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Application of Besov Spaces to Spline Approximation

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1. INTRODUCTION

In recent years, many papers have appeared in which error bounds for interpolating splines have been obtained. We shall present a survey of the results of these papers. We show that by expressing the error bound for the interpolation of a function f in terms of the norm of f in a Sobolev or a Besov space, we are able to give a common setting to these results. Furthermore, by applying the theory of intermediate spaces, we obtain new error bounds which fill gaps between the bounds previously established.

We remark that Besov spaces and the theory of intermediate spaces have found fruitful applications in other areas of approximation theory (cf. Butzer and Berens [9] and Löfström [19]). While our paper is perhaps the first application of these techniques to the theory of splines, Besov spaces and the theory of intermediate spaces have found important applications in numerical analysis in the study of initial-value problems (cf. Peetre and Thomée [28], Hedstrom [14], Löfström [19], and Widlund [37]).

To briefly describe the contents of this paper, we first define in Section 2 Besov spaces and state results needed from the theory of intermediate spaces. We then apply these theorems in Section 3 to error bounds for interpolation by Lg-splines (cf. Jerome and Varga [17]). In Section 4, we apply these techniques to the special cases of splines, Hermite splines, and periodic splines on a uniform mesh. Finally, in Section 5 we discuss error bounds for splines of best approximation, thereby generalizing recent results of de Boor [7].

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2. Besov Spaces

The Besov spaces arose from attempts to unify the various definitions of fractional-order Sobolev spaces (see Taibleson [35] and the survey article by Nikol'skii [21]). Since the Besov spaces are most easily described from the point of view of the theory of interpolation between Banach spaces, we begin with a brief description of that theory. For a more complete discussion of interpolation of Banach spaces, see Butzer and Berens [9], Grisvard [12], Lions [18], and Peetre [24] and [25].

Let X_0 and X_1 be two Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively, which are contained in a linear Hausdorff space $\mathscr{X}$, such that the identity mapping of X_i in $\mathscr{X}$ is continuous, for $i = 0, 1$. If $X_0 + X_1 \equiv \{f \in \mathscr{X} : f = f_0 + f_1, \text{ where } f_i \in X_i, i = 0, 1\}$, then (cf. Butzer and Berens [9, p. 165]) $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces with respect to the norms

$$\|f\|_{X_0 \cap X_1} \equiv \max\{\|f\|_0, \|f\|_1\},$$

$$\|f\|_{X_0 + X_1} \equiv \inf\{\|f_0\|_0 + \|f_1\|_1\},$$

the infimum being taken over all decompositions $f = f_0 + f_1$ with $f_i \in X_i$, $i = 0, 1$. Moreover, it follows that

$$X_0 \cap X_1 \subset X_i \subset X_0 + X_1 \subset \mathscr{X}, \qquad i = 0, 1, \tag{2.1}$$

where *inclusion throughout this paper is understood to mean that the identity mapping is continuous.* We say that a Banach space $X \subset \mathscr{X}$ is an *intermediate space* of X_0 and X_1 if it satisfies the inclusion

$$X_0 \cap X_1 \subset X \subset X_0 + X_1 \subset \mathscr{X}, \tag{2.2}$$

analogous to (2.1).

We now give Peetre's real-variable method (cf. Butzer and Berens [9, p. 167] and Peetre [25]) for constructing intermediate spaces of X_0 and X_1. For each positive t, and each $f \in (X_0 + X_1)$, define

$$K(t, f) = \inf_{f = f_0 + f_1} \{\|f_0\|_0 + t\|f_1\|_1\}. \tag{2.3}$$

Then, for any θ with $0 < \theta < 1$ and any q with $1 \leqslant q \leqslant \infty$, let $(X_0, X_1)_{\theta,q}$ be the set of all elements $f \in (X_0 + X_1)$ for which the norm

$$\|f\|_{(X_0, X_1)_{\theta,q}} \equiv \begin{cases} \left(\int_0^\infty (t^{-\theta} K(t, f))^q \dfrac{dt}{t}\right)^{1/q}, & 1 \leqslant q < \infty, \\ \sup_{t>0} t^{-\theta} K(t, f), & q = \infty, \end{cases} \tag{2.4}$$

is finite. The following result is known (cf. Butzer and Berens [9, p. 168] and Peetre [25]).

THEOREM 2.1. *For* $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, $(X_0, X_1)_{\theta,q}$ *is a Banach space which is an intermediate space of* X_0 *and* X_1, *and, thus, satisfies* (2.2). *In particular,* $(X, X)_{\theta,q} = X$.

Next, let Y_0 and Y_1 be two Banach spaces continuously contained (with respect to the identity mapping) in a linear Hausdorff space $\mathscr{Y}$, and let T denote any linear transformation from $(X_0 + X_1)$ to $(Y_0 + Y_1)$ for which

$$\| Tf \|_i \leqslant M_i \| f \|_i, \qquad \forall f \in X_i, \quad i = 0, 1,$$

i.e., T is a bounded linear transformation from X_i to Y_i with norm at most M_i, $i = 0, 1$. Again, the following result is known (cf. Butzer and Berens [9, p. 180] and Peetre [25]):

THEOREM 2.2. *For* $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, T *is a bounded linear transformation from the intermediate space* $(X_0, X_1)_{\theta,q}$ *to the intermediate space* $(Y_0, Y_1)_{\theta,q}$, *whose norm*

$$M \equiv \sup_{\| f \|_{(X_0, X_1)_{\theta,q}} = 1} \| Tf \|_{(Y_0, Y_1)_{\theta,q}} \quad \textit{satisfies} \quad M \leqslant M_0^{1-\theta} M_1^{\theta}.$$

Because this paper is devoted to interpolation and approximation by splines on finite intervals of the real line, it is necessary to define the Sobolev spaces $W_p^m[a, b]$ and the Besov spaces $B_p^{\sigma,q}[a, b]$. For m a positive integer and $1 \leqslant p \leqslant \infty$, the Sobolev space $W_p^m[a, b]$ is defined to be the collection of all real-valued functions $f(x)$ defined on the finite interval $[a, b]$ for which the generalized derivatives $D^j f$, $(D^j \equiv d^j/dx^j)$, $j = 0, 1, \ldots, m$, are all in $L_p[a, b]$. Equivalently, $W_p^m[a, b]$ is the collection of all real-valued functions $f(x)$ defined on $[a, b]$ for which $f \in C^{m-1}[a, b]$, $D^{m-1} f$ is absolutely continuous, and $D^m f \in L_p[a, b]$. It is well known that $W_p^m[a, b]$ is a Banach space with respect to the norm

$$\| f \|_{W_p^m[a,b]} \equiv \sum_{j=0}^{m} \| D^j f \|_{L_p[a,b]} \,.$$

To define the Besov space $B_p^{\sigma,q}[a, b]$, let $\omega_p^m(t, f)$ where m is a positive integer and $1 \leqslant p \leqslant \infty$, be the *m-th modulus of continuity* of $f \in L_p[a, b]$, i.e.,

$$\omega_p^m(t, f) \equiv \sup_{|y| \leqslant t} \left\| \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} f(x + \nu y) \right\|_{L_p[a,b]}, \tag{2.5}$$

where for y fixed, the L_p-norm is taken over the set of x in $[a, b]$ such that $x + \nu y \in [a, b]$ for all $\nu = 0, 1, \ldots, m$. The Besov space $B_p^{\sigma,q}[a, b]$, $0 < \sigma < m$, $1 \leqslant q \leqslant \infty$, $1 \leqslant p \leqslant \infty$, consists of all functions $f \in L_p[a, b]$ for which the norm

$$\|f\|_{B_p^{\sigma,q}[a,b]} \equiv \begin{cases} \|f\|_{L_p[a,b]} + \left(\int_0^1 (t^{-\sigma}\omega_p{}^m(t,f))^q \dfrac{dt}{t}\right)^{1/q}, & 1 \leqslant q < \infty, \\ \|f\|_{L_p[a,b]} + \sup\limits_{t>0} t^{-\sigma}\omega_p{}^m(t,f), & q = \infty, \end{cases} \tag{2.6}$$

is finite, and $B_p^{\sigma,q}[a, b]$ is a Banach space.

It would appear from (2.6) that $B_p^{\sigma,q}[a, b]$ depends on m, but actually it does not, and we have the following equivalent norms (cf. Grisvard [12], Butzer and Berens [9, p. 250], and Peetre [26]):

$$c_1 \|f\|_{B_p^{\sigma,q}[a,b]} \leqslant \begin{Bmatrix} \|f\|_{W_p^{[\sigma]}[a,b]} + \left(\int_0^1 (t^{[\sigma]-\sigma}\omega_p{}^1(t, D^{[\sigma]}f))^q \dfrac{dt}{t}\right)^{1/q}, & 1 \leqslant q < \infty, \\ \|f\|_{W_p^{[\sigma]}[a,b]} + \sup\limits_{t>0} t^{[\sigma]-\sigma}\omega_p{}^1(t, D^{[\sigma]}f), & q = \infty, \end{Bmatrix} \leqslant C_2 \|f\|_{B_p^{\sigma,q}[a,b]} \tag{2.7}$$

for noninteger σ (where $[\sigma]$ denotes the integral part of σ) and

$$c_1 \|f\|_{B_p^{\sigma,q}[a,b]} \leqslant \begin{Bmatrix} \|f\|_{W_p^{\sigma-1}[a,b]} + \left(\int_0^1 (t^{-1}\omega_p{}^2(t, D^{\sigma-1}f))^q \dfrac{dt}{t}\right)^{1/q}, & 1 \leqslant q < \infty, \\ \|f\|_{W_p^{\sigma-1}[a,b]} + \sup\limits_{t>0} \dfrac{\omega_p{}^2(t, D^{\sigma-1}f)}{t}, & q = \infty, \end{Bmatrix} \leqslant C_2 \|f\|_{B_p^{\sigma,q}[a,b]} \tag{2.8}$$

for integral σ. Note that (2.7) imposes a generalized Hölder condition on $D^{[\sigma]}f$ for $f \in B_p^{\sigma,q}[a, b]$, while (2.8) imposes analogously a Zygmund condition (cf. Zygmund [38, p. 43]) on $D^{\sigma-1}f$.

As we shall see in Sections 4 and 5, many error bounds for spline approximation have been obtained in terms of the Hölder classes $C^{\nu,\alpha}[a, b]$. A function f is said to be in $C^{\nu,\alpha}[a, b]$, $\nu = 0, 1, \ldots$, $0 < \alpha \leqslant 1$, if $f \in C^\nu[a, b]$ and the following norm is finite:

$$\|f\|_{C^{\nu,\alpha}[a,b]} \equiv \max_{x\in[a,b]} |f(x)| + \max_{x,y\in[a,b]} \frac{|f^{(\nu)}(x) - f^{(\nu)}(y)|}{|x - y|^\alpha}.$$

It is clear from (2.7) that if $0 < \alpha < 1$, then

$$C^{\nu,\alpha}[a, b] = B_\infty^{\nu+\alpha,\infty}[a, b]. \tag{2.9}$$

For $\alpha = 1$, we see from the definitions of the norms that

$$W_\infty^{\nu+1}[a, b] \subset C^{\nu,1}[a, b] \subset B_\infty^{\nu+1,\infty}[a, b], \tag{2.10}$$

where the inclusions of (2.10) are again to be interpreted in the sense of continuous imbeddings, i.e., for any $f \in W_\infty^{\nu+1}[a, b]$,

$$\|f\|_{B_\infty^{\nu+1,\infty}[a,b]} \leqslant C_1 \|f\|_{C^{\nu,1}[a,b]} \leqslant C_2 \|f\|_{W_\infty^{\nu+1}[a,b]}.$$

We now characterize the spaces intermediate between Sobolev spaces $W_p{}^m[a, b]$ and Besov spaces $B_p^{\sigma,q}[a, b]$. (Cf. Grisvard [12] and Peetre [24]. Although these results of [12] and [24] were proved for R^n, it follows from the extension and imbedding theorems of Besov [3, 4] that they hold for finite intervals as well.)

