Compact Operators, Invariant Subspaces, and Spectral Synthesis

JOHN FROELICH*

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

Communicated by the Editors

Let $\mathscr{L}$ be a subspace lattice which contains a sequence $\{P_n\}$ of commuting projections such that for any subsequence $\{P_{n_i}\}$, $VP_{n_i} = I$ and $AP_{n_i} = 0$. Then $\mathrm{Alg}(\mathscr{L}) \cap \mathscr{K}(H) = 0$. Suppose G is a compact abelian group and $E \subseteq G$ is a closed set. There is a family $\mathscr{L}_E$ of commutative subspace lattices for which $\mathrm{Alg}(\mathscr{L}_E) \cap \mathscr{K}(H) \neq 0$ precisely when E is a set of multiplicity in the sense of harmonic analysis. By showing that the graph of $\leqslant$ is a set of uniqueness in $2^\infty \times 2^\infty$ we obtain a "thin set" proof that $\mathrm{Alg}(2^\infty, \leqslant, m_{1/2})$ contains no nonzero compact operators.

For an important class of sets $K \subseteq G \times G$

$$A_{\min}(K) = A_{\max}(K)$$

iff K is a set of spectral synthesis in the group $G \times G$. It follows that if E_1 and E_2 are sets of spectral synthesis in G then $\mathrm{Alg}(P_{E_1}) \bar{\otimes} \mathrm{Alg}(P_{E_2}) = \mathrm{Alg}(P_{E_1} \otimes P_{E_2})$ iff $E_1 \times E_2$ is a set of spectral synthesis in $G \times G$ and if semigroups Σ_1 and Σ_2 are sets of spectral synthesis in G then $\mathrm{Alg}(\Sigma_1) \bar{\otimes} \mathrm{Alg}(\Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$ iff $\Sigma_1 \times \Sigma_2$ is a set of spectral synthesis in $G \times G$. The operator algebra $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$ is synthetic iff for all n, $\mathrm{Alg}(P_1 \otimes \cdots \otimes P_n)$ is synthetic. This implies that the operator algebras $\mathrm{Alg}(2^\infty, \leqslant, m_p)$ are synthetic. © 1988 Academic Press, Inc.

INTRODUCTION

The theory of non-self-adjoint operator algebras and their invariant subspace lattices has been of continual interest since the late 1940s [28]. In [15] P. R. Halmos drew attention to the more tractable class of reflexive operator algebras. The deepest structural results for reflexive operator algebras with a commutative subspace lattice were later obtained by Arveson [2]. Apart from introducing new tools for the analysis of such algebras the paper contributed significantly to the general theory of

1

operator algebras and established important connections with harmonic analysis and lattice theory.

The present work concerns itself with reflexive algebras and their invariant subspace lattices by establishing new connections with harmonic analysis, in particular the uniqueness problem for trigonometric series, and by developing further the subject of spectral synthesis in reflexive algebras with a commutative subspace lattice.

In Section 1 we explore the question of when the operator algebra associated with a commutative subspace lattice contains a nonzero compact operator. By viewing an operator as a distribution on a product group $G \times G$, the theory in [2] allows one to construct, for each closed $E \subseteq G$, a commutative subspace lattice $\mathscr{L}_E$ for which $\mathrm{Alg}(\mathscr{L}_E) \cap \mathscr{K}(H) \neq 0$ precisely when E is a set of multiplicity in the sense of harmonic analysis. This approach also leads to a number of theorems and examples describing the possible quantity and quality of compact operators in a reflexive algebra with a comutative subspace lattice. By showing that the graph of $\leqslant$ is a set of uniqueness in $2^{\infty} \times 2^{\infty}$ we obtain another proof that $\mathrm{Alg}(2^{\infty}, \leqslant, m_{1/2})$ contains no nonzero compact operators.

The second section is concerned with the phenomena of "noncommutative spectral synthesis." It is shown that for an important class of sets $K \subseteq G \times G$, $\mathscr{A}_{\min}(K) = \mathscr{A}_{\max}(K)$ iff K is a set of spectral synthesis in the group G. A consequence of our analysis is that the tensor product formula for reflexive algebras [19] implies that the cartesian product of two sets of spectral synthesis is again a set of spectral synthesis. The latter statement is at present a difficult unsolved problem in harmonic analysis.

Finally, we establish a general spectral synthesis theorem for infinite tensor products of operator algebras. This implies that the operator algebras $\mathrm{Alg}(2^{\infty}, \leqslant, m_p)$ are synthetic.

1. COMPACT OPERATORS IN REFLEXIVE ALGEBRAS WITH A COMMUTATIVE SUBSPACE LATTICE

1.1. *The invariant Subspace Lattice of a Compact Operator*

Little is known about the invariant subspace lattice of a general Hilbert space operator. In the opposite direction one may take a subspace lattice $\mathscr{L}$ and ask whether there is an operator A such that $\mathscr{L} \subseteq \mathrm{Lat}(A)$ (transitive lattice problem) [28, p. 78].

DEFINITION 1.1.1. A subspace lattice $\mathscr{L}$ is transitive for compact operators if $\mathscr{L} \subseteq \mathrm{Lat}(K)$ and K compact implies $K = 0$.

Remark. $\mathscr{L}$ is transitive for compact operators iff $\mathrm{Alg}(\mathscr{L}) \cap \mathscr{K}(H) = \{0\}$.

We first present a coordinate-free proof of a theorem which appeared in [11]. This serves as motivation for the main developments of Section 1. Let VE_α and ΛE_α denote the supremum and infimum, respectively, of a family $\{E_\alpha\}$ of projections on a Hilbert space H.

LEMMA 1.1.2. *Let $\{E_n\}$ be a sequence of commuting projections such that $VE_{n_i} = I$ for any subsequence $\{E_{n_i}\}$. If $x \in H$ satisfies* $\liminf \|E_n x\| = 0$ *then $x = 0$.*

Proof. Let $\varepsilon > 0$ and extract a subsequence $\{E_{n_j}\}$ such that $\|E_{n_j} x\|^2 \leqslant \varepsilon/2^j$ for each j. Let $P_1 = E_{n_1}$, $P_2 = E_{n_2} P_1^\perp$, ..., $P_j = E_{n_j} P_{j-1}^\perp$, ... Then the P_j's are mutually orthogonal projections and $\sum P_j = I$. Note that $P_j \leqslant E_{n_j}$. Since $\|x\|^2 = \sum \|P_j x\|^2 \leqslant \sum \|E_{n_j} x\|^2 \leqslant \sum \varepsilon/2^j = \varepsilon$ we have that $\|x\|^2 \leqslant \varepsilon$. Hence $x = 0$.

LEMMA 1.1.3. *Let $\{E_n\}$ be a sequence of commuting projections such that $\Lambda E_{n_j} = 0$ for any subsequence $\{E_{n_j}\}$. If $E_n x_n = x_n$ and $\{x_n\}$ converges to x then $x = 0$*

Proof. The projections $\{1 - E_n\}$ satisfy the hypotheses of Lemma 1.1.2. Therefore it is enough to show that $\liminf \|(1 - E_n) x\| = 0$. Pick $\varepsilon > 0$. Since $x_n \to x$ we can find n such that $\|x_n - x\| < \varepsilon$.
Therefore

$$\|(1 - E_n) x\| = \|(1 - E_n)(x - x_n)\| \leqslant \|x - x_n\| < \varepsilon.$$

THEOREM 1.1.4. *Let $\mathscr{L}$ be a commutative subspace lattice. Suppose that $\mathscr{L}$ contains a sequence $\{E_n\}$ of commuting projections such that $VE_{n_i} = 1$ and $\Lambda E_{n_i} = 0$ for any subsequence $\{E_{n_i}\}$. Then $\mathscr{L}$ is transitive for compact operators.*

Proof. Let K be a compact operator such that $\mathscr{L} \subseteq \mathrm{Lat}(K)$ and let $x \in H$. The sequence $\{E_n K^* x\}$ is bounded in norm and so there is a subsequence $\{E_{n_j} K^* x\}$ such that $KE_{n_j} K^* x$ is norm convergent, since K is compact. Also $KE_{n_j} K^* x \in E_{n_j}(H)$. By Lemma 1.1.3, $KE_{n_j} K^* x \to 0$ in norm. Therefore $(KE_{n_j} K^* x, x) \to 0$. So $(E_{n_j} K^* x, K^* x) = (E_{n_j} K^* x, E_{n_j} K^* x) \to 0$. By Lemma 1.1.2, $K^* x = 0$, so $K^* = 0$ and $K = 0$.

As an immediate consequence we have

COROLLARY 1.1.5. *Let $(X, \leqslant, m)$ be a standard pre-ordered probability measure space and $\{B_n\}$ a sequence of decreasing Borel sets such that for any subsequence $\{B_{n_i}\}$, $m(\bigcup_{i=1}^\infty B_{n_i}) = 1$ and $m(\bigcap_{i=1}^\infty B_{n_i}) = 0$. Then $\mathrm{Alg}(X, \leqslant, m)$ contains no nonzero compact operators.*

Proof. The projections P_{B_n} satisfy the hypotheses of Theorem 1.1.4.

Corollary 1.1.5 may be used to settle a conjecture of Arveson [2, p. 498]. For the pre-ordered measure spaces $(2^\infty, \leqslant, m_p)$, $0 < p < 1$, take $B_n = \{(x_1, x_2, ...): x_n = 0\}$. It is an easy measure-theoretic exercise to verify that if $\{B_{n_i}\}$ is a subsequence then

$$m_p \left(\bigcup_{i=1}^{\infty} B_{n_i} \right) = 1$$

and

$$m_p \left(\bigcap_{i=1}^{\infty} B_{n_i} \right) = 0.$$

Therefore $\text{Alg}(2^\infty, \leqslant, m_p)$ contains no nonzero compact operators.

1.2. Connections with the Theory of Exceptional Sets in Harmonic Analysis

Arveson [2] was the first to point out that the operator algebra $\text{Alg}(X, \leqslant, m)$ contains a nonzero Hilbert–Schmidt operator iff the graph of the pre-order has positive product measure. More generally there is a close connection between the existence of compact operators in CSL algebras (reflexive algebras with a commutative subspace lattice) and the theory of exceptional sets ("thin sets") in harmonic analysis.

For a Hilbert space H, let $\mathscr{K}$ denote the set of compact operators in $\mathscr{L}(H)$ and let $\mathscr{C}_p$, $p < \infty$, be the Schatten p-class. Certain exceptional sets may be used to construct CSL algebras with the following properties:

   (i) $\text{Alg}(P) \cap \mathscr{K} \neq \{0\}$, but $\text{Alg}(P)$ contains no nonzero compact pseudo-integral operators.

   (ii) $\text{Alg}(P)$ contains a nonzero operator which is in $\mathscr{C}_p$ for all $p > 2$ but no nonzero Hilbert–Schmidt operators.

   (iii) $\text{Alg}(P) \cap \mathscr{K} \neq \{0\}$, but $\text{Alg}(P) \cap \mathscr{C}_p = \{0\}$ for all $p < \infty$.

