \(X_n \setminus F(\alpha)\) can then be partitioned (into \(r\) subsets each of size not less than \(r - 1\)) in \(S(n - r, r, r - 1)\) ways. Moreover, since there are \(r!\) ways of tying the \(r\) elements of \(F(\alpha) (\text{one each})\) to the \(r\) subsets of \(X_n \setminus F(\alpha)\) so as to make each block stationary, then there are exactly
\[\left(\begin{array}{c} n \\ r \end{array}\right) S(n - r, r, r - 1). r!\]
idempotents in \(J_r.\) Thus taking the sum over \(r\) from 1 to \(p\) yields the required result.
We conclude with the following Tables 1 and 2 which record the total number of elements and the total number of idempotents in $R_n$, respectively, in each $j$-class $J$, for some values of $n$ and $r$.

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References