THEOREM 2.3. *If* $1 \leqslant p_0,\ p_1 \leqslant \infty$, *and* $0 \leqslant \theta \leqslant 1$ *are such that* $1/p = (1 - \theta)/p + \theta/p_1$, *then*

$$(L_{p_0}[a, b], L_{p_1}[a, b])_{\theta,p} = L_p[a, b], \tag{2.11}$$

$$(W_{p_0}^m[a, b], W_{p_1}^m[a, b])_{\theta,p} = W_p{}^m[a, b]. \tag{2.12}$$

If $0 < \theta < 1$, $1 \leqslant p, q \leqslant \infty$, *then*

$$(L_p[a, b], W_p{}^m[a, b])_{\theta,q} = B_p^{\theta m,q}[a, b]. \tag{2.13}$$

Furthermore, if $\sigma_0 \neq \sigma_1$, $0 < \theta < 1$, $1 \leqslant q_0,\ q_1 \leqslant \infty$, *and* $1 \leqslant p \leqslant \infty$, *then identifying equivalent norms, we have*

$$(B_p^{\sigma_0,q_0}[a, b], B_p^{\sigma_1,q_1}[a, b])_{\theta,q} = B_p^{\sigma,q}[a, b], \qquad \sigma = \theta\sigma_1 + (1 - \theta)\,\sigma_0, \tag{2.14}$$

$$(B_{p_0}^{\sigma_0,q_0}[a, b], B_{p_1}^{\sigma_1,q_1}[a, b])_{\theta,p} = B_p^{\sigma,p}[a, b], \qquad \sigma = \theta\sigma_1 + (1 - \theta)\,\sigma_0, \tag{2.15}$$

$$1/p = (\theta/p_1) + [(1 - \theta)/p_0],$$

and for integer values of σ_i, *either of the spaces* $B_p^{\sigma_i,q_i}[a, b]$ *in* (2.14), (2.15) *may be replaced by* $W_p^{\sigma_i}[a, b]$.

To conclude this section, we state some important imbedding results due to Besov [3, 4] and Peetre [24, 26], which will be used in the next sections.

THEOREM 2.4. *If* $1 \leqslant p \leqslant \infty$ *and* m *is a positive integer, then*

$$B_p^{m,1}[a, b] \subset W_p{}^m[a, b] \subset B_p^{m,\infty}[a, b]. \tag{2.16}$$

If $1 \leqslant q_1 < q_2 \leqslant \infty$, $1 \leqslant p \leqslant \infty$, *and* $0 < \sigma$, *then*

$$B_p^{\sigma,q_1}[a, b] \subset B_p^{\sigma,q_2}[a, b]. \tag{2.17}$$

If $0 < \sigma_2 < \sigma_1$, $1 \leqslant q_1, q_2 \leqslant \infty$, *and* $1 \leqslant p \leqslant \infty$, *then*

$$B_p^{\sigma_1,q_1}[a, b] \subset B_p^{\sigma_2,q_2}[a, b]. \tag{2.18}$$

Furthermore, if $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, *and*

$$\sigma_1 - \frac{1}{p_1} = \sigma_2 - \frac{1}{p_2},$$

then

$$B_{p_1}^{\sigma_1,q_1}[a, b] \subset B_{p_2}^{\sigma_2,q_2}[a, b], \tag{2.19}$$

and if $\sigma_1 = 1/p_1 - 1/p_2 > 0$, *then*

$$B_{p_1}^{\sigma_1,1}[a, b] \subset L_{p_2}[a, b]. \tag{2.20}$$

A proof of (2.19) for $\sigma_1 - (1/p_1) > \sigma_2 - (1/p_2)$ and of (2.20) for $\sigma_1 > (1/p_1) - (1/p_2)$ is given in Besov [4]. For the embeddings (2.19) and (2.20) under the weaker hypotheses of Theorem 2.4 the only proof we know of is in Peetre [26] which is in Swedish. We, therefore, present a proof in the Appendix.

3. INTERPOLATING SPLINES

In this section, we first review results on error bounds for interpolating Lg-splines in a single variable, and then apply the theory of interpolation spaces of Section 2 to these results.

For n a positive integer, let M be a linear differential operator of the form

$$M \equiv \sum_{j=0}^{n} a_j(x)\, D^j,$$

where for some positive constant τ, $a_n(x) \geqslant \tau > 0$ for all $x \in [a, b]$, and where $a_j \in C^j[a, b]$, $j = 0, 1,..., n$. Next, let $\Lambda = \{\lambda_i\}_{i=1}^k$ be any set of linearly independent, bounded linear functionals on the Sobolev space $W_2{}^n[a, b]$, and let $r = (r_1, r_2, ..., r_k)$ denote any vector of real Euclidean k-space, E^k.

A function $s \in W_2^n[a, b]$ is called an *Lg-spline* (cf. Jerome and Schumaker [16] and Jerome and Varga [17]) interpolating r with respect to Λ, provided that it solves the following minimization problem:

$$\| Ms \|_{L_2[a,b]} = \inf\{\| Mf \|_{L_2[a,b]} : f \in U_\Lambda(r)\}, \tag{3.1}$$

where

$$U_\Lambda(r) \equiv \{f \in W_2^n[a, b] : \lambda_i(f) = r_i , i = 1, 2,..., k\}.$$

The class of all Lg-splines s satisfying (3.1) for some $r \in E^k$ is denoted by $\mathrm{Sp}(M, \Lambda)$.

Based on the results of Golomb [10], Jerome and Schumaker [16] have proved

THEOREM 3.1. *Given any $r \in E^k$, there exists an $s \in W_2^n[a, b]$ satisfying* (3.1). *A function $s \in U_\Lambda(r)$ satisfies* (3.1) *if and only if*

$$\int_a^b Ms \cdot Mg \, dx = 0 \qquad \textit{for all} \quad g \in U_\Lambda(0).$$

Moreover, any two solutions of (3.1), *corresponding to a fixed $r \in E^k$, differ by a function in the null space $\mathcal{N}$ of M, and* (3.1) *possesses a unique solution if and only if $\mathcal{N} \cap U_\Lambda(0) = \{0\}$. Finally,* $\mathrm{Sp}(M, \Lambda)$ *is a linear subspace of $W_2^n[a, b]$ of dimension $k + \dim\{\mathcal{N} \cap U_\Lambda(0)\}$.*

In order to obtain error estimates for interpolating Lg-splines, we place extra restrictions on $\Lambda = \{\lambda_i\}_{i=1}^k$. Let Δ (possibly empty), called the *partition* of $[a, b]$, be the set of all $x \in [a, b]$ for which there exists a $\lambda \in \Lambda$ such that $\lambda(f) = f(x)$. If Δ is not empty, we define, as in Jerome and Varga [17], $\bar{\Delta}$ as the maximum length of the subintervals into which $[a, b]$ is decomposed by points of Δ, and we similarly define $\underline{\Delta}$ as the corresponding minimum length. If Δ is not empty, and $x \in \Delta \cap (a, b)$, let $i(x)$ be defined as the maximal positive integer such that there exists a $\lambda_k \in \Lambda$ for which

$$\lambda_k(f) = D^k f(x) \tag{3.2}$$

for *each* $k = 0, 1,..., i(x) - 1$. In other words, $i(x)$ is the number of consecutive derivative point functionals of Λ associated with the point $x \in \Delta \cap (a, b)$. If x is a or b, define $i(x)$ as the total number of values of k, not necessarily consecutive, for which (3.2) is valid. With this notation, we define $\gamma(\Lambda)$ by

$$\gamma(\Lambda) = \sum_{x \in \Delta} i(x)$$

if Δ is not empty, and we define $\gamma(\Lambda) = 0$ if Δ is empty.

Based on arguments used in Schultz and Varga [30], the following interpolation error bounds extend slightly the results of Jerome and Varga [17].

THEOREM 3.2. *Let $\Lambda = \{\lambda_i\}_{i=1}^k$ be such that $\gamma(\Lambda) \geqslant n$ and such that $\mathscr{N} \cap U_\Lambda(0) = \{0\}$. If $f \in W_2^n[a, b]$, and $s \in \mathrm{Sp}(M, \Lambda)$ is the unique* Lg-*spline which interpolates f with respect to Λ, i.e., $\lambda_i(s) = \lambda_i(f)$ for $i = 1, 2, \ldots, k$, then, for $\bar{\Delta}$ sufficiently small,*

$$\| D^j(f - s)\|_{L_\infty[a,b]} \leqslant K_j(\bar{\Delta})^{n-j-1/2} \|f\|_{W_2^n[a,b]}, \qquad j = 0, 1, \ldots, n-1,$$

and

$$\| D^j(f - s)\|_{L_\tau[a,b]} \leqslant K_j'(\bar{\Delta})^{n-j} \|f\|_{W_2^n[a,b]}, \qquad j = 0, 1, \ldots, n, \qquad 1 \leqslant \tau \leqslant 2, \tag{3.3}$$

where K_j and K_j' are independent of Λ and f.

We remark that Theorem 3.2 remains valid with the hypothesis $\mathscr{N} \cap U_\Lambda(0) = \{0\}$ deleted. In this case, the Lg-spline $s \in \mathrm{Sp}(M, \Lambda)$ which satisfies (3.1) is not uniquely defined. However, with the hypothesis $\mathscr{N} \cap U_\Lambda(0) = \{0\}$, the bounded linear operator $T: W_2^n[a, b] \to L_2[a, b]$ defined by $Tf = f - s$ is then well defined, and this fact is needed in subsequent discussions.

We now assume that Λ is such that the following second integral relation (cf. Ahlberg, Nilson and Walsh [1, p. 205]) is valid:

$$\int_a^b (Mf - Ms)^2 \, dx = \int_a^b (f - s)\, M^*Mf \, dx, \tag{3.4}$$

where $f \in W_2^{2n}[a, b]$ and s is an Lg-spline which interpolates f with respect to Λ. The relation (3.4) is known to be valid (cf. Schultz and Varga [30, Theorem 5]) if a and b are points of Δ with $i(a) = i(b) = n$.

Again, the following result slightly extends the results of Jerome and Varga [17].

THEOREM 3.3. *Let $\Lambda = \{\lambda_i\}_{i=1}^k$ be such that $\gamma(\Lambda) \geqslant n$, $\mathscr{N} \cap U_\Lambda(0) = \{0\}$, and such that the second integral relation* (3.4) *is valid. If $f \in W_2^{2n}[a, b]$ and if $s \in \mathrm{Sp}(M, \Lambda)$ is the unique* Lg-*spline which interpolates f with respect to Λ, then for $\bar{\Delta}$ sufficiently small,*

$$\| D^j(f - s)\|_{L_\infty[a,b]} \leqslant K_j(\bar{\Delta})^{2n-j-1/2} \|f\|_{W_2^{2n}[a,b]}, \qquad j = 0, 1, \ldots, n-1, \tag{3.5}$$

and

$$\| D^j(f - s)\|_{L_\tau[a,b]} \leqslant K_j'(\bar{\Delta})^{2n-j} \|f\|_{W_2^{2n}[a,b]}, \qquad j = 0, 1, \ldots, n, \qquad 1 \leqslant \tau \leqslant 2,$$

where K_j and K_j' are independent of Λ and f.

Theorem 3.3 can be extended if the exact continuity class of $\mathrm{Sp}(M, \Lambda)$, which depends on Λ, is known. For example, if $i(x) \leqslant l$ for every $x \in \Delta \cap (a, b)$, where $1 \leqslant l \leqslant n$, then it is known (Jerome and Varga [17, Corollary 2.4]) that

$$\mathrm{Sp}(M, \Lambda) \subset C^{2n-2-(l-1)}[a, b].$$

Based on the results of Perrin [29], one can further extend the result of [17] to obtain

THEOREM 3.4. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be such that $\gamma(\Lambda_i) \geqslant n$, $\mathcal{N} \cap U_{\Lambda_i}(0) = \{0\}$, and such that the second integral relation* (3.4) *is valid for all i. Further, assume that $\bar{\Delta}_i/\underline{\Delta}_i \leqslant \mu$ for all $i = 1, 2,\ldots$, and that $\mathrm{Sp}(M, \Lambda_i) \subset C^{m-1}[a, b]$ for all $i = 1, 2,\ldots$, where $n \leqslant m < 2n$. If $f \in W_2^{2n}[a, b]$ and if $s_i \in \mathrm{Sp}(M, \Lambda_i)$ is the unique* Lg-*spline which interpolates f with respect to Λ_i, then there exists an i_0 such that for all $i \geqslant i_0$,*

$$\| D^j(f - s_i)\|_{L_\infty[a,b]} \leqslant K_j(\bar{\Delta}_i)^{2n-j-1/2} \|f\|_{W_2^{2n}[a,b]}, \qquad j = 0, 1,\ldots, m,$$

and

$$\| D^j(f - s_i)\|_{L_\tau[a,b]} \leqslant K_j'(\bar{\Delta}_i)^{2n-j} \|f\|_{W_2^{2n}[a,b]}, \quad j = 0, 1,\ldots, m, \quad 1 \leqslant \tau \leqslant 2,$$

where K_j and K_j' are independent of the Λ_i and f.