Our reference for harmonic analysis on a locally compact abelian group will be [14]. In order to avoid some technical difficulties we shall suppose throughout this section that G is a separable compact abelian group so that $\hat{G} = \Gamma$ is a countable discrete abelian group.

Let $\mu \in M(G)$ and let $f \in L^2(G)$. The convolution map $C_\mu : f \to f * \mu$ defines a bounded linear operator on $L^2(G)$. The following proposition is well known.

PROPOSITION 1.2.1. (i) *The operator C_μ is in the Schatten p-class iff* $\hat{\mu} \in l^p(\Gamma)$.

   (ii)   *The operator C_μ is compact iff $\hat{\mu}$ vanishes at infinity.*

Proof. Recall the formula

$$(C_\mu f)(x) = \int_G f(x-y)\,\mu(dy).$$

For each $\gamma \in \Gamma$

$$(C_\mu \gamma)(x) = \int_G \gamma(x-y)\,\mu(dy)$$

$$= \int_G \gamma(x)\,\gamma(-y)\,\mu(dy)$$

$$= \hat{\mu}(\gamma)\,\gamma(x).$$

Since the characters form an orthonormal basis for $L^2(G)$ they diagonalize C_μ. The proposition follows.

PROPOSITION 1.2.2. *The convolution operator C_μ is a pseudo-integral operator whose kernel $\{\mu^x\}$ is given by*

$$\mu^x(E) = \mu(-E+x)$$

for any Borel set $E \subseteq G$.

Proof. Recall that if $T: G \to G$ is any measurable map and $\lambda \in M(G)$ then we can obtain a measure λ_T defined by $\lambda_T(E) = \lambda(T^{-1}(E))$ for any Borel set $E \subseteq G$, and which has the property that

$$\int_G f(T(y))\,\lambda(dy) = \int_G f(y)\,\lambda_T(dy)$$

whenever either side exists.

Fix $x \in G$ and let $T_x: G \to G$ be given by $T_x(y) = x - y$. Set $\mu^x = \mu_{T_x}$. Then

$$(C_\mu f)(x) = \int_G f(x-y)\,\mu(dy)$$

$$= \int_G f(T_x(y))\,\mu(dy)$$

$$= \int_G f(y)\,\mu_{T_x}(dy)$$

$$= \int_G f(y)\,\mu^x(dy).$$

Furthermore it is easy to check that

$$\mu^x(E) = \mu(-E + x)$$

for any Borel set $E \subseteq G$.

It is known that for any nondiscrete group G there is a measure μ singular with respect to the Haar measure on G whose Fourier transform vanishes at infinity. Therefore we have the following corollary, which answers a question of A. Sourour [31, p. 352].

COROLLARY 1.2.3. *There exists a compact pseudo-integral operator which is not an integral operator.*

Proof. Take $\mu \in M(G)$ singular with respect to the Haar measure on G and with Fourier transform vanishing at infinity. Then C_μ is a compact pseudo-integral operator. For each $x \in G$, μ^x is singular with respect to the Haar measure on G. By uniqueness of the kernel $\{\mu^x\}$ a.e. it follows that C_μ is not an integral operator.

In our context $A(G)$ is the algebra of absolutely convergent Fourier series on G. The space of pseudomeasures $PM(G)$ is identified with $l^\infty(\Gamma)$ via the Fourier transform so that if $F \in PM(G)$ and $\sum_{\gamma \in \Gamma} a_\gamma \gamma(x) \in A(G)$, then

$$F\left(\sum_{\gamma \in \Gamma} a_\gamma \gamma(x)\right) = \sum_{\gamma \in \Gamma} \hat{F}(-\gamma) a_\gamma.$$

Formally we can write

$$F = \sum_{\gamma \in \Gamma} \hat{F}(-\gamma) \gamma(x)$$

for a pseudomeasure on G or $F = \sum_{\alpha, \beta \in \Gamma} \hat{F}(-\alpha, -\beta) \alpha(x) \beta(y)$ for a pseudomeasure on the product group $G \times G$. The following theorem is fundamental.

THEOREM 1.2.4. *Let $A: L^2(G) \to L^2(G)$ be a linear operator with matrix $[c_{\alpha\beta}]_{\alpha, \beta \in \Gamma}$ with respect to the character basis of $L^2(G)$. Define a pseudomeasure*

$$F_A = \sum c_{-\alpha\beta} \alpha(x) \beta(y).$$

Then $\operatorname{supp}(A) = \operatorname{supp}(F_A)$ [2, Definition 2.2.4].

Proof. Note that $c_{\alpha\beta} = (A\beta(y), \alpha(x))$. We'll first show that for $u(x), v(y) \in A(G)$,

$$(Av, \bar{u}) = F_A(u(x)\,v(y)).$$

Let $u(x) = \sum_{\alpha \in \Gamma} a_\alpha \alpha(x)$ and $v(y) = \sum_{\beta \in \Gamma} b_\beta \beta(y)$. The inner product

$$(Av, \bar{u}) = \left(A \left(\sum_{\beta \in \Gamma} b_\beta \beta(y) \right), \sum_{\alpha \in \Gamma} \bar{a}_\alpha - \alpha(x) \right)$$

$$= \sum_{\alpha, \beta} a_\alpha b_\beta (A\beta(y), -\alpha(x))$$

$$= \sum_{\alpha, \beta} a_\alpha b_\beta c_{-\alpha\beta}$$

$$= \sum_{\alpha, \beta} \hat{F}(-\alpha, -\beta)\, a_\alpha b_\beta = F(u(x)\,v(y)).$$

In order to prove the theorem we must show that for any $(x, y) \in G \times G$, $(x, y) \notin \operatorname{supp}(A)$ iff $(x, y) \notin \operatorname{supp}(F_A)$. If $(x, y), \notin \operatorname{supp} A$, then there exist open sets U_1 and U_2 about x and y, respectively, such that for any pair of functions u and $v \in A(G)$ with supports in U_1 and U_2, respectively, $(Av, u) = 0$. Now pick open sets $P_1 \subseteq U_1$ and $P_2 \subseteq U_2$ about x and y, respectively, such that $\bar{P}_1 \subseteq U_1$ and $\bar{P}_2 \subseteq U_2$. Then [30, Theorem 2.6.2] we can find functions s_1 and s_2 in $A(G)$ such that

$$s_1(x) = \begin{cases} 1 & \text{if } x \in \bar{P}_1 \\ 0 & \text{if } x \notin U_1 \end{cases}$$

$$s_2(y) = \begin{cases} 1 & \text{if } y \in \bar{P}_2 \\ 0 & \text{if } y \notin U_2. \end{cases}$$

We can also assume that the support of $s_i \subseteq U_i$ for $i = 1, 2$. Now let $f \in A(G \times G)$ have support in $P_1 \times P_2$. Let $T_1 \subseteq T_2 \subseteq \cdots \subseteq \Gamma \times \Gamma$ be an ascending sequence of finite subsets of $\Gamma \times \Gamma$ such that $\bigcup_{n=1}^{\infty} T_n = \Gamma \times \Gamma$. If we write $f(x, y) = \sum_{\alpha, \beta} f_{\alpha\beta} \alpha(x)\,\beta(y)$ then $\sum_{\alpha, \beta \in T_n} f_{\alpha\beta} \alpha(x)\,\beta(y)$ converges to $f(x, y)$ in the norm of $A(G \times G)$. Hence $s_1(x)\,s_2(y) \sum_{\alpha, \beta \in T_n} f_{\alpha\beta} \alpha(x)\,\beta(y)$ converges in norm to $s_1(x)\,s_2(y)\,f(x, y) = f(x, y)$. Therefore $F_A(\sum_{\alpha, \beta \in T_n} f_{\alpha\beta} s_1(x)\,\alpha(x)\,s_2(y)\,\beta(y))$ converges to $\overline{F_A(f)}$. But $F_A(\sum_{\alpha, \beta} f_{\alpha\beta} s_1(x)\,\alpha(x)\,s_2(y)\,\beta(y)) = \sum_{\alpha, \beta \in T_n} f_{\alpha\beta}(As_2(y)\,\beta(y), \overline{s_1(x)\,\alpha(x)})$. Since $s_1(x)\,\alpha(x)$ is supported in U_1 and $s_2(y)\,\beta(y)$ is supported in U_2 the last sum is 0. Hence $(x, y) \notin \operatorname{supp}(F_A)$. The other implication follows from the fact that for any open set $E \subseteq G$, the collection of functions in $A(G)$ which are supported in E is dense in $L^2(E)$.

DEFINITION 1.2.5 [14]. A pseudomeasure $F \in PM(G)$ is called a *pseudofunction* if $\hat{F}$ vanishes at infinity.

DEFINITION 1.2.6. (i) A closed set $E \subseteq G$ is called a *set of multiplicity* (*M-set*) if it supports a nonzero pseudofunction.

(ii) A closed set $E \subseteq G$ is called a *set of multiplicity in the strict sense* (*M_0-set*) if it supports a nonzero measure which is a pseudofunction.

Remark. Sets of multiplicity originally arose in connection with problems of uniqueness of trigonometric series and have been extensively studied.

THEOREM 1.2.7. *Let $E \subseteq G \times G$ be a closed set. Then*

(i) *E is an M-set if it supports a nonzero compact operator on $L^2(G)$.*

(ii) *E is an M_0-set if it supports a nonzero compact pseudo-integral operator on $L^2(G)$.*

Proof. (i) Suppose that E supports a nonzero compact operator K. Then supp $F_K \subseteq E$. $\hat{F}_K(-\alpha, -\beta) = c_{-\alpha\beta} \to 0$ (see Theorem 1.2.4) as α, β go to infinity. Hence E is an M-set.

(ii) By (i), F_K is a pseudofunction. But F_K is given by the measure ($=$ kernel) which defines K. Hence E is an M_0-set.

It is of interest to use Theorem 1.2.7 to give a second proof that $\text{Alg}(2^\infty, \leqslant, m_{1/2})$ contains no nonzero compact operators. This will follow immediately if we can show that the graph of $\leqslant$ is a set of uniqueness in the group $2^\infty \times 2^\infty$.

Take 2^∞ to be the group $\{-1, 1\} \times \{-1, 1\} \times \cdots$. The graph of $\leqslant$ is

$$\{((x_1, x_2, ...), (y_1, y_2, ...)): x_n \leqslant y_n \text{ for all } n\}.$$

It is not hard to see that if $\phi: G \to G$ is a topological group isomorphism and $E \subseteq G$ is a set of uniqueness then $\phi(E)$ is a set of uniqueness. Let F be the group $\{-1, 1\} \times \{-1, 1\}$ and let $S \subset F$ be

$$\{(x, y): x \leqslant y\}.$$

Then to show that the graph of $\leqslant$ is a set of uniqueness it suffices to show that $S \times S \times \cdots$ is a set of uniqueness in the group $F \times F \times \cdots$.