We now apply the theory of interpolation spaces of Section 2 to the results of Theorems 3.2–3.4. If $\Lambda = \{\lambda_i\}_{i=1}^{k}$ is such that $\gamma(\Lambda) \geqslant n$ and $\mathcal{N} \cap U_\Lambda(0) = \{0\}$, define the linear transformation T on $W_2{}^n[a, b]$ by

$$Tf = f - s,$$

where s is the unique Lg-spline in $\mathrm{Sp}(M, \Lambda)$ which interpolates f with respect to Λ. From (3.3) of Theorem 3.2, we have that

$$\| D^j(f - s)\|_{L_\tau[a,b]} \leqslant K_j(\bar{\Delta})^{n-j} \|f\|_{W_2{}^n[a,b]}, \qquad j = 0, 1,\ldots, n, \qquad 1 \leqslant \tau \leqslant 2.$$

From the definition of the Sobolev norm $\| \cdot \|_{W_\tau{}^n[a,b]}$ in Section 2, and the fact that $\bar{\Delta} \leqslant b - a$, the above inequalities give us

$$\|f - s\|_{W_\tau{}^n[a,b]} \leqslant K \|f\|_{W_2{}^n[a,b]}, \qquad 1 \leqslant \tau \leqslant 2,$$

as well as

$$\|f - s\|_{L_\tau[a,b]} \leqslant K'(\bar{\Delta})^n \|f\|_{W_2{}^n[a,b]}, \qquad 1 \leqslant \tau \leqslant 2.$$

Thus, by choosing $X_0 = W_2{}^n[a, b] = X_1 = \mathscr{X}$, $Y_0 = \mathscr{Y} = L_\tau[a, b]$, and

$Y_1 = W_\tau{}^n[a, b]$, we see from the above inequalities that T is a bounded linear mapping from X_i to Y_i, $i = 0, 1$, with norms bounded above by

$$M_0 = K'(\bar{\Delta})^n, \qquad M_1 = K, \quad \text{respectively.}$$

Clearly, since, from (2.13) of Theorem 2.3, the Besov space $B_\tau^{\sigma,q}[a, b]$ is the intermediate space $(L_\tau[a, b], W_\tau{}^n[a, b])_{\theta,q}$, where $0 < \sigma = \theta n < n$ and $1 \leqslant q \leqslant \infty$, a direct application of Theorem 2.2 yields that

$$\| f - s \|_{B_\tau^{\sigma,q}[a,b]} \leqslant K(\sigma, q)(\bar{\Delta})^{n-\sigma} \| f \|_{W_2{}^n[a,b]}, \qquad 0 < \sigma < n, \qquad 1 \leqslant \tau \leqslant 2,$$

for any $1 \leqslant q \leqslant \infty$, where $K(\sigma, q)$ is independent of f and Λ. In a completely similar way, we deduce

PROPOSITION 3.1. *Let $\Lambda = \{\lambda_i\}_{i=1}^k$ be such that $\gamma(\Lambda) \geqslant n$, $\mathcal{N} \cap U_\Lambda(0) = \{0\}$, and such that the second integral relation of* (3.4) *is valid. If $f \in W_2{}^n[a, b]$ and if $s \in \mathrm{Sp}(M, \Lambda)$ is the unique* Lg-*spline which interpolates f with respect to Λ, then for $\bar{\Delta}$ sufficiently small,*

$$\| f - s \|_{B_\tau^{\sigma,q}[a,b]} \leqslant K(\sigma, q)(\bar{\Delta})^{n-\sigma} \| f \|_{W_2{}^n[a,b]}, \qquad 0 < \sigma < n, \tag{3.6}$$

for all q, $1 \leqslant q \leqslant \infty$, where $1 \leqslant \tau \leqslant 2$, and where $K(\sigma, q)$ is independent of $\bar{\Delta}$. If, moreover, $f \in W_2^{2n}[a, b]$, then

$$\| f - s \|_{B_\tau^{\sigma,q}[a,b]} \leqslant K'(\sigma, q)(\bar{\Delta})^{2n-\sigma} \| f \|_{W_2^{2n}[a,b]}, \qquad 0 < \sigma < n, \tag{3.7}$$

for all q and τ, $1 \leqslant q \leqslant \infty$, $1 \leqslant \tau \leqslant 2$, where $K'(\sigma, q)$ is independent of f and Λ.

The next result, however, is of more interest, in that we interpolate between the spaces on the right sides of (3.6) and (3.7), using the fact from (2.14) that

$$(W_2{}^n[a, b], W_2^{2n}[a, b])_{\theta,q} = B_2^{\sigma,q}[a, b], \quad \sigma = \theta n + (1 - \theta)\, 2n, \quad 1 \leqslant q \leqslant \infty.$$

More precisely, let $X_0 = W_2{}^n[a, b] = \mathscr{X}$, $X_1 = W_2^{2n}[a, b]$, and $\mathscr{Y} = Y_0 = Y_1 = B_\tau^{\sigma,q}[a, b] = Y$. Then, applying Theorem 2.2 to the inequalities of Proposition 3.1, gives

THEOREM 3.5. *Let $\Lambda = \{\lambda_i\}_{i=1}^k$ be such that $\gamma(\Lambda) \geqslant n$, $\mathcal{N} \cap U_\Lambda(0) = \{0\}$, and such that the second integral relation of* (3.4) *is valid. If $f \in B_2^{\sigma_2,r}[a, b]$, $n < \sigma_2 < 2n$, and if $s \in \mathrm{Sp}(M, \Lambda)$ is the unique* Lg-*spline which interpolates f with respect to Λ, then for $\bar{\Delta}$ sufficiently small,*

$$\| f - s \|_{B_\tau^{\sigma_1,q}[a,b]} \leqslant K(q, \sigma_1, \sigma_2)(\bar{\Delta})^{\sigma_2-\sigma_1} \| f \|_{B_2^{\sigma_2,r}[a,b]}, \tag{3.8}$$

where $1 \leqslant \tau \leqslant 2$, $0 < \sigma_1 < n$, *and* $1 \leqslant q, r \leqslant \infty$, *and where* $K(q, \sigma_1, \sigma_2)$ *is independent of* f *and* Λ. *Moreover, in the limiting cases* $f \in W_2^n[a, b]$, $f \in W_2^{2n}[a, b]$, *the inequalities* (3.6) *and* (3.7) *are valid.*

We remark that the results of Theorem 3.5 extend those of Jerome and Varga [17] and Schultz and Varga [30], in that the error bounds for Lg-spline interpolation are obtained for functions which are in $W_2^n[a, b]$, but not in $W_2^{2n}[a, b]$. In particular, these error bounds apply to functions in $W_2^l[a, b]$, where l is any positive integer satisfying $n \leqslant l \leqslant 2n$, since, from (2.16) of Theorem 2.4, we have

$$W_2^l[a, b] \subset B_2^{l,\infty}[a, b].$$

We state this as

COROLLARY 3.1. *With the hypotheses of Theorem* 3.5, *let* $f \in W_2^l[a, b]$, $n \leqslant l \leqslant 2n$. *Then,*

$$\| f - s \|_{B_\tau^{q,\sigma}[a,b]} \leqslant K(q, \sigma, l)(\bar{\Delta})^{l-\sigma} \| f \|_{W_2^l[a,b]}, \qquad 1 \leqslant \tau \leqslant 2,$$

where $0 < \sigma < n$ *and* $1 \leqslant q \leqslant \infty$. *In particular,*

$$\| D^j(f - s)\|_{L_\tau[a,b]} \leqslant K(\bar{\Delta})^{l-j} \| f \|_{W_2^l[a,b]}, \qquad j = 0, 1,..., n, \qquad 1 \leqslant \tau \leqslant 2.$$

For another result of interest which can be obtained as a special case of Theorem 3.5, we utilize (2.9), (2.10), and (2.19). We have

COROLLARY 3.2. *With the hypothesis of Theorem* 3.5, *let* $f \in C^{l,\alpha}[a, b]$, *where* l *is a positive integer*, $0 < \alpha \leqslant 1$, *and* $n \leqslant l + \alpha \leqslant 2n$. *Then*

$$\| D^j(f - s)\|_{L_\tau[a,b]} \leqslant K(\bar{\Delta})^{l+\alpha-j} \| f \|_{C^{l,\alpha}[a,b]},$$

for all $j = 0, 1,..., n$, $1 \leqslant \tau \leqslant 2$.

It is also worth noting that the exponents of $\bar{\Delta}$ obtained in Theorem 3.5 are, in general, *best possible*. This follows from the results of Birkhoff, Schultz and Varga [6] concerning Hermite piecewise polynomial interpolation, i.e. the special case in which $M = D^n$ and Λ consists only of point functionals of the form

$$\lambda_i(f) = D^j f(x_i), \qquad j = 0, 1,..., n - 1,$$

for all $i = 0, 1,..., N + 1$, where $a = x_0 < x_1 < \cdots < x_{N+1} = b$.

To extend further the results of Theorem 3.5, we use the embedding results of (2.19) and (2.20) of Theorem 2.4. Specifically, from (2.20), we know that

$$B_2^{\sigma_1,1}[a,b] \subset L_p[a,b], \qquad \text{for} \quad 2 < p, \qquad \sigma_1 = \frac{1}{2} - \frac{1}{p}.$$

Hence, it follows that there exists a positive constant C such that

$$\| f - s \|_{L_p[a,b]} \leqslant C \| f - s \|_{B_2^{\sigma_1,1}[a,b]}.$$

Thus, from (3.8) with $\sigma_1 = 1/2 - 1/p$, we have

PROPOSITION 3.2. *With the hypotheses of Theorem* 3.5, *let* $f \in B_2^{\sigma_2,r}[a,b]$, $n < \sigma_2 < 2n$. *Then,*

$$\| f - s \|_{L_p[a,b]} \leqslant K(q, \sigma_2, p)(\bar{\Delta})^{\sigma_2 - 1/2 + 1/p} \| f \|_{B_2^{\sigma_2,r}[a,b]}, \tag{3.9}$$

$2 \leqslant p \leqslant \infty$, *where* $1 \leqslant r \leqslant \infty$.

Similarly, using (2.19) of Theorem 2.4, we have that

$$\| f - s \|_{B_p^{\sigma,q}[a,b]} \leqslant C \| f - s \|_{B_2^{\sigma_1,q}[a,b]} \qquad \text{where} \qquad \sigma_1 - \frac{1}{2} = \sigma - \frac{1}{p},$$

$2 \leqslant p \leqslant \infty$. From Theorem 3.5, it follows that

$$\| f - s \|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta})^{\sigma_2 - \sigma_1} \| f \|_{B_2^{\sigma_2,r}[a,b]}.$$

Consequently, since $\sigma_1 = \sigma + \frac{1}{2} - 1/p$, we have

THEOREM 3.6. *With the hypotheses of Theorem* 3.5, *let* $f \in B_2^{\sigma_2,r}[a,b]$, *where* $n < \sigma_2 < 2n$. *Then, in addition to* (3.9),

$$\| f - s \|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta})^{\sigma_2 - \sigma - 1/2 + 1/p} \| f \|_{B_2^{\sigma_2,r}[a,b]} \tag{3.10}$$

for any $0 < \sigma < n - \frac{1}{2} + 1/p$, *where* $2 \leqslant p \leqslant \infty$.

Thus far, the results based on interpolation spaces involve bounds for $\| f - s \|_{B_p^{\sigma,q}[a,b]}$ for $0 < \sigma < n$, and one would desire similar results for $\sigma > n$. This can, in fact, be achieved, based on the results of Perrin [29] in Theorem 3.4.

THEOREM 3.7. *With the hypotheses of Theorems* 3.4 *and* 3.5, *let* $f \in B_2^{\sigma_2,r}[a,b]$, *where* $n < \sigma_2 < 2n$. *Then, if* $2 \leqslant p \leqslant \infty$, *there exists an* i_0 *such that for* $i > i_0$,

$$\| f - s_i \|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta}_i)^{\sigma_2 - \sigma - 1/2 + 1/p} \| f \|_{B_2^{\sigma_2,r}[a,b]}$$

for any positive

$$\sigma < \left(n - 1 + \frac{2}{p}\right)\left(\frac{2n - \sigma_2}{n}\right) + \frac{m}{n}(\sigma_2 - n).$$

Moreover, in the limiting cases $f \in W_2{}^n[a, b]$, $f \in W_2^{2n}[a, b]$, *we have, respectively,*

$$\|f - s_i\|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta}_i)^{n-\sigma-1/2+1/p} \|f\|_{W_2{}^n[a,b]}, \qquad 0 < \sigma < n - 1 + \frac{2}{p},$$

$$\|f - s_i\|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta}_i)^{2n-\sigma-1/2+1/p} \|f\|_{W_2^{2n}[a,b]}, \qquad 0 < \sigma < m.$$

All the results of this section basically depend upon the error estimates for Lg-spline interpolation of Theorems 3.2–3.4. These same error bounds have been extended to more general operators M and linear functionals Λ (cf. Lucas [20], Varga [36], and Jerome and Pierce [15]), so that our interpolation results, Theorems 3.6–3.7, *remain valid* for these more general M and Λ, with *no change in the arguments*. We have presented the simplest error bounds for Lg-spline interpolation so as to make the discussion as brief and clear as possible.