For each n define an inclusion

$$i_n: A(F) \to A(F \times F \times \cdots)$$

as follows. If $f \in A(F)$ then

$$i_n(f)(x_1, x_2, \ldots) = f(x_n).$$

Let D be a nonzero pseudofunction supported in $S \times S \times \cdots$. By multiplying D, if necessary, by a suitable multiple of a character we may assume that $\hat{D}(1) = 1$. Then the composition $D \circ i_n$ is a pseudomeasure on F which is supported by S. The group F has four characters

$$\pi_1 = 1, \pi_2, \pi_3, \pi_4$$

and their inclusions

$$i_n(\pi_1) = \pi_1^{(n)}, \qquad i_n(\pi_2) = \pi_2^{(n)}$$

$$i_n(\pi_3) = \pi_3^{(n)}, \qquad i_n(\pi_4) = \pi_4^{(n)}$$

are characters on $F \times F \times \cdots$. Since F is a finite group the pseudomeasure $D \circ i_n$ is actually a function on F,

$$(D \circ i_n)(x) = 1 + a_n \pi_2(x) + b_n \pi_3(x) + c_n \pi_4(x),$$

where

$$a_n = \hat{D}(\pi_2^{(n)})$$

$$b_n = \hat{D}(\pi_3^{(n)})$$

$$c_n = \hat{D}(\pi_4^{(n)}).$$

The action of $D \circ i_n$ on $f \in A(F)$ is given by

$$\int_F f(x)(D \circ i_n)(x)\, dx.$$

In order for $\text{supp}(D \circ i_n) \subseteq S$ we must have

$$D \circ i_n((1, -1)) = 1 \pm a_n \pm b_n \pm c_n = 0.$$

Since D is a pseudofunction

$$|\hat{D}(\pi_2^{(n)})|, |\hat{D}(\pi_3^{(n)})|, |\hat{D}(\pi_4^{(n)})|$$

are arbitrarily small for sufficiently large n and this shows that

$$1 \pm a_n \pm b_n \pm c_n = 0$$

cannot hold for sufficiently large n. Hence $S \times S \cdots$ is a set of uniqueness in $F \times F \times \cdots$.

It is natural to imagine about the extent to which the converses of the assertions of Theorem 1.2.7 are true. For example, suppose that E is an M_0-set. Can a measure $N \in PF(E)$ be chosen that gives a nonzero compact operator on $L^2(G)$? In general no, since E must be marginally nonnull [2, p. 453]. Also N must not be concentrated on a marginally null set for then the associated pseudo-integral operator would be 0. The following is a partial converse of (ii).

THEOREM 1.2.8.   *Suppose that a closed set $E \subseteq G \times G$ supports a measure μ in $PF(E)$ which is not concentrated on a marginally null set. Then E supports a nonzero pseudo-integral operator T such that $F_T \in PF(E)$.*

Proof.   Suppose that μ satisfies the above conditions. By a simple extension of [14, Proposition 1.5.1] we may assume that $\mu \geqslant 0$. Let μ_1 be the maginal of μ on the x-axis ($= G \times 0$). By the Lebesgue decomposition theorem $\mu_1 = f(x)\,dx + S$, where $S \perp dx$ and $f(x) \in L^1(G)$. We now pick a set $A \subseteq G$ such that $\int_A f\,dx > 0$, $\chi_A \cdot f \in L^\infty(G)$, and $S(A) = 0$. Let v be the restriction of the measure μ to the set $A \times G$. Then $v \neq 0$ and v has an L^∞ marginal on the x-axis. Since $v \ll \mu$, [14] again implies that $v \in PF(E)$. Hence without loss of generality we may assume that μ has an L^∞ marginal on the x-axis and is a positive measure. Then there is a constant C_1 such that $\mu_1(Q) \leqslant C_1(dx)(Q)$ for any Borel set $Q \subseteq G$.

Next we examine μ_2, the y-marginal of μ. Again $\mu_2 = g(y)\,dy + S$, where $S \perp dy$, $g \in L^1(G)$. We can find a set B such that $\int_B g\,dy > 0$, $\chi_B\, g \in L^\infty(G)$, and $S(B) = 0$. Let v be the restriction of μ to $G \times B$. Then $v \neq 0$ and $v \in PF(E)$. Also $v_2 \in L^\infty(G)$. So there is a constant C_2 such that for any Borel set $Q \subseteq G$, $v_2(Q) \leqslant C_2(dy)(Q)$. By construction $v_1(Q) \leqslant \mu_1(Q) \leqslant C_1(dx)(Q)$. Therefore [2, Theorem 1.5.1] T_v is a nonzero pseudo-integral operator. Since

$$F_{T_v} = v,$$

$$F_{T_v} \in PF(E).$$

Combining Theorems 1.2.7 and 1.2.8 yields

THEOREM 1.2.9.   *Let E be a closed set in $G \times G$. Then E supports a nonzero pseudo-integral operator T_μ such that $F_{T_\mu} \in PF(E)$ iff E is an M_0-set which supports a $\mu \in PF(E)$ which is not concentrated on a marginally null set.*

Before we proceed to the constructions (I)–(III) we need a few technical observations.

If D is pseudomeasure on G and $f \in L^2(G)$ then the convolution map $f \to f * D$ defines a bounded operator on $L^2(G)$ denoted by C_D. If

supp $D = E$ then supp $C_D = \phi(G \times E)$, where $\phi(x, y) = (x, x - y)$, for consider the pseudomeasure $1 \otimes D$ on $G \times G$ where $1 \otimes D(u(x) v(y)) = (\int_G u(x)\, dx) \cdot D(v(y))$. Evidently supp $1 \otimes D = G \times E$.

Define $F(v(x, y)) = 1 \otimes D(v(x, x - y)) = 1 \otimes D(v(\phi(x, y)))$. Then supp $F = \phi(G \times E)$. An elementary computation gives

$$\hat{F}(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha + \beta \neq 0 \\ \hat{D}(-\beta) & \text{otherwise} \end{cases}$$

and

$$\hat{F}_{C_D}(\alpha, \beta) = (C_D - \beta, \alpha) = \begin{cases} 0 & \text{if } \alpha + \beta \neq 0 \\ \hat{D}(-\beta) & \text{otherwise.} \end{cases}$$

Therefore $F = F_{C_D}$ as pseudomeasures. So supp $C_D =$ supp $F_{C_D} =$ supp $F = \phi(G \times E)$.

Note that the Haar measure of $\phi(G \times E) = 0$ iff the Haar measure of E is 0, since ϕ is a measure-preserving homeomorphism of $G \times G$.

Next we describe a method for constructing closed partial orders on G. Consider a relation R on $G \cup G$ (disjoint union) whose graph is shown in Fig. 1. Thus R consists of the diagonal together with an arbitrary closed subset of $G \times G$ placed in the upper right-hand corner. It is not hard to verify that R is a closed pre-order. Now let $E \subseteq G$ be a closed set and replace "ANY CLOSED SET" by $\phi(G \times E)$. We will denote the resulting partial order by P_E.

If F is a pseudomeasure on G and supp $F \subseteq E$, then the operator

$$\begin{pmatrix} 0 & C_F \\ 0 & 0 \end{pmatrix} \in \text{Alg}(P_E) \text{ and acts on } L^2(G \cup G).$$

Here, $\text{Alg}(P_E) = \{\begin{pmatrix} M_1 & T \\ 0 & M_2 \end{pmatrix} : M_1, M_2$ are multiplication operators and supp $T \subseteq \phi(G \times E)\}$. This method of constructing partial orders and the
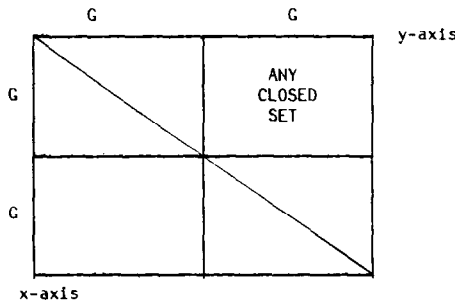

FIG. 1. A class of partial orders.

resulting CSL algebras was introduced by Arveson in [2] and used by him to give his example of a CSL algebra which is not synthetic.

LEMMA 1.2.10. *Let $E \subseteq G$ be a closed set. Then*

   (i) $\phi(G \times E)$ *is an M-set iff E is an M-set.*

   (ii) $\phi(G \times E)$ *is an M_0-set iff E is an M_0-set.*

Proof. (i) Suppose that $\phi(G \times E)$ is an M-set. Since ϕ is a group homeomorphism it follows that $G \times E$ is an M-set. Hence there is a pseudofunction $D_1 \neq 0$ such that $\text{supp } D_1 \subseteq G \times E$. There exists $(\alpha, \beta) \in \hat{G} \times \hat{G}$ such that $D_1(\alpha(x) \beta(y)) \neq 0$. Define a pseudofunction D_2 on G by

$$D_2(v(y)) = D_1(\alpha(x) v(y)).$$

Then $D_2 \neq 0$ and $\text{supp } D_2 \subseteq E$. Hence E is an M-set.

For the converse let D be a nonzero pseudofunction with $\text{supp } D \subseteq E$. Then the pseudofunction $1 \otimes D$ has $\text{supp}(1 \otimes D) \subseteq G \times E$.

The proof of (ii) is similar.

THEOREM 1.2.11. *Let $E \subseteq G$ be closed. Then*

   (i) $\text{Alg}(P_E)$ *contains a nonzero compact operator iff E is an M-set.*

   (ii) $\text{Alg}(P_E)$ *contains a nonzero compact pseudo-integral operator iff E is an M_0-set.*

   (iii) $\text{Alg}(P_E) \cap \text{Schatten p-class} \neq 0$ *iff E supports a nonzero pseudo measure whose Fourier transform is in l^p.*

Proof. (i) Assume that $\text{Alg}(P_E) \cap \mathscr{K} \neq 0$. Then by Theorem 1.2.7, $\phi(G \times E)$ is an M-set and so by Lemma 1.2.10 E is an M-set. Conversely if E is an M-set then there is a nonzero pseudofunction D with $\text{supp } D \subseteq E$. Then

$$\begin{pmatrix} 0 & C_D \\ 0 & 0 \end{pmatrix}$$

is a nonzero compact operator in $\text{Alg}(P_E)$.

The proof of (ii) is similar to (i). For the proof of (iii) one shows that if $T \in \text{Alg}(P_E) \cap \mathscr{C}_p$ then each of the pseudofunctions

$$v(y) \to F_T(\alpha(x) v(x - y))$$

$$\alpha \in \hat{G} \text{ fixed}$$

has support in E and Fourier transform in l^p. If $T \neq 0$ then at least one of these is nonzero.

One further consequence of our analysis is worth noting. The operators in $A_{\max}(\phi(G \times E))$ correspond to bounded matrices whose "diagonals" are pseudomeasures supported in E.

Let us consider the circle group T as the unit interval $[0, 1]$ and suppose that $0 < \xi < \frac{1}{2}$. Write $E_0 = [0, 1]$, $E_1 = [0, \xi] \cup [1 - \xi, 1]$ and in general obtain E_n from E_{n-1} by removing the middle $(1 - 2\xi)$th from all intervals which compose E_{n-1}. Set $E_\xi = \bigcap_{n=0}^{\infty} E_n$. Cantor's middle-third set occurs when $\xi = \frac{1}{3}$. It is known that E_ξ is a set of uniqueness iff $1/\xi$ is a Pisot number [13]. Hence we have the following

COROLLARY 1.2.12. *Let* $0 < \xi < \frac{1}{2}$. *Then* $\mathrm{Alg}(P_{E_\xi})$ *contains a nonzero compact operator iff* $1/\xi$ *is not a Pisot number.*

The existence of operator algebras having properties (I)–(III) *now follows from the existence of the appropriate thin sets.*

(I)   *There is an M-set which is not an* M_0-*set* [14, *Theorem* 4.4.2].

(II)   *Salem* [6, 2.6] *has constructed a closed set* $E \subseteq T$, *the circle group, of Lebesgue measure zero and a nonzero measure* μ *with* supp $\mu = E$ *such that* $\hat{\mu} \in l^p$ *for all* $p > 2$. *Hence the operator* C_μ *is in the Schatten p-class for all* $p > 2$ *and so*

$$\begin{pmatrix} 0 & C_\mu \\ 0 & 0 \end{pmatrix} \in \mathrm{Alg}(P_E)$$

and in the Schatten p-class for all $p > 2$. *Since the measure of* $\phi(G \times E) = 0$, $\mathrm{Alg}(P_E)$ *contains no nonzero Hilbert–Schmidt operator.*

(III) LEMMA. *Suppose that* $E \subseteq G$ *is a closed set. If for each natural number* n *the Haar measure of*

$$\underbrace{E + \cdots + E}_{n\,\text{times}}$$

is 0 *then* E *supports no nonzero pseudomeasure whose Fourier transform is in some* l^p.

Proof. Suppose $\hat{F} \in l^p$ and supp $F \subseteq E$. Pick n so that $\hat{F} \in l^{2n}$. Then

$$\mathrm{supp}\, F^n = \mathrm{supp}\, \underbrace{F * \cdots * F}_{n\,\text{times}} \subseteq \underbrace{E + \cdots + E}_{n\,\text{times}}.$$

Since $\frac{1}{2} = n(1/2n)$, Holder's inequality gives $\widehat{F^n} = (\hat{F})^n \in l^2$. This says that $F^n \in L^2(G)$ and is concentrated on a set of measure 0. Hence $F^n = 0$ so $F = 0$.

We now make use of the existence of a closed set $E \subseteq T$ such that the Lebesgue measure of

$$\underbrace{E + \cdots + E}_{n \text{ times}} = 0$$

for all n and which supports a nonzero measure μ whose Fourier transform vanishes at infinity [14, Proposition 6.3.13]. Then $\text{Alg}(P_E)$ contains a nonzero compact operator. If $\text{Alg}(P_E)$ contained a nonzero operator in some Schatten p-class then E would support a nonzero pseudomeasure whose Fourier transform is in l^p. The above lemma shows that this is not possible.

1.3. *More Thin Sets*

THEOREM 1.3.1.   *Let $(X, \leqslant, m)$ be a standard pre-ordered measure space and suppose that the product measure of the graph of the pre-order is zero. If K is a compact operator in $\text{Alg}\, L(X, \leqslant, m)$, then K is quasinilpotent.*

Proof.   Assume K is not quasinilpotent. Let

$$\lambda \neq 0 \in \sigma(K)$$

and let

$$E = \int_{\text{Appropriate Contour}} \frac{1}{(\lambda - K)} \, dz$$

be the Riesz idempotent corresponding to λ [28, Theorem 2.10]. Then $E \neq 0$ and since $(\lambda - K)^{-1}$ is in the uniformly closed algebra generated by K and 1 so is E. Thus $E \in \text{Alg}\, L(X, \leqslant, m)$. Furthermore the range of E is finite dimensional so E is a Hilbert–Schmidt operator. Thus $E = 0$. This contradiction proves that $\sigma(K) = \{0\}$ so K is quasinilpotent.

This result also follows from a theorem of Hopenwasser since in this case $L(X, \leqslant, m)$ will contain no atoms [16].

We will give examples to show that the nilpotency index of K can be any natural number greater than 2 (in previous examples it was 2). Then we construct an example where K is quasinilpotent but not nilpotent.

Let $E \subseteq [0, 2\pi]$ be an M_0-set such that for each natural number n, the Lebesgue measure of

$$\underbrace{E + \cdots + E}_{n \text{ times}}$$

is zero. Fix n and assume that $E \subseteq [2\pi/n, 4\pi/n]$. Let μ be a nonzero probability measure supported in E whose Fourier transform vanishes at infinity. Set $+_n E = \{e_1 + \cdots + e_n \in [0, 2\pi] : \text{for each } i, \ e_i = 0 \text{ or } e_i \in E\}$. In

$[0, 2\pi] \times [0, 2\pi]$ let $P = \{(x, x + e): e \in +_n E$ and $(x, x + e) \in [0, 2\pi] \times [0, 2\pi]\} \cup \{(x, x): x \in [0, 2\pi]\}$. Then P is a partial order. Suppose that $(x, x + e_1 + \cdots + e_n)$ and $(x + e_1 + \cdots + e_n, x + e_1 + \cdots + e_n + e'_1 + \cdots + e'_n)$ are in P. Then there can be at most n nonzero numbers among $e_1, ..., e_n, e'_1, ..., e'_n$, and $x + e_1 + \cdots + e_n + e'_1 + \cdots + e'_n \leqslant 2\pi$. Therefore $(x, x + (e_1 + \cdots + e_n + e'_1 + \cdots + e'_n)) \in P$. We may visualize P as in Fig 2. Thus for each point $e \in +_n E$ we have a "fiber" parallel to the diagonal emanating from that point. Next we show that P is closed and hence standard. Suppose

$$(x^i, x^i + e^i_1 + \cdots + e^i_n) \to (x, y),$$

where each $(x^i, x^i + e^i_1 + \cdots + e^i_n) \in P$. Then $x_i \to x$ and by passing to subsequences we may assume that

$$e^i_1 \to e_1$$
$$\vdots$$
$$e^i_n \to e_n.$$

Since E is closed it is easy to see that for any k, $e_k \in E$ or $e_k = 0$. Furthermore $x^i + e^i_1 + \cdots + e^i_n \leqslant 2\pi$ so that $x + e_1 + \cdots + e_n \leqslant 2\pi$. Hence $(x, y) = (x, x + e_1 + \cdots + e_n) \in P$. Thus P is closed. Let $P_x = \{e: (x, e) \in P\}$. The Lebesgue measure of P_x is zero for each x since $P_x \subseteq_n^+ E$. Therefore by Fubini's theorem the Lebesgue measure of P is zero.

Define a pseudo-integral operator $T_\mu \in \text{Alg}(P)$ by

$$\mu^x = x + \mu \qquad \text{for} \quad x \in [0, (n-1)(2\pi/n)]$$

(translate the measure μ by x), it being understood that for $x \in [(n-2)(2\pi/n), (n-1)(2\pi/n)]$ we only use that part of μ which "lies" in $[0, 2\pi]$. It is evident that supp $T_\mu \subseteq P$ and supp T_μ may be visualized as in Fig. 3.

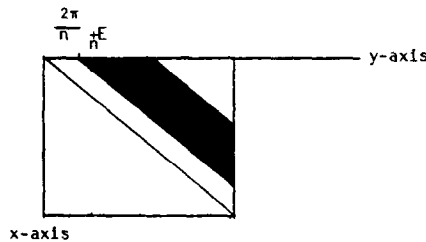Since T_μ is the "compression" of a compact convolution operator to
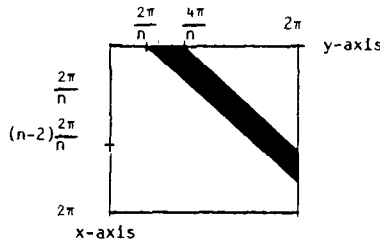

FIG. 2. The partial order P.

FIG. 3.　The support of T_μ.

certain "squares" it follows that T_μ is compact. Also we may assume that $2\pi/n \in \operatorname{supp} \mu$ so that

$$(T_\mu 1)(x) > 0, \qquad 0 \leqslant x < (n-1)(2\pi/n)$$

while

$$(T_\mu 1)(x) = 0 \qquad \text{otherwise}$$

and for any k, $2 \leqslant k \leqslant n-1$,

$$(T_\mu^k 1)(x) > 0, \qquad 0 \leqslant x < (n-k)(2\pi/n)$$

while

$$(T_\mu^k 1)(x) = 0 \qquad \text{otherwise.}$$

Evidently $T_\mu^n = 0$

To construct an algebra $\operatorname{Alg}(P)$ which contains a quasinilpotent compact operator which is not nilpotent and for which the measure of P is zero we pick for each $n \geqslant 2$ a CSL algebra $\operatorname{Alg}(P_n)$ such that the measure of P_n is zero and such that $\operatorname{Alg}(P_n)$ contains a compact operator K_n with nilpotency index equal to n. We may assume that $\|K_n\| \to 0$ and then take $K = \bigoplus_{n=2}^{\infty} K_n$ in the CSL algebra $\operatorname{Alg}(P) = \bigoplus_{n=2}^{\infty} \operatorname{Alg}(P_n)$. K is quasinilpotent but not nilpotent and P being a countable disjoint union of sets of measure zero has measure 0.

As a further illustration of how the pre-order P determines both the quantity and the quality of the compact operators in $\operatorname{Alg}(P)$, we have the following.

THEOREM 1.3.2.　*Suppose that $K \in \operatorname{Alg}(P)$ belongs to some Schatten p-class, $p < \infty$. If the measure of P is zero then K is nilpotent.*

Proof. Pick n such that K belongs to the Schatten $2n$-class. Then K^n is a Hilbert–Schmidt operator in $\text{Alg}(P)$. Hence $K^n = 0$.

Let P_1, P_2 be standard pre-orders on standard Borel measure spaces (X, μ) and (Y, v), respectively. The tensor product order $P_1 \otimes P_2$ on $(X \times Y, \mu \times v)$ is defined by $(x_1, y_1) \leqslant (x_2, y_2)$ iff $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. If the characteristic functions

$$\chi_{\tilde{E}_1}(x), x_{\tilde{E}_1}(x), \dots \text{ define } P_1$$

and

$$\chi_{\tilde{F}_1}(y), \chi_{\tilde{F}_2}(y), \dots \text{ define } P_2,$$

the tilde standing for complement, then the characteristic functions $\chi_{\tilde{E}_1 \times Y}$, $\chi_{\tilde{E}_2 \times Y}, \dots$ together with $\chi_{X \times \tilde{F}_1}, \chi_{X \times \tilde{F}_2}, \dots$ define $P_1 \otimes P_2$. This turns $(X \times Y, \mu \times v)$ into a standard pre-ordered measure space and we obtain the associated CSL algebra $\text{Alg}(P_1 \otimes P_2)$. We note that if $A \in \text{Alg}(P_1)$ and $B \in \text{Alg}(P_2)$ then the tensor product $A \otimes B$ is in $\text{Alg}(P_1 \otimes P_2)$. Let $\mathscr{C}_p$ denote the Schatten p-class.