4. Special Types of Interpolating Splines

In the previous section, the emphasis was on the L_2-theory. Here we shall present L_p-estimates, which may be obtained by applying the theory of Section 2 to estimates known for certain special types of splines. We begin with *Hermite L-spline interpolation*, where L_p-error bounds are known for all p with $2 \leqslant p \leqslant \infty$. We then turn to interpolation by cubic and quintic splines and to interpolation of periodic functions by polynomial splines on a uniform mesh. In these cases, we obtain L_p-error bounds, $2 < p < \infty$, from known L_2- and L_∞-error bounds.

For Hermite L-spline interpolation, let $\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$ be any partition of $[a, b]$, and let the *Hermite L-spline space* $H(M, \Delta)$ be specifically the Lg-spline space (cf. Section 3) $\mathrm{Sp}(M, \Lambda)$ in which the set Λ of linearly independent bounded linear functions on $W_2{}^n[a, b]$ is given by $\Lambda = \{\lambda_{i,j}\}_{i=0,j=0}^{N+1,n-1}$, where

$$\lambda_{i,j} \equiv D^j s(x_i), \qquad i = 0, 1,..., N + 1, \qquad j = 0, 1,..., n - 1.$$

The following is a known result of Swartz and Varga [34, Corollary 7.5]. (For special cases, see Birkhoff, Schultz, and Varga [6], and Hall [13]).

Theorem 4.1. *For any* $f \in W_p{}^k[a, b]$, $1 \leqslant k \leqslant 2n$, $1 \leqslant p \leqslant \infty$, *and any* *partition Δ of $[a, b]$ for which $\bar{\Delta}/\underline{\Delta} \leqslant B$, let s be the unique interpolant of f in $H(M, \Delta)$ in the sense that*

$$D^j s(x_i) = D^j(\mathscr{L}_{2n,i} f)(x_i), \qquad j = 0, 1,..., n-1, \qquad i = 0, 1,..., N+1,$$

where, for each i, $i = 0, 1,..., N+1$, there exist 2n specified consecutive knots $x_{j_i}, x_{j_i+1},..., x_{j_i+2n-1}$ with $x_i \in [x_{j_i}, x_{j_i+2n-1}]$, and where $\mathscr{L}_{2n,i} f$ is the Lagrange polynomial (of degree $2n-1$) interpolating to f at these knots, i.e., $(\mathscr{L}_{2n,i} f)(x_l) = f(x_l)$, $l = j_i, j_i+1,..., j_i+2n-1$. Then, for r and j with $\max(p, 2) \leqslant r \leqslant \infty$, $j = 0, 1,..., \min(k-1, n-1)$, *we have*

$$\| f - s \|_{W_r^j[a,b]} \leqslant K(\bar{\Delta})^{k-j-1/p+1/r} \| f \|_{W_p^k[a,b]} . \tag{4.1}$$

We can immediately apply Theorems 2.2 and 2.3 to obtain

COROLLARY 4.1. *For any $f \in B_p^{\sigma,q}[a, b]$, $1 < \sigma < 2n$, $1 \leqslant q \leqslant \infty$, and any partition Δ of $[a, b]$ for which $\bar{\Delta}/\underline{\Delta} \leqslant B$, let s be the unique interpolant in $H(M, \Delta)$ in the sense of Theorem* 4.1. *Then, if* $\max(p, 2) \leqslant r \leqslant \infty$,

$$\| f - s \|_{W_r^j[a,b]} \leqslant K(\bar{\Delta})^{\sigma-j-1/p+1/r} \| f \|_{B_p^{\sigma,q}[a,b]} \tag{4.2}$$

for any nonnegative integer j with $j < \min(k-1, n)$. *Furthermore,*

$$\| f - s \|_{B_r^{\tau,q'}[a,b]} \leqslant K(\bar{\Delta})^{\sigma-\tau-1/p+1/r} \| f \|_{B_p^{\sigma,q}[a,b]} \tag{4.3}$$

if $0 < \tau < \min(k-1, n)$, $1 \leqslant q' \leqslant \infty$.

One goal of this study of error estimates for piecewise-polynomial interpolants is to establish inequalities of the type (4.1)–(4.3) for *spline* functions. Theorem 3.6 is a step in this direction. For cubic and quintic splines, we can go beyond Theorem 3.6. In analogy with the Hermite L-spline space $H(M, \Delta)$, we now define the *spline space* $\mathrm{Sp}^{(n)}(\Delta)$ (which corresponds (cf. Section 3) to the Lg-spline space $\mathrm{Sp}(M, \Lambda)$ with $M \equiv D^n$, and Λ particularly chosen). If $\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$ is a partition of $[a, b]$, then $\mathrm{Sp}^{(n)}(\Delta)$ is the collection of all real-valued functions s on $[a, b]$ such that s is a polynomial of degree at most $2n-1$ on each subinterval (x_i, x_{i+1}), $i = 0, 1,..., N$, and such that $s \in C^{2n-2}[a, b]$. In the case of interpolation by a cubic ($n = 2$) or quintic ($n = 3$) spline without any restriction on $\bar{\Delta}/\underline{\Delta}$, we have de Boor's result [8]:

THEOREM 4.2. *For any $f \in C^k[a, b]$, $n \leqslant k \leqslant 2n-1$, where $n = 2$ or 3, let s be the unique interpolant of f in $\mathrm{Sp}^{(n)}(\Delta)$, i.e.,*

$$\begin{aligned} s(x_i) &= f(x_i), \qquad i = 0, 1,..., N+1, \\ D^j s(a) &= D^j f(a), \qquad D^j s(b) = D^j f(b), \qquad j = 1,..., n-1. \end{aligned} \tag{4.4}$$

Then,

$$\|f - s\|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{k-j}\, \omega_\infty{}^1(\bar{\Delta}, D^k f), \qquad j = 0, 1,..., n.$$

Sharma and Meir [32] obtained Theorem 4.2 for the case $n = 2$, as well as for that of a periodic quintic spline, $n = 3$, with every knot a double knot. Special cases of Theorem 4.2 for cubic splines were obtained earlier by Atkinson [2] and Birkhoff and de Boor [5].

We shall not apply the theory of Section 2 directly to Theorem 4.2, but rather to its weaker form, Corollary 4.2. For our purposes, it is more convenient to have estimates in terms of the Hölder classes $C^{\nu,\alpha}[a, b]$, rather than in terms of the moduli of continuity $\omega_\infty{}^1(\bar{\Delta}, D^k f)$. As a direct consequence of Theorem 4.2, we have

COROLLARY 4.2. *For any* $f \in C^{\nu,\alpha}[a, b]$, $\nu = n, n + 1,..., 2n - 1$, $n = 2$ *or* 3, $0 < \alpha \leqslant 1$, *let s be its unique interpolant in* $\mathrm{Sp}^{(n)}(\Delta)$. *Then*

$$\|f - s\|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{\nu+\alpha-j}\, \|f\|_{C^{\nu,\alpha}[0,1]}, \qquad j = 0, 1,..., n.$$

Furthermore, for any $f \in W_\infty{}^n[a, b]$,

$$\|f - s\|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{n-j}\, \|f\|_{W_\infty{}^n[a,b]}, \qquad j = 0, 1,..., n.$$

Because the space $\mathrm{Sp}^{(n)}(\Delta)$ is a special Lg-spline space $\mathrm{Sp}(M, \Lambda)$ with $M \equiv D^n$ and because of (4.4), it follows that the hypotheses of Theorem 3.5 are fulfilled, i.e., $\gamma(\Lambda) \geqslant n$, $\mathcal{N} \cap U_\Lambda(0) = \{0\}$, and the second integral relation (3.4) is valid. Consequently, Corollaries 3.1 and 3.2 are applicable. Thus, we may couple Corollary 4.2 with Corollaries 3.1 and 3.2, via Theorems 2.2 and 2.3. The result is

THEOREM 4.3. *For* $f \in C^{n-1}[a, b]$ *and* $n = 2$ *or* 3, *let s be its unique interpolant in* $\mathrm{Sp}^{(n)}(\Delta)$ (*cf.* (4.4)). *Then, for f in* $W_p{}^k[a, b]$, $k = n, n + 1,..., 2n$, $2 \leqslant p \leqslant \infty$, *we have*

$$\|f - s\|_{W_p{}^j[a,b]} \leqslant K(\bar{\Delta})^{k-j}\, \|f\|_{W_p{}^k[a,b]}, \qquad j = 0, 1,..., n,$$

and for $f \in B_p^{\tau,r}[a, b]$, $n < \tau < 2n$, $1 \leqslant r \leqslant \infty$, $2 \leqslant p \leqslant \infty$, *we have*

$$\|f - s\|_{B_p^{\sigma,q}[a,b]} \leqslant K(\bar{\Delta})^{\tau-\sigma}\, \|f\|_{B_p^{\tau,r}[a,b]}, \qquad 0 < \sigma < n, \qquad 1 \leqslant q \leqslant \infty.$$

It is worthwhile to compare Theorems 4.2 and 4.3. If $f \in B_\infty^{\tau,\infty}[a, b]$ where $n < \tau < 2n$ and τ is not an integer, then Theorem 4.3 and the imbedding $B_\infty^{j,1}[a, b] \subset W_\infty{}^j[a, b]$ of (2.16) give

$$\|f - s\|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{\tau-j}\, \|f\|_{B_\infty^{\tau,\infty}[a,b]}, \qquad j = 0, 1,..., n.$$

This, however, is, in general, less sharp than the result $k = [\tau]$ of Theorem 4.2, viz.,

$$\| f - s \|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{[\tau]-j}\, \omega_\infty{}^1(\bar{\Delta}, D^{[\tau]}f),$$

since, from (2.7), the ratio $\omega_\infty{}^1(t, D^{[\sigma\tau]}f)/t^{\tau-[\tau]} \| f \|_{B_\infty^{\tau,\infty}[a,b]}$ is bounded for all t, and this ratio may in fact tend to zero as $t \to 0$. On the other hand, if $f \in B_\infty^{k+1,\infty}[a, b]$, $k + 1 = n + 1, n + 2,..., 2n - 1$, the imbedding (2.16) applied to the second inequality of Theorem 4.3 gives

$$\| f - s \|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{k+1-j} \| f \|_{B_\infty^{k+1,\infty}[a,b]}, \qquad j = 0, 1,..., n, \tag{4.5}$$

and this inequality is *not* implied by the analogous inequality

$$\| f - s \|_{W_\infty{}^j[a,b]} \leqslant K(\bar{\Delta})^{k-j}\, \omega_\infty{}^1(\bar{\Delta}, D^k f) \tag{4.6}$$

of Theorem 4.2. To show this, consider the function

$$f(x) = x^{k+1} \ln x, \qquad 0 \leqslant x \leqslant 1,$$

where k is a positive integer with $n \leqslant k \leqslant 2n - 2$. Then $f \in C^k[0, 1]$ and

$$\omega_\infty{}^1(t, D^k f) \sim k!\, t \ln(1/t) \qquad \text{as} \qquad t \to 0.$$

But, because f is an element of $B_\infty^{k+1,\infty}[0, 1]$, it follows that the upper bound of (4.5) is asymptotically better than that of (4.6) by the factor $\ln(1/\bar{\Delta})$.

We now define the Besov space $B_p^{\sigma,q}[\bigcirc]$ of periodic functions. Given a function f in $L_p[0, 2\pi]$, we extend it periodically to $(-\infty, +\infty)$, and say that its extension is in $L_p[\bigcirc]$. Then, the m-th modulus of continuity is defined (cf. (2.5)) by

$$\omega_p{}^m(t, f) = \sup_{|y| \leqslant t} \left| \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} f(x + \nu y) \right|_{L_p[0,2\pi]}$$

where the L_p-norm is taken over *all* $x \in [0, 2\pi]$. The Besov space $B_p^{\sigma,q}[\bigcirc]$, $0 < \sigma < m, 1 \leqslant q \leqslant \infty, 1 \leqslant p \leqslant \infty$, consists of all such periodic $f \in L_p[\bigcirc]$ for which the norm of (2.6), with the above definition of $\omega_p{}^m(t, f)$, is finite. We remark that $C[\bigcirc]$, $C^{\nu,\alpha}[\bigcirc]$, and $W_p{}^j[\bigcirc]$ denote, respectively, the space of continuous 2π-periodic functions, the Hölder spaces of 2π-periodic functions, and the Sobolev spaces of 2π-periodic functions.