THEOREM 1.3.3. *The algebra* $\text{Alg}(P_1 \otimes P_2) \cap \mathscr{C}_p \neq \{0\}$ *iff* $\text{Alg}(P_1) \cap \mathscr{C}_p \neq \{0\}$ *and* $\text{Alg}(P_2) \cap \mathscr{C}_p \neq \{0\}$.

Proof. Suppose that $A \in \text{Alg}(P_1) \cap \mathscr{C}_p$, $B \in \text{Alg}(P_2) \cap \mathscr{C}_p$ and $A, B \neq 0$. Then $A \otimes B \neq 0$ and $A \otimes B \in \text{Alg}(P_1 \otimes P_2) \cap \mathscr{C}_p$.

Conversely suppose that $\text{Alg}(P_1 \otimes P_2)$ contains a nonzero $K \in \mathscr{C}_p$. We will assume the μ, v are finite measures. The σ-finite case follows easy from this. Then K has as an invariant subspace each of the projections $P_{E_n \times Y}$. There exist $f \in L^2(X)$, $g \in L^2(Y)$ such that $w(x, y) = K(f(x) g(y)) \neq 0$. Assume that $S = \{w > 0\}$ has positive $\mu \times v$ measure. Then $M_{\chi_S} \in \text{Alg}(P_1 \otimes P_2)$ and $M_{\chi_S} K \in \mathscr{C}_p$. Define a map A from $L^2(X)$ to $L^2(X \times Y)$ by $[A(h)](x, y) = h(x) g(y)$ and a map B from $L^2(X \times Y)$ to $L^2(X)$ by $[B(z)](x) = \int_Y z(x, y) v(dy)$. Consider now the composition $B(M_{\chi_S} K) A$ on $L^2(X)$. This map is nonzero as can be verified easily and is in $\mathscr{C}_p \cap \text{Alg}(P_1)$ since $M_{\chi_S} K$ is. Thus $\text{Alg}(P_1) \cap \mathscr{C}_p \neq \{0\}$. Similarly $\text{Alg}(P_2) \cap \mathscr{C}_p \neq \{0\}$. This completes the proof.

This theorem can be thought of as an analogue of the fact that in a compact abelian group the product of two M-sets is an M-set. Now let $\text{Alg}(P_1) \bar{\otimes} \text{Alg}(P_2)$ denote the ultraweak closure of the linear span of all tensors $A \otimes B$ with $A \in \text{Alg}(P_1)$ and $B \in \text{Alg}(P_2)$. We shall see in Section 2 that the unsolved problem $\text{Alg}(P_1) \bar{\otimes} \text{Alg}(P_2) = \text{Alg}(P_1 \otimes P_2)$ is related to a difficult unsolved problem in harmonic analysis, i.e., whether the product of two sets of spectral synthesis is again a set of spectral synthesis.

1.4. $\mathscr{C}_p$-Density in Reflexive Operator Algebras

A very interesting and promising lattice-theoretic structural question was proposed by Arveson in [2, p. 469] and is still not completely answered. Let us call a subspace lattice $\mathscr{L}$ strongly reflexive if there is a set S of compact operators on H such that $\mathscr{L} = \text{Lat}(S)$. When is a separately acting commutative subspace lattice strongly reflexive? Consequently Hopenwasser, Laurie, and Moore [19] characterized those commutative subspace latices (CSLs) for which the Hilbert–Schmidt operators are dense in Alg($\mathscr{L}$) as the completely distributive CSLs. Laurie and Longstaf contributed further to this circle of ideas in [27] and proposed problem 2. Some ideas from the previous section allow us to settle this for $p < \infty$. We will need to use the following deep "non-self-adjoint density theorem."

THEOREM 1.4.1 (Arveson and Davidson [26, Theorem 4.1]). *Let A be a reflexive algebra on a separable Hilbert space with* Lat A *commutative. Let I be an ideal in A such that $EIE \neq 0$ for all semi-invariant projections E of A. Then the ultraweak closure of I is A.*

LEMMA 1.4.2. *Let B any algebra of operators and let $\bar{B}$ be its closure in the ultraweak topology. Suppose there is an n such that for all $T_1, T_2, ..., T_n \in B$, $T_1 T_2 \cdots T_n = 0$. Then for any $T_1, T_2, ..., T_n \in \bar{B}$, $T_1 T_2 \cdots T_2 = 0$.*

Proof. Consider the function $f(T_1, T_n, ..., T_n) = T_1 T_2 \cdots T_n$. If we fix $(n-1)$ variables then the map

$$T \to f(T_1, ..., T, ..., T_n)$$

is continuous in the ultraweak topology. The result follows at once.

Let $\mathscr{C}_p$ be the Schatten p-class, $p < \infty$.

THEOREM 1.4.3. *Let A be a reflexive operator algebra with a commutative subspace lattice. If $A \cap \mathscr{C}_p$ is dense in A then $A \cap$ trace class operators is dense in A.*

Proof. Let E be a semi-invariant projection for A and let I be the ideal of trace class operators in A. We will show that $EIE \neq 0$. We may assume that p is a natural number.

Let B be the ideal of $\mathscr{C}_p$ operators in the algebra EAE. If every product of p such operators is zero then by Lemma 1.4.2, $\bar{B}$ the ultraweak closure of B has the same property. But since $1 \in EAE$ is in $\bar{B}$ this leads to a contradiction. Hence there exist $T_1, T_2, ..., T_p$ in $EAE \cap \mathscr{C}_p$ such that $T_1 T_2 \cdots T_p \neq 0$. It is well known that $T_1 T_2 \cdots T_p$ is in trace class [29, Theorem 2.3.10]. Hence $EIE \neq 0$.

1.5. *Compact Perturbation of a CSL Algebra*

Let $\mathscr{A}$ be an algebra of operators on a Hilbert space H and let K be the ideal of compact operators in $\mathscr{L}(H)$. The algebra $\mathscr{A} + \mathscr{K} = \{A + K: A \in \mathscr{A}, K \in \mathscr{K}$ compact$\}$ is called the algebra of compact perturbations of $\mathscr{A}$.

Compact perturbations of operator algebras have been the subject of considerable study for various reasons [4, 7, 10]. One problem is that of determining whether $\mathscr{A} + \mathscr{K}$ is norm closed in $\mathscr{L}(H)$. It is known that compact perturbations of C^*-algebras are norm closed and furthermore that if $\mathscr{A}$ has an "approximate identity" of compact operators then $\mathscr{A} + \mathscr{K}$ is norm closed [26, Proposition 7.1]. In particular the algebra of compact perturbations of a nest algebra is norm closed [10].

In this chapter we will give an example of a CSL algebra $\text{Alg}(P)$ such that $\text{Alg}(P) + \mathscr{K}$ is *not* norm closed.

DEFINITION 1.5.1 [14].   Let E be a closed set in the circle group T. We set

$$I(E) = \{f \in A(T): E \subseteq f^{-1}(0)\}$$

and $N(E) =$ the annihilator of $I(E)$ in $\text{PM}(T)$.

A closed set E is called a set of spectral synthesis iff $\text{PM}(E) = N(E)$. Let $\eta(E) = \inf\{\limsup |S(n)|/\|S\|_{\text{PM}}: S \in N(E), S \neq 0\}$.

In [14] a closed set $E \subseteq T$ is constructed which has the following properties:

  (i) $\eta(E) = 0$,

  (ii) E is what is known as a $U_1 -$ set, i.e., $\text{PF}(T) \cap N(E) = \{0\}$,

  (iii) E is a set of spectral synthesis (cf. [14, Lemma 4.3.6, Proof of 4.3.3, and Lemma 4.3.7).

From (ii) and (iii) we immediately infer that $\text{PF}(E) = \{0\}$. Hence E is not a set of multiplicity.

Through a number of simple observations we now proceed to show that $\text{Alg}(P_E) + \mathscr{K}$ is not norm closed.

First recall that $\text{Alg}(P_E)$ can be described as the algebra of all $\begin{pmatrix} M_1 & T \\ 0 & M_2 \end{pmatrix}$, where $T \in \mathscr{A}_{\max}(\phi(T \times E))$, the set of all operators supported in $\phi(T \times E)$, and M_1, M_2 are in the algebra of multiplication operators on $L^2(T)$. It is easy to see that $\text{Alg}(P_E) + \mathscr{K}$ is norm closed iff $\mathscr{A}_{\max}(\phi(T \times E)) + \mathscr{K}$ is norm closed as a set.

Now if $\pi: \mathscr{L}(H) \to \mathscr{L}(H)/\mathscr{K}(H)$ is the quotient map into the Calkin algebra and $\mathscr{A}$ is a collection of operators then $\mathscr{A} + \mathscr{K}$ is norm closed iff $\pi(\mathscr{A})$ is closed in the Calkin algebra. Elementary Banach space theory shows that this is equivalent to the restriction of π to $\mathscr{A}$ being bounded

below in case $\mathscr{A} \cap \mathscr{K} = \{0\}$. Let $q: \mathrm{PM}(T) \to \mathrm{PM}(T)/\mathrm{PF}(T)$ be the quotient map.

LEMMA 1.5.2. *Let S be a pseudomeasure on T and let C_S be the associated convolution operator. Then*

$$\|\pi(C_S)\| = \|q(S)\|_{\mathrm{PM}(T)/\mathrm{PF}(T)}.$$

Furthermore $\|q(S)\|_{\mathrm{PM}(T)/\mathrm{PF}(T)} = \limsup_{|n| \to \infty} |\hat{S}(n)|$.

Proof. Let K be a compact operator with $\|C_S - K\| < \varepsilon + \|\pi(C_S)\|$. Then the diagonal of K can be viewed as a pseudofunction Q on T. It is evident that $\|q(S)\| \leqslant \|S - Q\|_{\mathrm{PM}(T)} \leqslant \|C_S - K\| < \varepsilon + \|\pi(C_S)\|$. Hence $\|q(S)\| \leqslant \|\pi(C_S)\|$.

To prove the other inequality suppose that $Q \in \mathrm{PF}(T)$ is such that $\|S - Q\|_{\mathrm{PM}(T)} < \varepsilon + \|q(S)\|$. Then $\|C_S - C_Q\| = \|S - Q\|_{\mathrm{PM}(T)} < \varepsilon + \|q(S)\|$. Since C_Q is a compact operator $\|\pi(C_S)\| \leqslant \|C_S - C_Q\| < \varepsilon + \|q(S)\|$. Hence $\|\pi(C_S)\| \leqslant \|q(S)\|$. The last statement is straightforward.

Since E is not a set of multiplicity, Theorem 1.2.7 implies that $\mathscr{A}_{\max}(\phi(T \times E)) \cap \mathscr{K} = \{0\}$. We will show that the map

$$\pi: \mathscr{A}_{\max}(\phi(T \times E)) \to \mathscr{A}_{\max}(\phi(T \times E))/\mathscr{K}$$

is not bounded below. Since $\eta(E) = 0$ there is a sequence $\{S_n\}$ of pseudomeasures supported in E such that $\|S_n\|_{\mathrm{PM}} = 1$ and such that $\limsup_{m \to \infty} |S_n(m)| \to 0$ as $n \to \infty$. Consider the sequence $\{C_{S_n}\}$ of convolution operators. For any n, $C_{S_n} \in \mathscr{A}_{\max}(\phi(T \times E))$ and $\|C_{S_n}\| = \|S_n\|_{\mathrm{PM}} = 1$. However, $\|\pi(C_{S_n})\| = \|q(S_n)\| \to 0$. Hence π is not bounded below on $\mathscr{A}_{\max}(\phi(T \times E))$. This completes the construction.