For a partition $0 \leqslant x_1 < x_2 < \cdots < x_N < 2\pi$, $N \geqslant 1$, of the interval $[0, 2\pi]$, let Δ denote the periodic extension of this partition to $(-\infty, +\infty)$, with $\bar{\Delta}$ and $\underline{\Delta}$ denoting, respectively, the maximum and minimum lengths of the subintervals (x_i, x_{i+1}) (with $x_{N+1} = x_1 + 2\pi$). Then $\mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ is defined as the collection of all $s \in C^{2n-2}(-\infty, +\infty)$ which are of period 2π and which coincide with a polynomial of degree at most $2n - 1$ on each subinterval (x_i, x_{i+1}).

For periodic cubic splines, we have the following result:

THEOREM 4.4. *Let $\{\Delta_i\}_{i=1}^{\infty}$ be periodic partitions of $(-\infty, +\infty)$ such that $\bar{\Delta}_i/\underline{\Delta}_i \leqslant \mu$ for $i = 1, 2,\ldots$. For $f \in C[\bigcirc]$, let $s_i \in \mathrm{Sp}_{\Delta_i}^{(2)}[\bigcirc]$ be its unique cubic spline interpolant, i.e.,*

$$f(x_j) = s_i(x_j), \qquad \forall x_j \in \Delta_i .$$

Then, for all i,

$$\|f - s_i\|_{L_\infty[0,2\pi]} \leqslant K\|f\|_{L_\infty[0,2\pi]}, \tag{4.7}$$

$$\|f - s_i\|_{W_\infty^j[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{4-j}\|f\|_{W_\infty^4[\bigcirc]}, \qquad j = 0, 1, 2, 3, \tag{4.8}$$

$$\|f - s_i\|_{L_\infty[0,2\pi]} \leqslant K\omega_\infty^1(\bar{\Delta}_i, f), \tag{4.9}$$

$$\|f - s_i\|_{W_\infty^j[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{3-j}\,\omega_\infty^1(\bar{\Delta}_i, D^3 f), \qquad j = 0, 1, 2, 3. \tag{4.10}$$

The estimates (4.8) and (4.10) are due to Birkhoff and de Boor [5], and (4.9) is due to Sharma and Meir [32]. Nord [22] has improved the constant in (4.9). Clearly, (4.7) follows directly from (4.9).

If we apply the theorems of Section 2 to the estimates of Theorem 4.4, we obtain the following

THEOREM 4.5. *Let $\{\Delta_i\}_{i=1}^{\infty}$ be periodic partitions of $(-\infty, +\infty)$ such that $\bar{\Delta}_i/\underline{\Delta}_i \leqslant \mu$ for $i = 1, 2,\ldots$. For $f \in C[\bigcirc]$, let $s_i \in \mathrm{Sp}_{\Delta_i}^{(2)}[\bigcirc]$ be its unique cubic spline interpolant. Then if $2 \leqslant p \leqslant r \leqslant \infty$, we have, respectively, for f in $W_p^4[\bigcirc]$ or $B_p^{\sigma,q}[\bigcirc]$,*

$$\|f - s_i\|_{W_r^j[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{4-j-1/p+1/r}\|f\|_{W_p^4[\bigcirc]}, \qquad j = 0, 1, 2, 3, \quad i = 1, 2,\ldots. \tag{4.11}$$

$$\|f - s_i\|_{L_r[0,2\pi]} \leqslant K(\bar{\Delta}_i)^{\sigma-1/p+1/r}\|f\|_{B_p^{\sigma,q}[\bigcirc]}, \qquad i = 1, 2,\ldots. \tag{4.12}$$

where either $4/p < \sigma < 4$ and $1 \leqslant q \leqslant \infty$, or $\sigma = 4/p$ and $1 \leqslant q \leqslant p$. Moreover, for $f \in B_p^{\sigma,q}[\bigcirc]$,

$$\|f - s_i\|_{W_r^3[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{\sigma-3-1/p+1/r}\|f\|_{B_p^{\sigma,q}[\bigcirc]}, \tag{4.13}$$

where either $3 + (2/p) < \sigma < 4$ *and* $1 \leqslant q \leqslant \infty$, *or* $\sigma = 3 + (2/p)$ *and* $1 \leqslant q \leqslant p$. *Furthermore, if* $2 \leqslant p \leqslant r \leqslant \infty$ *and* f *is in* $W_p^4[\bigcirc]$ *or in* $B_p^{\sigma,q}[\bigcirc]$, *we have, respectively,*

$$\|f - s_i\|_{B_r^{\tau,q'}[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{4-\tau-1/r+1/p} \|f\|_{W_p^4[\bigcirc]}, \quad 0 < \tau < 3, \quad 1 \leqslant q' \leqslant \infty, \tag{4.14}$$

$$\|f - s_i\|_{B_r^{\tau,q'}[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{\sigma-\tau-1/r+1/p} \|f\|_{B_p^{\sigma,q}[\bigcirc]}, \tag{4.15}$$

where either (i) $\sigma = 4/p$, $q = p$, $0 < \tau < (3/p) + (1/r)$, $1 \leqslant q' \leqslant \infty$, (ii) $\sigma = 4/p$, $q = p$, $\tau = (3/p) + (1/r)$, $p \leqslant q' \leqslant \infty$, (iii) $4/p < \sigma < 4$, $1 \leqslant q, q' \leqslant \infty$, $0 < \tau < \mu(\sigma)$, *where* (*see Fig.* 1)

$$\mu(\sigma) = \min\left(3, \frac{3}{p} + \frac{1}{r} + \left(\frac{3 - \frac{3}{p} - \frac{1}{r}}{3 - \frac{2}{p}}\right)\left(\sigma - \frac{4}{p}\right)\right),$$

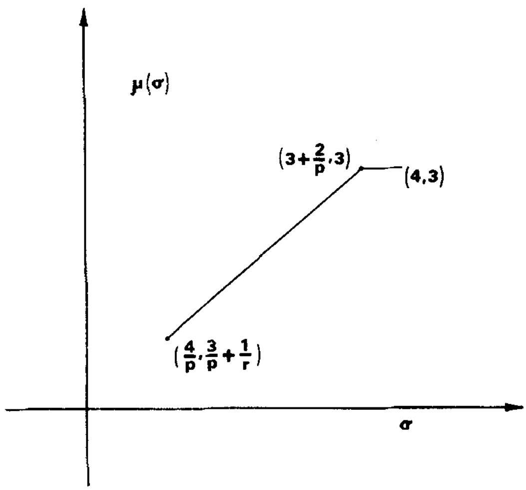

FIGURE 1

or (iv) $4/p < \sigma < 3 + (2/p)$, $\tau = \mu(\sigma)$, $1 \leqslant q, q' \leqslant \infty$.

Proof. Because Theorem 3.4 is applicable in the periodic case, (4.11) is a direct consequence of Theorem 2.2, (2.11) and (2.12) repeatedly applied to (4.8), and both inequalities of Theorem 3.4. Similarly, (4.14) follows from (2.13) of Theorem 2.3 applied to the cases $j = 0$ and $j = 3$ of (4.11). The special case of (4.12),

$$\|f - s_i\|_{L_p[0,2\pi]} \leqslant K(\bar{\Delta}_i)^{4/p} \|f\|_{B_p^{4/p,p}[\bigcirc]}, \qquad 2 \leqslant p \leqslant \infty, \tag{4.16}$$

follows from an application of Theorem 2.2 and (2.15) to (4.7) and to the case $\tau = 2, j = 0$, and $n = 2$ of (3.3). The case of (4.12),

$$\|f - s_i\|_{L_p[0,2\pi]} \leqslant K(\bar{\Delta}_i)^\sigma \|f\|_{B_p^{\sigma,q}[\bigcirc]}, \tag{4.17}$$

$2 \leqslant p \leqslant \infty, 4/p < \sigma < 4, 1 \leqslant q \leqslant \infty$, is obtained by applying Theorem 2.2 and (2.15) to (4.16) and to (4.11) with $j = 0$ and $r = p$. The other cases of (4.12) follow directly from (4.17) and Theorem 2.4.

We conclude directly from (4.10) that

$$\|f - s_i\|_{W_\infty^3[\bigcirc]} \leqslant K\|f\|_{W_\infty^3[\bigcirc]}. \tag{4.18}$$

By Theorem 2.2, (2.12) and (2.15), the estimate intermediate between (4.18) and the case $j = 3$ in Theorem 3.4 is

$$\|f - s_i\|_{W_r^3[\bigcirc]} \leqslant K(\bar{\Delta}_i)^{\sigma-3-1/p+1/r} \|f\|_{B_p^{\sigma,q}[\bigcirc]}, \tag{4.19}$$

$2 \leqslant p \leqslant r \leqslant \infty, \sigma = 3 + (2/p)$. Inequality (4.19) is a special case of (4.13); the case of (4.13) with $2 \leqslant p \leqslant r \leqslant \infty$, $1 \leqslant q \leqslant p$, $\sigma = 3 + (2/p)$ follows from (4.19) and (2.17). The other cases of (4.13) are proved by an application of Theorem 2.2 and (2.14) to (4.19) and to (4.11) with $j = 3$.

We still have to prove (4.15). By Theorem 2.2 and (2.15), the result intermediate between (4.7) and (3.3) with $n = j = 2$, $\tau = 2$ is

$$\|f - s_i\|_{B_p^{4/p,p}[\bigcirc]} \leqslant K\|f\|_{B_p^{4/p,p}[\bigcirc]}. \tag{4.20}$$

Case (ii) of (4.15) follows from (4.20), (2.17), and (2.19). Case (i) of (4.15) is the estimate intermediate between case (ii) of (4.15) and the case $\sigma = 4/p$ of (4.12). We obtain case (iv) of (4.15) by an application of Theorem 2.2 and (2.14) to case (ii) of (4.15) and to (4.13) with $\sigma = 3 + (2/p)$. Finally, case (iii) of (4.15) is obtained by writing either the estimate intermediate between case (iv) of (4.15) and case (i) of (4.15) or the one intermediate between case (iv) of (4.15) and (4.14). This completes the proof of Theorem 4.5.

If the partition Δ of $[a, b]$ is uniform, it is possible to obtain error bounds of the type given in Theorems 4.1–4.5 for polynomial splines ($M \equiv D^n$) of arbitrary order. We first consider, as in Theorem 4.1, the case where the derivatives of f at the end-points of $[a, b]$ are approximated via Lagrange interpolation polynomials. Then, we turn to the case of periodic splines on a uniform partition. Swartz and Varga [34], using results of Swartz [33], have obtained the following result.

THEOREM 4.6. *Given $f \in W_p^k[a, b]$ with $1 \leqslant k \leqslant 2n$, $1 \leqslant p \leqslant \infty$, and a* *uniform partition* Δ *of* $[a, b]$, *let* s *be the unique interpolant of* f *in* $\mathrm{Sp}^{(n)}(\Delta)$ *in the sense that*

$$s(x_i) = f(x_i), \qquad i = 0, 1,..., N+1,$$
$$D^j s(a) = D^j(\mathscr{L}_{2n,0} f)(a), \qquad j = 1,..., n-1,$$

where $\mathscr{L}_{2n,0} f$ *is the Lagrange polynomial* (*of degree* $2n - 1$) *interpolating to* f *at the knots* $x_0, x_1, ..., x_{2n-1}$, *and similar relations hold at* $x = b$. *Then, if* $\max(p, 2) \leqslant r \leqslant \infty$, $0 \leqslant j \leqslant k - 1$, *we have*

$$\| f - s \|_{W_r{}^j[a,b]} \leqslant K(\bar{\Delta})^{k-j-1/p+1/r} \| f \|_{W_p{}^k[a,b]} .$$

As the extension of this result to Besov spaces is clear, we omit its details.

When a periodic function is interpolated on a *uniform* mesh by a periodic polynomial spline, we can prove stronger results than those of Theorem 4.5. The basic estimates are due to Golomb [11] and Ahlberg, Nilson, and Walsh [1]. The error bounds in [1] involve the Hölder classes $C^{m,\alpha}[\bigcirc]$ and are given in the following theorem. Similar estimates, but in less generality, are contained in Golomb [11] and Schurer [31].

THEOREM 4.7. *Let* Δ *be a uniform partition of* $(-\infty, +\infty)$. *For* $f \in C[\bigcirc]$, *let* $s \in \mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ *be its unique interpolant. Then, for* $f \in C^{m,\alpha}[\bigcirc]$, $0 \leqslant m < 2n$, $0 < \alpha \leqslant 1$, *we have the estimate*

$$\| D^j(f - s) \|_{L_\infty[0,2\pi]} \leqslant K(\bar{\Delta})^{m+\alpha-j} \| f \|_{C^{m,\alpha}[\bigcirc]}, \qquad j = 0, 1,..., m, \tag{4.21}$$

and, for $f \in C[\bigcirc]$, *we have*

$$\| f - s \|_{L_\infty[0,2\pi]} \leqslant K \| f \|_{L_\infty[0,2\pi]} . \tag{4.22}$$

The other result on which we base our study of periodic splines is due to Golomb [11]. Golomb's estimates are in terms of the Hilbert space $H^\sigma[\bigcirc]$, $\sigma > 0$, of periodic functions

$$f(x) = \sum_j c_j e^{ijx}$$

for which the norm

$$\| f \|_{H^\sigma[\bigcirc]} \equiv | c_0 | + \Big(\sum_j | i |^{2\sigma} | c_j |^2 \Big)^{1/2}$$

is finite. It is clear from (2.6) that $H^\sigma[\bigcirc] = B_2^{\sigma,2}[\bigcirc]$, with equivalence of norms. Stated in terms of Besov spaces, Golomb's result is as follows.

THEOREM 4.8. *For* $f \in B_2^{\sigma,2}[\bigcirc]$, $\sigma > 1/2$, *let* $s \in \mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ *be its unique spline interpolant, where* Δ *is a uniform partition of* $(-\infty, +\infty)$, *and* $2n \geqslant \sigma$. *Then, we have*

$$\| D^j(f - s)\|_{L_2[0,2\pi]} \leqslant K(\bar{\Delta})^{\sigma-j} \| f \|_{B_2^{\sigma,2}[\bigcirc]}, \qquad 0 \leqslant j < \sigma - 1/2, \tag{4.23}$$

$$\| D^j(f - s)\|_{L_\infty[0,2\pi]} \leqslant K(\bar{\Delta})^{\sigma-j-1/2} \| f \|_{B_2^{\sigma,2}[\bigcirc]}, \qquad 0 \leqslant j < \sigma - 1/2. \tag{4.24}$$

The restriction $\sigma > 1/2$ in Theorems 4.8 is necessary for two reasons. On the one hand, the condition $f \in B_2^{\sigma,2}[\bigcirc]$, $\sigma > 1/2$, implies that f is continuous, while there exist functions in $B_2^{\sigma,2}[\bigcirc]$, $0 < \sigma \leqslant 1/2$, which are not even bounded (cf. Golomb [11]). On the other hand, for a fixed uniform periodic partition Δ of $(-\infty, +\infty)$, there exists a sequence of trigonometric polynomials $\{f_j\}$ such that

$$\| f_j \|_{B_2^{1/2,2}[\bigcirc]} \leqslant K, \tag{4.25}$$

while the corresponding interpolating periodic $s_j \in \mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ satisfy

$$\| s_j \|_{L_2[0,2\pi]} \to \infty. \tag{4.26}$$

In fact, we may assume without loss of generality that $\Delta = \{j/2\pi N\}$, $j = 0, \pm 1, \pm 2, \ldots$. Then, $s_j \in \mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ which interpolates

$$f_j(x) = \sum_{k=2}^{j} \frac{1}{k \log k} e^{iNkx}$$

is the constant function

$$s_j(x) = \sum_{k=2}^{j} \frac{1}{k \log k}.$$

Clearly, f_j and s_j satisfy (4.25) and (4.26), respectively.

We collect the results intermediate between Theorems 4.7, 4.8, and the periodic version of Theorem 3.2.

THEOREM 4.9. *For* $f \in C[\bigcirc]$, *let* $s \in \mathrm{Sp}_\Delta^{(n)}[\bigcirc]$ *be its unique spline interpolant, where* Δ *is a uniform partition of* $(-\infty, +\infty)$. *Then, for* $f \in W_p^{2n}[\bigcirc]$, $2 \leqslant p \leqslant \infty$, *we have*

$$\| f - s \|_{L_p[0,2\pi]} \leqslant K(\bar{\Delta})^{2n} \| f \|_{W_p^{2n}[\bigcirc]}, \tag{4.27}$$

$$\| f - s \|_{W_p^{2n-1}[\bigcirc]} \leqslant K \cdot \bar{\Delta} \| f \|_{W_p^{2n}[\bigcirc]}, \tag{4.28}$$

$$\| f - s \|_{B_p^{\tau,q}[\bigcirc]} \leqslant K(\bar{\Delta})^{2n-\tau} \| f \|_{W_p^{2n}[\bigcirc]}, \tag{4.29}$$

where $0 < \tau < 2n - 1$, $1 \leqslant q \leqslant \infty$. *Furthermore, if* $2 \leqslant p \leqslant \infty$ *and* $f \in B_p^{\sigma,r}[\bigcirc]$, $1 \leqslant r \leqslant \infty$, *we have*

$$\|f - s\|_{L_p[0,2\pi]} \leqslant K(\bar{\Delta})^{\sigma} \|f\|_{B_p^{\sigma,r}[\bigcirc]}, \qquad \frac{1}{p} < \sigma < 2n, \tag{4.30}$$

$$\|f - s\|_{W_p^{2n-1}[\bigcirc]} \leqslant K(\bar{\Delta})^{\sigma-2n+1} \|f\|_{B_p^{\sigma,p}[\bigcirc]}, \qquad 2n - 1 + \frac{1}{p} < \sigma < 2n, \tag{4.31}$$

$$\|f - s\|_{B_p^{\sigma,p}[\bigcirc]} \leqslant K \|f\|_{B_p^{\sigma,p}[\bigcirc]}, \qquad \sigma = 2n/p, \tag{4.32}$$

$$\|f - s\|_{B_p^{\tau,q}[\bigcirc]} \leqslant K(\bar{\Delta})^{\sigma-\tau} \|f\|_{B_p^{\sigma,r}[\bigcirc]}, \qquad \frac{1}{p} < \sigma < 2n, \quad 0 < \tau < h(\sigma), \tag{4.33}$$

where $h(\sigma)$ *is given by* (*see Fig*. 2)

$$h(\sigma) = \begin{cases} \left(1 + \dfrac{1}{2n-1}\right)\left(\sigma - \dfrac{1}{p}\right), & \dfrac{1}{p} < \sigma \leqslant \dfrac{2n}{p}, \\[2ex] \sigma - \dfrac{\sigma - \dfrac{2n}{p}}{(p-1)(2n-1)}, & \dfrac{2n}{p} < \sigma \leqslant 2n - 1 + \dfrac{1}{p}, \\[2ex] 2n - 1, & 2n - 1 + \dfrac{1}{p} < \sigma \leqslant 2n. \end{cases}$$

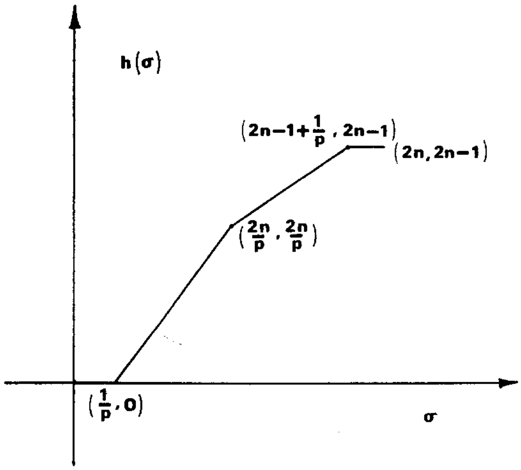

FIGURE 2

Proof. Inequalities (4.27) and (4.28) both follow from (4.21) and Theorem 3.4 by means of Theorem 2.2, (2.11), and (2.12). By (2.13) of Theorem 2.3, the estimate (4.29) is a direct consequence of (4.27) and (4.28).

By (2.15) of Theorem 2.3, the estimate intermediate between (4.22) and (4.23) with $j = 0$ is

$$\| f - s \|_{L_p[0,2\pi]} \leqslant C(\bar{\Delta})^{\sigma} \| f \|_{B_p^{\sigma,p}[\bigcirc]}, \qquad 2 \leqslant p \leqslant \infty, \qquad 1/p \leqslant \sigma \leqslant 4n/p. \tag{4.34}$$

Inequality (4.30) now follows from (4.27) and (4.34) by (2.14) and Theorem 2.2. Inequality (4.31) is a direct consequence of (2.15) and Theorem 2.2 applied to (4.23), (4.21), and Theorem 3.4. The estimate (4.32) is proved by an application of Theorem 2.2 and (2.15) to (4.22) and (3.3).

In order to prove (4.33), we apply Theorem 2.2 and (2.14) to (4.30) and (4.32) if $1/p < \sigma \leqslant 2n/p$, to (4.31) and (4.32) if $2n/p < \sigma \leqslant 2n - 1 + 1/p$, and to (4.29) and (4.30) if $2n - 1 + (1/p) < \sigma < 2n$. Q.E.D.

In Theorem 4.9 we considered the operator $Tf = f - s$, where $f \in W_p^{2n}[\bigcirc]$ or $f \in B_p^{\sigma,r}[\bigcirc]$, $2 \leqslant p \leqslant \infty$, and $Tf \in L_p[0, 2\pi]$ or $Tf \in B_p^{\sigma,q}[\bigcirc]$. It is also possible to consider T as an operator from $W_{p_1}^{2n}[\bigcirc]$ or $B_{p_1}^{\sigma,r}[\bigcirc]$, $2 \leqslant p_1 \leqslant \infty$, to $L_{p_2}[0, 2\pi]$ or $B_{p_2}^{\tau,r}[\bigcirc]$, $p_1 \leqslant p_2 \leqslant \infty$. The easiest way to obtain such a generalization is to combine the inequalities of Theorem 4.9 with the following inequalities:

$$\| f - s \|_{B_{p_2}^{\tau,q}[\bigcirc]} \leqslant K \| f - s \|_{B_{p_1}^{\tau+1/p_1-1/p_2,q}[\bigcirc]}, \qquad p_1 \leqslant p_2 \leqslant \infty, \qquad \tau > 0, \tag{4.35}$$

$$\| f - s \|_{L_{p_2}[0,2\pi]} \leqslant K \| f - s \|_{B_{p_1}^{1/p_1-1/p_2,1}[\bigcirc]}, \qquad p_1 \leqslant p_2 \leqslant \infty. \tag{4.36}$$

The estimates (4.35) and (4.36) are merely restatements of the imbeddings (2.19) and (2.20) of Theorem 2.4.

We would find, though, that we would not be able to derive all cases of (4.24) in this manner. Therefore, we give the estimates intermediate between (4.24) and those obtained from Theorem 4.8 by using (4.35) and (4.36). In the following theorem, we write out only the generalization of (4.33).