2. SPECTRAL SYNTHESIS IN REFLEXIVE ALGEBRAS

2.1. *Spectral Synthesis in $\mathscr{A}_{\max}(\phi(G \times E))$*

In this section we show how, for compact G, harmonic analysis in $A(G \times G)$ "embeds" in that of $\mathscr{C}_1(L^2(G))$. This allows us to transfer results in the spectral synthesis of pseudomeasures to CSL algebras.

THEOREM 2.1.1. *There is an injective contraction from $A(G \times G)$ into $\mathscr{C}_1(L^2(G))$ $(f \mapsto T_f)$ such that $\mathrm{tr}(CT_f) = F_C(f)$ for any bounded linear operator C on $L^2(G)$. Here "tr" denotes the usual trace function on $\mathscr{C}_1(L^2(G))$.*

Proof. Let $f(x, y) = \sum_{\alpha, \beta \in \Gamma} a_{\alpha\beta} \alpha(x) \beta(y)$ be an element of $A(G \times G)$. Write C as the matrix $[c_{ij}]$. Then $F_C(f) = \sum_{\alpha, \beta \in \Gamma} F_C(-\alpha, -\beta) a_{\alpha\beta} = \sum_{\alpha, \beta \in \Gamma} c_{-\alpha\beta} a_{\alpha\beta} = \sum_{\alpha, \beta \in \Gamma} c_{\alpha\beta} a_{-\alpha\beta}$. So if we define $(T_f)_{\beta\alpha} = a_{-\alpha\beta}$ then $\text{tr}(CT_f) = F_C(f)$.

COROLLARY 2.1.2. *If the net $\{B_\alpha\}$ converges to B ultraweakly then $\{F_{B_\alpha}\}$ converges to F_B in the weak* topology on pseudomeasures.*

Remark. For Hilbert space operators "tr" extends the distributional action of F_B on $A(G \times G)$ to $\mathscr{C}_1(L^2(G))$.

We now indicate a general method for obtaining closed sets $K \subseteq G \times G$ for which $\mathcal{C}_{\min}(K) \neq \mathcal{C}_{\max}(K)$ [2]. Let K be a closed set in $G \times G$ which is not synthetic in the sense of harmonic analysis. Then there is a nonzero pseudomeasure D with supp $D \subseteq K$ and a nonzero $f \in A(G \times G)$ such that f vanishes on K and $D(f) \neq 0$. Hence D is not the w*-limit of measures supported in K. Now if $D = F_B$ for some Hilbert space operator B then $\mathcal{C}_{\min}(K) \neq \mathcal{C}_{\max}(K)$. This is because, by Corollary 2.1.2, B cannot be the ultraweak limit of a net of pseudo-integral operators.

Indeed suppose for example that $f(y) = \sum_n a_n e^{iny}$ is in $A(T)$, $\sum_n \hat{D}(-n) e^{iny}$ is in PM(E), and that f vanishes on E with $D(f) = \sum_n \hat{D}(-n) a_n \neq 0$. Then the convolution operator C_D is supported in $\varphi(T \times E)$, the function $f(x - y)$ vanishes on $\varphi(T \times E)$, and

$$F_{C_D}(f(x - y)) = \text{tr}(C_D \cdot T_{f(x-y)})$$

$$= \text{tr} \begin{pmatrix} \ddots & & & & 0 \\ & \hat{D}(-1) & & & \\ & & \hat{D}(0) & & \\ & & & \hat{D}(1) & \\ 0 & & & & \ddots \end{pmatrix} \begin{pmatrix} \ddots & & & & 0 \\ & a_1 & & & \\ & & a_0 & & \\ & & & a_{-1} & \\ 0 & & & & \ddots \end{pmatrix} = D(f) \neq 0.$$

For any pseudo-integral operator G supported in $\varphi(T \times E)$ we have $0 = F_G(f(x - y)) = \text{tr}(G \cdot T_{f(x-y)})$. Hence $\mathcal{C}_{\max}(\varphi(T \times E)) \neq \mathcal{C}_{\min}(\varphi(T \times E))$. Thus the CSL algebra Alg(P_E) is not synthetic. In fact we will show next that if a closed set E is contained in G, a separable locally compact abelian group, then Alg(P_E) is synthetic iff E is a set of spectral synthesis in the sense of harmonic analysis. This amounts to proving that $\mathcal{C}_{\max}(\varphi(G \times E)) = \mathcal{C}_{\min}(\varphi(G \times E))$ iff E is a set of spectral synthesis. Owing to the possible noncompactness of G, we shall encounter some technical difficulties which we have hitherto been able to avoid.

From now on therefore, G will denote a separable locally compact abelian group. It is known that G contains an open subgroup isomorphic

to $R^n \times K$ for some n and compact abelian group K [30, Theorem 2.4.1]. Hence G is σ-compact. It is also known that G is metrizable.

DEFINITION 2.1.3. Let $h: G \times G \to \mathbb{C}$. Then h is said to be concentrated on a neighborhood of the diagonal if there is a compact neighborhood K of 0 in G such that for each α, $h(\alpha, \beta)$, as a function of β, is concentrated on $\alpha + K$.

The next lemma says that if $h(\alpha, \beta)$ is a bounded continuous function which is concentrated on a neighborhood of the diagonal, then the "kernel" $h(\alpha, \beta)$ defines a bounded integral operator on $L^2(G)$, and moreover, this integral operator is the strong limit of a sequence of linear combinations of "diagonals" of $h(\alpha, \beta)$.

LEMMA 2.1.4. Let $h: G \times G \to \mathbb{C}$ be continuous, bounded, and concentrated on a neighborhood of the diagonal. Then the associated integral operator Int h is bounded on $L^2(G)$ and is the strong limit of a sequence of operators of the form

$$f(\alpha) \to \sum_i a_i h(\alpha, \alpha + \beta_i) f(\alpha + \beta_i).$$

Proof. First we show that a certain sequence of linear combinations of diagonals of $h(\alpha, \beta)$ converges strongly. It will then be apparent that this strong limit is Int h, the integral operator associated with h. Let K be a neighborhood of 0 as in Definition 2.1.3. Pick a sequence $\{\mathscr{P}_n\}$ of partitions of K such that the diameter of $\mathscr{P}_n$ goes to 0. Fix $\mathscr{P}_n$, let $S_i \in \mathscr{P}_n$, and pick $\beta_i \in S_i$. Define T_n on $L^2(G)$ by the formula

$$(T_n f)(\alpha) = \sum_{S_i \in \mathscr{P}_i} m(S_i) h(\alpha, \alpha + \beta_i) f(\alpha + \beta_i).$$

One calculates that $\|T_n\| \leqslant m(K) \|h\|_\infty$. We will show that T_n converges strongly on a dense subspace of $L^2(G)$. Let f be a continuous function with compact support J in G. Then it is possible to find a compact set $L \subseteq G$ such that for $\alpha \notin L$, $J \cap \alpha + K = \varnothing$. This implies that $f \in \text{domain}(\text{Int } h)$, since $(\text{Int } h) f = 0$ on the complement of L.

If $x \in L$ then

$$((\text{Int } h) f)(\alpha) = \int_{\alpha + K} h(\alpha, \beta) f(\beta) \, d\beta$$

$$= \int_K h(\alpha, \alpha + \beta) f(\alpha + \beta) \, d\beta$$

$$= \sum_{S_i \in \mathscr{P}_n} \int_{S_i} h(\alpha, \alpha + \beta) f(\alpha + \beta) \, d\beta.$$

Also

$$(T_n f)(\alpha) = \sum_{S_i \in \mathscr{P}_n} m(S_i) \, h(\alpha, \alpha + \beta_i) \, f(\alpha + \beta_i)$$

$$= \sum_{S_i \in \mathscr{P}_n} \int_{S_i} h(\alpha, \alpha + \beta_i) \, f(\alpha + \beta_i) \, d\beta.$$

Hence

$$\| (\text{Int } h) \, f - T_n f \|_\infty$$

$$= \left\| \sum_{S_i \in \mathscr{P}_n} \int_{S_i} h(\alpha, \alpha + \beta) \, f(\alpha + \beta) - h(\alpha, \alpha + \beta_i) \, f(\alpha + \beta_i) \, d\beta \right\|_\infty$$

$$\leqslant \sum_{S_i \in \mathscr{P}_n} \left\| \int_{S_i} h(\alpha, \alpha + \beta) \, f(\alpha + \beta) - h(\alpha, \alpha + \beta_i) \, f(\alpha + \beta_i) \, d\beta \right\|_\infty$$

We can estimate this last sum by observing that

$$\left\| \int_{S_i} h(\alpha, \alpha + \beta) \, f(\alpha + \beta) - h(\alpha, \alpha + \beta_i) \, f(\alpha + \beta_i) \, \beta \right\|_\infty$$

$$\leqslant \left\| \int_{S_i} h(\alpha, \alpha + \beta) [f(\alpha + \beta) - f(\alpha + \beta_i)] \, d\beta \right\|_\infty$$

$$+ \left\| \int_{S_i} [h(\alpha, \alpha + \beta) - h(\alpha, \alpha + \beta_i)] \, f(\alpha + \beta_i) \, d\beta \right\|_\infty .$$

Now f is uniformly continuous. Furthermore since h is continuous it is uniformly continuous on the ccompact set $L \times (K + L)$. Hence for sufficiently large n (making the distance between any $\beta \in S_i$ and β_i sufficiently small)

$$\left\| \int_{S_i} h(\alpha, \alpha + \beta) [f(\alpha + \beta) - f(\alpha + \beta_i)] \, d\beta \right\|_\infty$$

$$+ \left\| \int_{S_i} [h(\alpha, \alpha + \beta) - h(\alpha, \alpha + \beta_i)] \, f(\alpha + \beta_i) \, d\beta \right\|_\infty$$

$$\leqslant (\|h\|_\infty \cdot \varepsilon \, \|f\|_\infty) \, m(S_i)$$

for any prescribed ε. Thus

$$\| (\text{Int } h) f - T_n f \|_\infty \leqslant \varepsilon \cdot (\|h\|_\infty + \|f\|_\infty) \, m(K)$$

for sufficiently large n. Therefore $T_n f$ converges to $(\text{Int } h) f$ in the norm of $L^2(G)$. Since the continuous functions with compact support are dense in

$L^2(G)$ it follows that T_n converges strongly to some bounded operator T on $L^2(G)$. It is not hard to see that

$$(Tf)(\alpha) = \int_G h(\alpha, \beta) f(\beta) \, d\beta.$$

Let $S \in \text{PM}(G \times G)$. Then if $f \in L^1(\Gamma \times \Gamma)$, $S(\hat{f}) = \int_\Gamma \int_\Gamma \hat{S}(\alpha, \beta) f(\alpha, \beta) \, d\alpha \, d\beta$. We can fix α and define $S_\alpha \in \text{PM}(G)$ by the formula $S_\alpha(\hat{f}) = \int_\Gamma \hat{S}(\alpha, \beta) f(\beta) \, d\beta$ for $f \in L^1(\Gamma)$, provided $\hat{S}$ is bounded everywhere.