THEOREM 4.10. *For $f \in C[\bigcirc]$, let $s \in \mathrm{Sp}_{\Delta}^{(n)}[\bigcirc]$ be its unique spline interpolant, where Δ is a uniform partition of $(-\infty, +\infty)$. Then, for $f \in B_{p_1}^{\sigma,r}[\bigcirc]$, $1/p_1 < \sigma < 2n$, $2 \leqslant p_1 \leqslant \infty$, $1 \leqslant r \leqslant \infty$, we have*

$$\| f - s \|_{B_{p_2}^{\tau,q}[\bigcirc]} \leqslant K(\bar{\Delta})^{\sigma-\tau-1/p_1+1/p_2} \| f \|_{B_{p_1}^{\sigma,r}[\bigcirc]},$$

where $p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q \leqslant \infty$, *and* $0 < \tau < g(\sigma)$; $g(\sigma)$ *is given by* (*see Fig.* 3)

$$g(\sigma) = \begin{cases} \left(1 + \dfrac{p_1}{p_2(2n-1)}\right)\left(\sigma - \dfrac{1}{p_1}\right), & \dfrac{1}{p_1} < \sigma \leqslant \dfrac{2n}{p_1}, \\ \dfrac{2n-1}{p_1} + \dfrac{1}{p_2} - \left(1 - \dfrac{p_1}{p_2(p_1-1)(2n-1)}\right)\left(\sigma - \dfrac{2n}{p_1}\right), & \\ \qquad \dfrac{2n}{p_1} < \sigma \leqslant 2n - 1 + \dfrac{1}{p_1}, & \\ 2n-1, \quad 2n - 1 + \dfrac{1}{p_1} < \sigma \leqslant 2n. \end{cases}$$

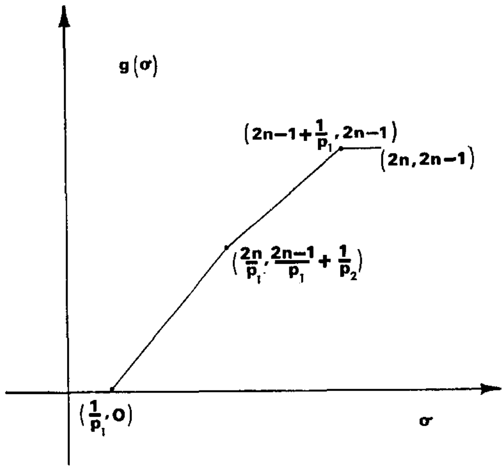

FIGURE 3

5. SPLINES OF BEST APPROXIMATION

For a family $\mathscr{F}$ of splines, we introduce the functional

$$E_p(f) \equiv \inf_{s \in \mathscr{F}} \| f - s \|_{L_p[a,b]}, \qquad 1 \leqslant p \leqslant \infty,$$

on $L_p[a, b]$. In this section, we discuss the behavior of $E_p(f)$ as $\bar{\Delta} \to 0$, for f in a Besov space. We begin by stating a result of de Boor [7] in this direction.

THEOREM 5.1. *Let* $\Delta : a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{N+1} = b$ *be a k-extended partition of* $[a, b]$, *i.e., for some integer* $k \geqslant 2$, $x_j < x_{j+k-1}$ *for all* $j = 0, 1, \ldots, N - k + 2$. *Consider the family* $\mathscr{F} = S_\Delta^{(k)}$ *of all real functions s* *on $[a, b]$ which coincide with a polynomial of degree at most $k-1$ on each subinterval (x_j, x_{j+1}), $j = 0, 1,\ldots, N$, and such that if x_j has multiplicity ν_j with respect to Δ, then $s \in C^{k-\nu_j-1}$ in a neighborhood of x_j. Then for any $f \in C^{j,\alpha}[a, b]$, $j = 0, 1,\ldots, k-1$, $0 < \alpha \leqslant 1$, we have*

$$E_\infty(f) \leqslant K(\bar{\Delta})^{j+\alpha} \|f\|_{C^{j,\alpha}[a,b]}, \tag{5.1}$$

where K is a constant, independent of f and Δ.

It is interesting to remark that de Boor's error estimate (5.1) was obtained by linear projections of $C^{j,\alpha}[0, 1]$ onto $S_\Delta^{(k)}$.

If the family $\mathscr{F}$ is one of the classes previously considered, viz., $\mathrm{Sp}^{(n)}(\Delta)$, $H^{(n)}(\Delta)$, or $\mathrm{Sp}(M, \Lambda)$, then it is easy to obtain upper bounds for $E_p(f)$ from the results of Sections 3-4, since

$$E_p(f) \leqslant \|f - \tilde{s}\|_{L_p[a,b]},$$

where $\tilde{s}$ is a spline interpolant of f. If the functional E_p were linear, we could immediately apply the intermediate-space theory of Section 2. However, E_p, for $p \neq 2$, is a *nonlinear* functional. But, we can make use of the fact that E_p is *semilinear*, i.e.,

$$E_p(f_1 + f_2) \leqslant E_p(f_1) + E_p(f_2).$$

As we shall see in the following lemma, some of the theory of intermediate spaces may be applied to E_p. Although this lemma may be regarded as a folk-theorem, we include its proof for completeness.

LEMMA 5.1. *Let X be a Banach space, and let $T : L_p[a, b] \to X$ be any nonlinear mapping such that there exist constants β, M_0, M_1, and some positive integer m for which*

$$\| T(f_1 + f_2)\|_X \leqslant \beta\{\| Tf_1 \|_X + \| Tf_2 \|_X\}, \qquad \forall f_i \in L_p[a, b], \tag{5.2}$$

and

$$\begin{aligned} \| Tf\|_X &\leqslant M_0 \|f\|_{L_p[a,b]} \qquad \forall f \in L_p[a, b], \\ \| Tf\|_X &\leqslant M_1 \|f\|_{W_p{}^m[a,b]} \qquad \forall f \in W_p{}^m[a, b]. \end{aligned} \tag{5.3}$$

Then, for any $f \in B_p^{\sigma,q}[a, b]$, $0 < \sigma < m$, $1 \leqslant q \leqslant \infty$, we have

$$\| Tf\|_X \leqslant C_0(\theta, q)\, M_0^{1-\theta} M_1{}^\theta \|f\|_{B_p^{\sigma,q}[a,b]}, \qquad \theta = \sigma/m. \tag{5.4}$$

Similarly, if (5.2) *holds and if for some* τ, r *with* $0 < \tau < m$, $1 \leqslant r \leqslant \infty$, *one has*

$$\begin{aligned} \| Tf \|_X &\leqslant M_0 \| f \|_{B_p^{\tau,r}[a,b]}, \\ \| Tf \|_X &\leqslant M_1 \| f \|_{W_p{}^m[a,b]}, \end{aligned} \tag{5.5}$$

then for any $f \in B_p^{\sigma,q}[a, b]$, $\tau < \sigma < m$, $1 \leqslant q \leqslant \infty$, *we have*

$$\| Tf \|_X \leqslant C_0(\theta, q)\, M_0^{1-\theta} M_1{}^\theta \| f \|_{B_p^{\sigma,q}[a,b]}, \qquad \theta = \frac{(\sigma - \tau)}{(m - \tau)}. \tag{5.6}$$

Proof. Since the proofs of the inequalities (5.4) and (5.6) are similar, we shall only establish (5.4). Our basic tool is the inequality

$$K_1(t, Tf) \leqslant 2M_0 K_2(M_1 t/M_0, f), \qquad 0 < t < \infty, \tag{5.7}$$

where, as in (2.3), K_1 and K_2 are defined for $t > 0$ by

$$K_1(t, Tf) = \inf_{Tf = v_0 + v_1} (\| v_0 \|_X + t \| v_1 \|_X), \tag{5.8}$$

and

$$K_2(t, f) = \inf_{f = f_0 + f_1} (\| f_0 \|_{L_p[a,b]} + t \| f_1 \|_{W_p{}^m[a,b]}). \tag{5.9}$$

Suppose we have secured (5.7). Then, if $1 \leqslant q < \infty$ and $0 < \theta < 1$,

$$\begin{aligned} \left(\int_0^\infty (t^{-\theta} K_1(t, Tf))^q \frac{dt}{t} \right)^{1/q} &\leqslant 2M_0 \left(\int_0^\infty (t^{-\theta} K_2(M_1 t/M_0, f))^q \frac{dt}{t} \right)^{1/q} \\ &= 2M_0^{1-\theta} M_1{}^\theta \left(\int_0^\infty (t^{-\theta} K_2(t, f))^q \frac{dt}{t} \right)^{1/q}, \end{aligned} \tag{5.10}$$

and for $q = \infty$ and $0 < \theta < 1$,

$$\sup_{t>0} t^{-\theta} K_1(t, Tf) \leqslant 2M_0^{1-\theta} M_1{}^\theta \sup_{t>0} t^{-\theta} K_2(t, f). \tag{5.11}$$

From the definition (5.8), it is readily verified that

$$K_1(t, Tf) = \begin{cases} t \| Tf \|_X, & 0 < t \leqslant 1, \\ \| Tf \|_X, & t > 1, \end{cases} \tag{5.12}$$

so that the left sides of (5.10) and (5.11) are equal to $C(\theta, q) \| Tf \|_X$, where

$$C(\theta, q) \equiv \begin{cases} \left\{ \dfrac{1}{q} \left(\dfrac{1}{1-\theta} + \dfrac{1}{\theta} \right) \right\}^{1/q}, & 1 \leqslant q < \infty, \\ 1, & q = \infty. \end{cases}$$

Furthermore, it follows from (2.4) that the expressions on the right sides of (5.10) and (5.11) are simply

$$2M_0^{1-\theta}M_1^{\theta}\,\|f\|_{(L_p[a,b],W_p^m[a,b])_{\theta,q}}.$$

Consequently, (5.10) and (5.11) become

$$C(\theta,q)\,\|Tf\|_X \leqslant 2M_0^{1-\theta}M_1^{\theta}\,\|f\|_{(L_p[a,b],W_p^m[a,b])_{\theta,q}}. \tag{5.13}$$

Next, using (2.13) of Theorem 2.4, we have

$$(L_p[a,b],\,W_p^m[a,b])_{\theta,q} = B_p^{\theta m,q}[a,b], \qquad 0<\theta<1, \qquad 1\leqslant q\leqslant\infty.$$

Thus, using equivalence of norms, i.e.,

$$C_1(\theta,q)\,\|f\|_{B_p^{\theta m,q}[a,b]} \leqslant \|f\|_{(L_p[a,b],W_p^m[a,b])_{\theta,q}} \leqslant C_2(\theta,q)\,\|f\|_{B_p^{\theta m,q}[a,b]},$$

(5.13) can be expressed as

$$C(\theta,q)\,\|Tf\|_X \leqslant 2M_0^{1-\theta}M_1^{\theta}C_2(\theta,q)\,\|f\|_{B_p^{\theta m,q}[a,b]}, \quad 0<\theta<1, \quad 1\leqslant q\leqslant\infty,$$

i.e., the desired result of (5.4). Thus, (5.7) implies Lemma 5.1.

It remains to prove (5.7). Assume $f\in L_p[a,b]$. From the definition (5.9) it is clear that for each fixed positive t, there exist $f_0\in L_p[a,b]$ and $f_1\in W_p^m[a,b]$ with $f=f_0+f_1$ such that

$$\|f_0\|_{L_p[a,b]} + \frac{M_1 t}{M_0}\|f_1\|_{W_p^m[a,b]} \leqslant 2K_2(M_1t/M_0\,,f). \tag{5.14}$$

Therefore, using the hypotheses (5.2) and (5.3), and the relation (5.12), we find for $t>1$ that

$$\begin{aligned}K_1(t,Tf) &= \|Tf\|_X = \|T(f_0+f_1)\|_X \leqslant \beta\{\|Tf_0\|_X + \|Tf_1\|_X\}\\ &\leqslant \beta\{\|Tf_0\|_X + t\,\|Tf_1\|_X\} \leqslant \beta\{M_0\,\|f_0\|_{L_p[a,b]} + tM_1\,\|f_1\|_{W_p^m[a,b]}\}\\ &\leqslant 2\beta M_0K_2(M_1t/M_0\,,f),\end{aligned} \tag{5.15}$$

the last inequality following from (5.14). Similarly, for $0<t\leqslant 1$, we have

$$\begin{aligned}K_1(t,Tf) &= t\,\|T(f_0+f_1)\| \leqslant \beta\{\|Tf_0\|_X + t\,\|Tf_1\|_X\}\\ &\leqslant \beta\{M_0\,\|f_0\|_{L_p[a,b]} + tM_1\,\|f_1\|_{W_p^m[a,b]}\} \leqslant 2\beta M_0K_2(M_1t/M_0\,,f).\end{aligned} \tag{5.16}$$

The desired inequality (5.7) now follows directly from (5.15) and (5.16).

Q.E.D.

We now present our main theorem on best approximation by splines.

THEOREM 5.2. *Let $1 \leqslant p \leqslant r \leqslant \infty$, and let $\mathscr{F}$ be a collection of splines defined on the interval $[a, b]$. Suppose that for some positive integer m, $m > 1/p - 1/r$, and for all $f \in W_p{}^m[a, b]$, the following inequality is valid:*

$$E_r(f) \equiv \inf_{s \in \mathscr{F}} \| f - s \|_{L_r[a,b]} \leqslant K(\bar{\Delta})^{m-1/p+1/r} \| f \|_{W_p{}^m[a,b]} . \tag{5.17}$$

Then, for all $f \in B_p^{\sigma,q}[a, b]$, $0 \leqslant 1/p - 1/r < \sigma < m$, $1 \leqslant q \leqslant \infty$, we have

$$E_r(f) \equiv \inf_{s \in \mathscr{F}} \| f - s \|_{L_r[a,b]} \leqslant K(\bar{\Delta})^{\sigma-1/p+1/r} \| f \|_{B_p^{\sigma,q}[a,b]} . \tag{5.18}$$

Proof. First, suppose that $p = r$. From the trivial estimate

$$E_p(f) \leqslant \| f \|_{L_p[a,b]}$$

and the assumed inequality (5.17), the desired result (5.18) follows immediately by Lemma 5.1 with

$$Tf \equiv E_p(f) \qquad \text{and} \qquad X \equiv R. \tag{5.19}$$

If $r > p$, we use (2.20) of Theorem 2.4 to deduce that

$$E_r(f) \leqslant \| f \|_{L_p[a,b]} \leqslant C \| f \|_{B_p^{\sigma_1,1}[a,b]} , \qquad \sigma_1 = \frac{1}{p} - \frac{1}{r} . \tag{5.20}$$

But then, inequality (5.18) for $r > p$ follows from Lemma 5.1, using (5.19) and the inequalities (5.17) and (5.20). Q.E.D.

Since the theorems of Sections 3–4 provide estimates of the type (5.17), we can, of course, apply Theorem 5.2 to each of them. We give two such applications to illustrate the method.

As a consequence of Theorem 5.2 and (3.9), we have

COROLLARY 5.1. *Let $\mathscr{F}$ be the collection* $\mathrm{Sp}(M, \Lambda)$ *of* Lg-*splines on $[a, b]$ with respect to a family $\Lambda = \{\lambda_i\}_{i=1}^k$ of bounded linear functionals on $W_2{}^n[a, b]$ such that $\gamma(\Lambda) \geqslant n$, $\mathscr{N} \cap U_\Lambda(0) = \{0\}$, and such that the second integral relation* (3.4) *is valid. If $f \in B_2^{\sigma,q}[a, b]$, $0 < \sigma < 2n$, $1 \leqslant q \leqslant \infty$, and if $2 \leqslant p \leqslant \infty$, then*

$$E_p(f) \leqslant K(\bar{\Delta})^{\sigma-1/2+1/p} \| f \|_{B_2^{\sigma,q}[a,b]} . \tag{5.21}$$

We remark that the inequality (3.9) of Proposition 3.2 directly implies (5.21) if $n < \sigma < 2n$, since trivially $E_p(f) \leqslant \| f - s \|_{L_p[a,b]}$ where $s \in \mathrm{Sp}(M, \Lambda)$ is the unique Lg-spline interpolant of f. On the other hand, if $0 < \sigma < n$, the inequality (5.21) extends Proposition 3.2.

As a consequence of Theorem 5.2 and (4.27), we obtain

COROLLARY 5.2. *Let $\mathscr{F}$ be the collection* $\mathrm{Sp}^{(n)}[\bigcirc]$ *of periodic* $2n$*-splines on a uniform mesh. If* $f \in B_p^{\sigma,q}[\bigcirc]$, $0 < \sigma < 2n$, $2 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, *then*

$$E_p(f) \equiv \inf_{s \in Sp^{(n)}[\bigcirc]} \| f - s \|_{L_p[0,2\pi]} \leqslant K(\bar{\Delta})^\sigma \| f \|_{B_p^{\sigma,q}[\bigcirc]} .$$

Finally, we state a corollary which extends Theorem 5.1 of de Boor [7]. From the special case $j = k - 1$, $\alpha = 1$ of Theorem 5.1 and from Theorem 5.2, we obtain (cf. (2.10))

COROLLARY 5.3. *Let Δ be a k-extended partition of $[a, b]$. Then, for* $f \in B_\infty^{\sigma,q}[a, b]$, $0 < \sigma < k$, $1 \leqslant q \leqslant \infty$,

$$E_\infty(f) \equiv \inf_{s \in S_\Delta^{(k)}} \| f - s \|_{L_\infty[a,b]} \leqslant K(\bar{\Delta})^\sigma \| f \|_{B_\infty^{\sigma,q}[a,b]} . \tag{5.22}$$

We remark that if σ is not an integer, (5.22) is *not* an extension of (5.1), but is equivalent to it since

$$B_\infty^{\sigma,q}[a, b] \subset B_\infty^{\sigma,\infty}[a, b] = C^{[\sigma],\sigma-[\sigma]}[a, b].$$

However, if σ is an integer j, then (5.21) is stronger than (5.1) in the sense that it is valid for a larger class of functions since (cf. (2.10))

$$B_\infty^{j,\infty}[a, b] \not\subset C^{j-1,1}[a, b] \qquad \text{and} \qquad C^{j-1,1}[a, b] \subset B_\infty^{j,\infty}[a, b].$$

APPENDIX. PROOF OF (2.19) AND (2.20)

The proof is based on Peetre's characterization [27] of Besov spaces. This characterization is in terms of functions $\varphi(x)$ and $\Psi(x)$, $-\infty < x < \infty$, whose Fourier transforms $\hat{\varphi}$, $\hat{\Psi}$ are $C^\infty(-\infty, +\infty)$ functions such that the support of $\hat{\varphi}(\xi)$ is the set $\{\xi : 1/2 \leqslant |\xi| \leqslant 2\}$ and the support of $\hat{\Psi}(\xi)$ is the set $\{\xi : |\xi| \leqslant 2\}$. For every positive integer k, we define the function $\varphi_k(x)$ by the relation

$$\varphi_k(x) = 2^k \varphi(2^k x), \tag{A.1}$$

so that

$$\hat{\varphi}_k(\xi) = \hat{\varphi}(2^{-k}\xi).$$

The support of $\hat{\varphi}_k(\xi)$ is thus the set $\{\xi : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$. Further, the functions φ, Ψ are chosen so that

$$\hat{\Psi}(\xi) + \sum_{k=1}^{\infty} \hat{\varphi}_k(\xi) = 1. \tag{A.2}$$

It follows from (A.1) that

$$\| \varphi_k \|_{L_1(-\infty,\infty)} \leqslant C, \qquad \| \varphi_k \|_{L_\infty(-\infty,\infty)} \leqslant C' 2^k.$$

Consequently, from Hölder's inequality, we have that

$$\| \varphi_k \|_{L_r(-\infty,\infty)} \leqslant C'' 2^{k(1-1/r)}, \qquad 1 \leqslant r \leqslant \infty. \tag{A.3}$$

We shall also use a special case of a continuation theorem for Besov spaces. It is clear that if $f \in B_p^{\sigma,q}(-\infty, \infty)$, then the restriction of f to $[a, b]$ is in $B_p^{\sigma,q}[a, b]$. Conversely, Besov [3] has shown that if $f \in B_p^{\sigma,q}[a, b]$, then there exists an $F \in B_p^{\sigma,q}(-\infty, \infty)$ such that

$$F(x) = f(x), \qquad a \leqslant x \leqslant b,$$
$$\|f\|_{B_p^{\sigma,q}[a,b]} \leqslant C \| F \|_{B_p^{\sigma,q}(-\infty,\infty)} .$$

Consequently, we may restrict our attention to the space $B_p^{\sigma,q}(-\infty, \infty)$. This restriction is made in order to enable us to take the Fourier transform of f.

Peetre [26, 27] has shown that if $\sigma > 0$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant q \leqslant \infty$, then the norm in $B_p^{\sigma,q}(-\infty, +\infty)$ is equivalent to

$$N_p^{\sigma,q}(f) = \begin{cases} \| \Psi * f \|_{L_p(-\infty,\infty)} + \left\{ \sum\limits_{k=1}^{\infty} (2^{k\sigma} \| \varphi_k * f \|_{L_p(-\infty,\infty)})^q \right\}^{1/q}, & 1 \leqslant q < \infty, \\ \| \Psi * f \|_{L_p(-\infty,\infty)} + \sup\limits_k 2^{k\sigma} \| \varphi_k * f \|_{L_p(-\infty,\infty)} , & q = \infty, \end{cases}$$

where $\Psi * f$ denotes the convolution of Ψ and f. Hence, there exist constants C_1 and C_2 such that

$$C_1 N_p^{\sigma,q}(f) \leqslant \|f\|_{B_p^{\sigma,q}(-\infty,\infty)} \leqslant C_2 N_p^{\sigma,q}(f), \qquad f \in B_p^{\sigma,q}(-\infty, \infty). \tag{A.4}$$

We may therefore use $N_p^{\sigma,q}(f)$ as the norm in $B_p^{\sigma,q}(-\infty, \infty)$.

By (2.17), it is sufficient to prove the imbedding (2.19) only in the case $q_1 = q_2 = q$. We begin by estimating $\| \Psi * f \|_{L_{p_2}(-\infty,\infty)}$ and $\| \varphi_k * f \|_{L_{p_2}(-\infty,\infty)}$. After taking the Fourier transform of $\Psi * f$ and noting (A.2) and the location of the supports of $\hat{\Psi}$ and $\hat{\varphi}_k$, we find that

$$\Psi * f = \Psi * (\Psi * f) + \varphi_1 * (\Psi * f). \tag{A.5}$$

Similarly,

$$\begin{aligned} \varphi_1 * f &= \Psi * (\varphi_1 * f) + \sum_{\nu=0}^{1} \varphi_{1+\nu} * (\varphi_1 * f), \\ \varphi_k * f &= \sum_{\nu=-1}^{1} \varphi_{k+\nu} * (\varphi_k * f), \qquad k \geqslant 2. \end{aligned} \tag{A.6}$$

We now apply Young's inequality (cf. O'Neil [22]),

$$\| g * h \|_{L_{p_2}(-\infty,\infty)} \leqslant \| g \|_{L_r(-\infty,\infty)} \| h \|_{L_{p_1}(-\infty,\infty)}, \qquad 1/p_2 = 1/r + 1/p_1 - 1,$$

and inequality (A.3) to deduce that

$$\begin{aligned} \| \varphi_{k+\nu} * (\varphi_k * f) \|_{L_{p_2}(-\infty,\infty)} &\leqslant C\, 2^{-(k+\nu)(1/p_2 - 1/p_1)} \| \varphi_k * f \|_{L_{p_1}(-\infty,\infty)}, \\ \| \Psi * (\Psi * f) \|_{L_{p_2}(-\infty,\infty)} &\leqslant C \| \Psi * f \|_{L_{p_1}(-\infty,\infty)}, \\ \| \Psi * (\varphi_1 * f) \|_{L_{p_2}(-\infty,\infty)} &\leqslant C \| \varphi_1 * f \|_{L_{p_1}(-\infty,\infty)}. \end{aligned} \tag{A.7}$$

Consequently, if $\sigma_1 - (1/p_1) = \sigma_2 - (1/p_2)$, it follows from (A.4) through (A.7) that for $1 \leqslant q \leqslant \infty$,

$$\begin{aligned} \| f \|_{B_{p_2}^{\sigma_2,q}(-\infty,\infty)} &\leqslant C_2 N_{p_2}^{\sigma_2,q}(f) = C_2 \| \Psi * f \|_{L_{p_2}(-\infty,\infty)} \\ &\quad + C_2 \left(\sum_{k=1}^{\infty} (2^{k\sigma_2} \| \varphi_k * f \|_{L_{p_2}(-\infty,\infty)})^q \right)^{1/q} \\ &\leqslant C' \| \Psi * f \|_{L_{p_1}(-\infty,\infty)} + C' \left(\sum_{k=1}^{\infty} (2^{k\sigma_1} \| \varphi_k * f \|_{L_{p_1}(-\infty,\infty)})^q \right)^{1/q} \\ &= C' N_{p_1}^{\sigma_1,q}(f) \leqslant \frac{C'}{C_1} \| f \|_{B_{p_1}^{\sigma_1,q}(-\infty,\infty)}. \end{aligned}$$

This proves (2.19) for the case $1 \leqslant q < \infty$; the proof in the case $q = \infty$ is analogous.

We now turn to the proof of (2.20). If $B_p^{\sigma,q}(-\infty, \infty)$, $\sigma \leqslant 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is defined to be the collection of tempered distributions f with finite norm $N_p^{\sigma,q}(f)$, then it is easy to show that $B_p^{\sigma,q}(-\infty, \infty)$ is a Banach space (cf. Peetre [26, 27]). Furthermore, the above argument shows that the imbedding (2.19) is valid even when σ_1 or σ_2 is nonpositive. In particular, if $p_1 < p_2$ and $\sigma_1 = 1/p_1 - 1/p_2$, then

$$B_{p_1}^{\sigma_1,1}(-\infty, \infty) \subset B_{p_2}^{0,1}(-\infty, \infty). \tag{A.8}$$

However, we also have (cf. Peetre [26, 27])

$$B_{p_2}^{0,1}(-\infty, \infty) \subset L_{p_2} \subset B_{p_2}^{0,\infty}(-\infty, \infty). \tag{A.9}$$

The imbedding (2.20) now follows from (A.8) and (A.9). Q.E.D.

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