LEMMA 2.1.5. *Let $S \in \text{PM}(G \times G)$ have support in $G \times E$ for some closed set $E \subseteq G$. Assume that $\hat{S}$ is continuous. Then $\text{supp}(S_\alpha) \subseteq E$ for all $\alpha \in \Gamma$.*

Proof. Let $\{f_n\}$ be a bounded sequence of L^1 functions which converges in the weak* topology to the point mass located at α. Consider $S_n \in \text{PM}(G)$ defined by

$$S_n(\hat{g}(y)) = S(\hat{f}_n(x) \, \hat{g}(y)).$$

It is evident that $\text{supp} \, S_n \subseteq E$. We will show that S_n converges in the weak* topology to S_α. Now

$$S_n(\hat{g}(y)) = \int_\Gamma \int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, g(\beta) \, d\alpha \, d\beta$$

$$= \int_\Gamma g(\beta) \left(\int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, d\alpha \right) d\beta.$$

For fixed β

$$\lim_{n \to \infty} \int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, d\alpha = \hat{S}(\alpha, \beta).$$

Also

$$\left| g(\beta) \int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, d\alpha \right| \leq |g(\beta)| \left| \int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, d\alpha \right|$$

$$\leq |g(\beta)| \, \|\hat{S}\|_\infty \, \|f_n\|_1$$

$$\leq |g(\beta)| \, \|\hat{S}\|_\infty \cdot \sup_n \|f_n\|_1.$$

Lebesgue's dominated convergence theorem implies that

$$\int_\Gamma g(\beta) \left(\int_\Gamma \hat{S}(\alpha, \beta) f_n(\alpha) \, d\alpha \right) d\beta$$

converges to

$$\int_\Gamma g(\beta) \, \hat{S}(\alpha, \beta) \, d\beta = S_\alpha(\hat{g}).$$

Hence S_n converges in the weak* topology to S_α. Since supp $S_n \subseteq E$ it follows that supp $S_\alpha \subseteq E$ for each α.

Let $S \in \mathrm{PM}(G \times G)$. Define S_φ by the formula $S_\varphi(\hat{f}(x, y)) = S(\hat{f}(x, x - y))$. Now $\hat{f}(x, y) = \int_\Gamma \int_\Gamma f(\alpha, \beta) \, \alpha(-x) \, \beta(-y) \, d\alpha \, d\beta$ so

$$\hat{f}(x, x - y) = \int_\Gamma \int_\Gamma f(\alpha, \beta) \, \alpha(-x) \, \beta(-(x - y)) \, d\alpha \, d\beta$$

$$= \int_\Gamma \int_\Gamma f(\alpha, \beta)(\alpha + \beta)(-x)(-\beta)(-y) \, d\alpha \, d\beta$$

$$= \int_\Gamma \int_\Gamma f(\alpha + \beta, -\beta) \, \alpha(-x) \, \beta(-y) \, d\alpha \, d\beta.$$

So if we define $g(\alpha, \beta) = f(\alpha + \beta, -\beta)$ then $\hat{g}(x, y) = \hat{f}(x, x - y)$. Also $\|g\|_1 = \|f\|_1$. Hence $S_\varphi \in \mathrm{PM}(G \times G)$ and $S_\varphi(\hat{f}) = S(\hat{g})$. From this it follows that

$$\hat{S}_\varphi(\alpha, \beta) = \hat{S}(\alpha + \beta, -\beta).$$

Now let K be an operator on $L^2(G)$ with compact support. Then there is a compact set $E \subseteq G$ such that $\mathrm{supp}(K) \subseteq E \times E$. Define a kernel on $\Gamma \times \Gamma$ by the formula $k(\alpha, \beta) = (K P_E \beta, P_E \alpha)$ for $\alpha, \beta \in \Gamma$. It follows from [30, Theorem 1.2.6] that if (α_1, β_1) is sufficiently close to (α, β) then $\|P_E \alpha_1 - P_E \alpha\|_2 < \varepsilon$ and $\|P_E \beta_1 - P_E \beta\|_2 < \varepsilon$ for any prescribed ε. Hence $k(\alpha, \beta)$ is a continuous function. Now suppose that K is an integral operator with L^∞ kernel. That is,

$$(Ku)(x) = \int_G h(x, y) \, u(y) \, dy,$$

where h vanishes off a compact set and is in $L^\infty(G \times G)$. For f, g compactly supported in $L^\infty(\Gamma)$ we have

$$\int_\Gamma \int_\Gamma k(\alpha, \beta) f(\beta) \, \overline{g(\alpha)} \, d\alpha \, d\beta$$

$$= \int_\Gamma \int_\Gamma \int_G \int_G h(x, y) \, \beta(y) \, \overline{\alpha(x)} \, f(\beta) \, \overline{g(\alpha)} \, d\alpha \, d\beta \, dx \, dy$$

$$= \int_G \int_G \int_\Gamma \int_\Gamma h(x, y) \, \beta(y) \, \overline{\alpha(x)} \, f(\beta) \, \overline{g(\alpha)} \, d\alpha \, d\beta \, dx \, dy$$

$$= \int_G \int_G h(x, y) \, \hat{f}(-y) \, \overline{\hat{g}(-x)} \, dx \, dy$$

$$= (K \mathscr{F}^{-1} f, \mathscr{F}^{-1} g).$$

Here we have written $\mathscr{F}: L^2(G) \to L^2(\Gamma)$ for the Fourier transform and recalled that $(\mathscr{F}^{-1}f)(x) = \hat{f}(-x)$. We see that Int k represents the operator $\mathscr{F}K\mathscr{F}^{-1}$ on $L^2(\Gamma)$.

If K is not an integral operator choose a sequence $\{K_n\}$ of integral operators, as above, with common compact support such that K_n converges weakly to K. Then

$$\int_\Gamma \int_\Gamma k_n(\alpha, \beta)\, f(\beta)\, \overline{g(\alpha)}\, d\alpha\, d\beta = (K_n\mathscr{F}^{-1}f, \mathscr{F}^{-1}g) \to (K\mathscr{F}^{-1}f, \mathscr{F}^{-1}g)$$

for L^∞ functions f, g having compact support.

We also have that $\sup_n \|k_n(\alpha, \beta)\|_\infty < \infty$ so that $k_n(\alpha, \beta)f(\beta)\,\overline{g(\alpha)}$ is dominated by an L^1 function and converges pointwise to $k(\alpha, \beta)\,f(\beta)\,\overline{g(\alpha)}$. Lebesgue's dominated convergence theorem yields

$$\int_\Gamma \int_\Gamma k(\alpha, \beta)\, f(\beta)\, \overline{g(\alpha)}\, d\alpha\, d\beta = (K\mathscr{F}^{-1}f, \mathscr{F}^{-1}g).$$

Hence the kernel $k(\alpha, \beta)$ represents $\mathscr{F}K\mathscr{F}^{-1}$ on $L^2(\Gamma)$ whenever K has compact support. Note for future reference that the kernel is independent of E provided supp $K \subseteq E \times E$.

Now associate an element of $\mathrm{PM}(G \times G)$ with each compactly supported operator K on $L^2(G)$ by the formula

$$\hat{F}_K(\alpha, \beta) = k(\alpha, -\beta).$$

It is possible to show, in essentially the same way as in Section 1, that $\mathrm{supp}(F_K) = \mathrm{supp}(K)$.

We next collect some facts about a certain operator-valued integral. In the context of [1] let $X = \mathscr{L}(L^2(G))$ and $X_* = \mathscr{C}_1(L^2(G))$. Define an automorphism group U of $\mathscr{L}(L^2(G))$ by the formula

$$U_x A = T_x A T_{-x},$$

where T_x is translation on $L^2(G)$ and $A \in \mathscr{L}(L^2(G))$. The group U is ultraweakly continuous so that for each $f \in L^1(G)$ and $A \in \mathscr{L}(L^2(G))$ there is an operator $U_f A$ such that for any ultraweakly continuous linear functional ρ, $\rho(U_f A) = \int_G f(x)\,\rho(U_x^{-1}A)\,dx$. Furthermore if $\{f_n\}$ is a bounded approximate identity for $L^1(G)$ then $U_{f_n}A$ converges ultraweakly to A.

The first fact that we wish to establish is that if A has compact support then even though $U_f A$ may not have compact support there is a kernel on $\Gamma \times \Gamma$ which represents $\mathscr{F}(U_f A)\mathscr{F}^{-1}$. Let f, g be compactly supported L^∞ functions on Γ. Then

$$((U_f A)\,\mathscr{F}^{-1}f, \mathscr{F}^{-1}g) = \int_G f(x)(T_{-x}AT_x\mathscr{F}^{-1}f, \mathscr{F}^{-1}g)\,dx$$

$$= \int_G f(x)(AT_x\mathscr{F}^{-1}f, T_x\mathscr{F}^{-1}g)\,dx$$

$$= \int_G f(x)(\mathscr{F}AT_x\mathscr{F}^{-1}f, \mathscr{F}T_x\mathscr{F}^{-1}g)\,dx$$

$$= \int_G f(x)(\mathscr{F}A\mathscr{F}^{-1}\mathscr{F}T_x\mathscr{F}^{-1}f, \mathscr{F}T_x\mathscr{F}^{-1}g)\,dx.$$

Now $(\mathscr{F}T_x\mathscr{F}^{-1}f)(\beta) = \beta(x)f(\beta)$ and $(\mathscr{F}T_x\mathscr{F}^{-1}g)(\alpha) = \alpha(x)g(\alpha)$ so the last integral becomes

$$\int_G f(x)\int_\Gamma\int_\Gamma a(\alpha, \beta)\,\beta(x)\,f(\beta)\,\overline{\alpha(x)\,g(\alpha)}\,d\alpha\,d\beta\,dx$$

$$= \int_\Gamma\int_\Gamma a(\alpha, \beta)f(\beta)\,\overline{g(\alpha)}\left(\int_G f(x)(\beta - \alpha)(x)\,dx\right)d\alpha\,d\beta$$

$$= \int_\Gamma\int_\Gamma \hat{f}(\alpha - \beta)\,a(\alpha, \beta)\,f(\beta)\,\overline{g(\alpha)}\,d\alpha\,d\beta$$

where $a(\alpha, \beta)$ represents $\mathscr{F}A\mathscr{F}^{-1}$ on $L^2(\Gamma)$. Hence the kernel $k(\alpha, \beta) = \hat{f}(\alpha - \beta)\,a(\alpha, \beta)$ represents $\mathscr{F}(U_f A)\mathscr{F}^{-1}$ on $L^2(\Gamma)$. This last relation may be written as

$$k(\alpha + \beta, \beta) = \hat{f}(\alpha)\,a(\alpha + \beta, \beta).$$

Thus every diagonal of $U_f A$ is a scalar multiple of the corresponding diagonal of A. From this we may also conclude that if $\hat{f}$ has compact support then $k(\alpha, \beta)$ is concentrated on a neighborhood of the diagonal. We can now associate a pseudomeasure $F_{U_f A}$ with $U_f A$ by the formula

$$\hat{F}_{U_f A}(\alpha, \beta) = k(\alpha, -\beta).$$

It can be shown that $\operatorname{supp}(F_{U_f A}) = \operatorname{supp}(U_f A)$.

LEMMA 2.1.6. *Let E be a closed subset of G. Let $A \in \mathcal{C}_{\max}(\varphi(G \times E))$ and $f \in L^1(G)$. Then $U_f A \in \mathcal{C}_{\max}(\varphi(G \times E))$.*

Proof. There exists a family $\mathscr{T}$ of open product sets $E_1 \times E_2$, each disjoint from $\varphi(G \times E)$, such that $A \in \mathcal{C}_{\max}(\varphi(G \times E))$ iff $P_{E_1}AP_{E_2} = 0$ for all $E_1 \times E_2 \in \mathscr{T}$. For any ultraweakly continuous linear functional ρ, and any $E_1 \times E_2 \in \mathscr{T}$, $\rho(P_{E_1}(U_f A)\,P_{E_2}) = \int_G f(x)\,\rho(P_{E_1}T_{-x}AT_xP_{E_2})\,dx = \int_G f(x)\,\rho(T_{-x}P_{E_1+x}AP_{E_2+x}T_x)\,dx.$

Fix $x \in G$ and suppose that $(E_1 + x) \times (E_2 + x) \cap \varphi(G \times E) \neq 0$. Then $(e_1 + x, e_2 + x) = (a, a - b)$ fr some $e_1 \in E_1$, $e_2 \in E_2$, $a \in G$, $b \in E$. This gives $(e_1, e_2) = (a - x, a - x - b) \in \varphi(G \times E)$, contradicting $E_1 \times E_2 \cap \varphi(G \times E) = \varnothing$. Hence $P_{E_1 + x} A P_{E_2 + x} = 0$. Thus $P_{E_1}(U_f A) P_{E_2} = 0$ and so $U_f A \in \mathcal{C}_{\max}(\varphi(G \times E))$.

We are now ready to prove

THEOREM 2.1.7. *Let E be a closed subset of G. Then $\mathrm{Alg}(P_E)$ is synthetic iff E is a set of spectral synthesis in the sense of harmonic analysis.*

Proof. As we remarked previously, it suffices to show that $\mathcal{C}_{\min}(\varphi(G \times E)) = \mathcal{C}_{\max}(\varphi(G \times E))$ iff E is a set of spectral synthesis. So assume that E is a set of spectral synthesis. Let $A \in \mathcal{C}_{\max}(\varphi(G \times E))$. Since G is σ-compact there is an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact sets such that $G = \bigcup_{n=1}^{\infty} K_n$. The sequence $P_{K_n} A P_{K_n}$ converges strongly to A. Also $\mathcal{C}_{\max}(\varphi(G \times E))$ is a bimodule over the multiplication algebra so $P_{K_n} A P_{K_n} \in \mathcal{C}_{\max}(\varphi(G \times E))$. Hence it follows that we need only show that every compactly supported operator in $\mathcal{C}_{\max}(\varphi(G \times E))$ is in $\mathcal{C}_{\min}(\varphi(G \times E))$. So let A be a compactly supported element of $\mathcal{C}_{\max}(\varphi(G \times E))$. Take a bounded approximate identity for $L^1(G)$ such that for each $n, \hat{f}_n$ has compact support. Then $U_{f_n} A$ converges ultraweakly to A. Thus it suffices to show that $U_{f_n} A \in \mathcal{C}_{\min}(\varphi(G \times E))$. Recall that $\mathscr{F}(U_{f_n} K) \mathscr{F}^{-1} \in \mathscr{F}(\mathcal{C}_{\max}(\varphi(G \times E))) \mathscr{F}^{-1}$ is represented on $L^2(\Gamma)$ by a kernel $k(\alpha, \beta)$ which is concentrated on a neighborhood of the diagonal. Furthermore $\mathscr{F}(U_{f_n} A) \mathscr{F}^{-1}$ is the strong limit of a sequence of operators of the form

$$f(\alpha) \mapsto \sum_i a_i k(\alpha, \alpha + \beta_i) f(\alpha + \beta_i).$$

So we can reduce further to showing that the maps

$$f(\alpha) \mapsto k(\alpha, \alpha + \beta_i) f(\alpha + \beta_i)$$

are in $\mathscr{F} \mathcal{C}_{\min}(\varphi(G \times E)) \mathscr{F}^{-1}$. The pseudomeasure $F_{U_{f_n} A}$ is supported in $\varphi(G \times E)$. Hence the pseudomeasure $(F_{U_{f_n} A})_\varphi$ is supported in $G \times E$. By Lemma 2.1.5, for fixed α, the pseudomeasure $\widehat{(F_{U_{f_n} K})_\varphi}(\alpha, \beta)$ is supported in E. But by previous calculations $\widehat{(F_{U_{f_n} A})_\varphi}(\alpha, \beta) = \widehat{(F_{U_{f_n} A})}(\alpha + \beta, -\beta) = k(\alpha + \beta, \beta)$. Hence for fixed α, the L^∞ function $k(\alpha, \alpha + \beta_i)$ is the Fourier transform of a pseudomeasure D_i supported in E. Then the convolution operators C_{D_i} lie in $\mathcal{C}_{\max}(\varphi(G \times E))$. The map $f(\alpha) \mapsto f(\alpha + \beta_i)$ is $\mathscr{F} M_{\beta_i} \mathscr{F}^{-1}$, where M_{β_i} is multiplication by the character $\beta_i(x)$. The map $f(\alpha) \mapsto k(\alpha, \alpha + \beta_i) f(\alpha + \beta_i)$ is then $\mathscr{F} C_{D_i} M_{\beta_i} \mathscr{F}^{-1}$. If E is a set of spectral

synthesis then there is a net $\{\mu_\alpha\}$ of measures supported in E such that μ_α converges w* to D_i. As was observed in [2, Corollary to 2.4.3], the w*-closed linear span of a set of pseudomeasures coincides with the ultraweakly closed linear span of this set of pseudomeasures viewed as convolution operators. Hence there is a net $\{\mu_\alpha\}$ of measures supported in E such that C_{μ_α} converges ultraweakly to C_{D_i}. Therefore the pseudointegral operators $C_{\mu_\alpha} M_{\beta_i}$ converge ultraweakly to $C_{D_i} M_{\beta_i}$. Hence $\mathscr{F} C_{D_i} M_{\beta_i} \mathscr{F}^{-1} \in \mathscr{F} A_{\min}(\varphi(G \times E)) \mathscr{F}^{-1}$. Thus, $\mathcal{A}_{\max}(\varphi(G \times E)) = \mathcal{A}_{\min}(\varphi(G \times E))$.

For the converse we consider only the case where G is $R^n \times Q$ for some separable compact group Q, leaving the general case where G is a countable disjoint union of cosets of $R^n \times Q$ to the reader. Suppose that $E \subseteq G$ is not a set of spectral synthesis. Then there is a pseudomeasure D supported in E and $g \in L^1(\Gamma)$ such that $\hat{g}$ vanishes on E and $D(\hat{g}) \neq 0$. Let $\{f_n\}$ be an approximate identity for $L^1(\Gamma)$ and suppose that for all n, $\hat{f}_n$ has compact support. Define pseudomeasures D_n by the formula $D_n(\hat{g}) = D(\hat{f}_n \hat{g})$. It is clear that each D_n has compact support and that supp $D_n \subseteq E$ for all n. Furthermore for all $g \in L^1(\Gamma)$, $D_n(\hat{g}) \to D(\hat{g})$. So if $D_n(\hat{g}) = 0$ then $D(\hat{g}) = 0$. Therefore we may assume that D has compact support. Our plan now is to send $L^1(\Gamma \times \Gamma)$ into the trace class operators on $L^2(G)$ by "cutoff" of the Fourier transform $(f \mapsto T_f)$ so that for a fixed compactly supported operator B on $L^2(G)$, $\mathrm{tr}(BT_f) = F_B(\hat{f})$. Then if a net $\{B_\alpha\}$ has common compact support and converges ultraweakly to B, $\{F_{B_\alpha}\}$ will converge in the weak* topology to F_B. We will then use the pseudomeasure D to construct $B \in \mathcal{A}_{\max}(\varphi(G \times E))$ such that $B \notin \mathcal{A}_{\min}(\varphi(G \times E))$. Toward this end we need a routine generalization of [30, Theorem 2.7.6]. That is, suppose that f is a function on $T^n \times Q$, $0 < \delta < \pi$, and f is supported in $[-\pi + \delta, \pi - \delta]^n \times Q$. Then $f \in A(R^n \times Q)$ iff $f \in A(T^n \times Q)$. To prove this make use of the function $h(x_1) \cdots h(x_n)$, $(x_1, ..., x_n) \in R^n$, defined on $R^n \times Q$, and follow the proof in [30]. This result is also true if we regard T^n as $[-m\pi, m\pi]^n$ for some natural number m and insist that f vanish outside $[-m\pi + \delta, m\pi - \delta]^n \times Q$.

Let B be a compactly supported operator on $L^2(G)$. Then there is a compact set S such that supp $B \subseteq S \times S$. Let $c \in L^1(\Gamma)$ have the property $\hat{c} = 1$ on S and c vanishes outside some compact subset of G. If $f \in L^1(\Gamma \times \Gamma)$ then the convolution $(c(\alpha) c(\beta)) * (f(\alpha, \beta)) \in L^1(\Gamma \times \Gamma)$ and its Fourier transform $\hat{c}(x) \hat{c}(y) \hat{f}(x, y)$ vanishes off a compact subset of $G \times G$. The function $\hat{c}(x) \hat{c}(y) \hat{f}(x, y)$ will be denoted by $\hat{f}_c(x, y)$. Now define an integral operator T_f on $L^2(G)$ by the formula

$$(T_f u)(x) = \int_G \hat{f}_c(y, x) u(y) \, dy$$

for $u \in L^2(G)$. Since $\hat{f}_c$ has compact support there is a natural number m

such that supp f_c lies in the interior of $([-m\pi, m\pi]^n \times Q) \times ([-m\pi, m\pi]^n \times Q)$. Therefore $f_c \in A([-m\pi, m\pi]^n \times Q \times [-m\pi, m\pi]^n \times Q)$. This implies that f_c admits an absolutely convergent expansion in terms of the characters of the group $[-m\pi, m\pi]^n \times Q \times [-m\pi, m\pi]^n \times Q$ and hence that T_f is a trace class operator on $L^2([-m\pi, m\pi]^n \times Q)$. Therefore T_f is a trace class operator on $L^2(G)$. To prove that $\mathrm{tr}(BT_f) = F_B(f)$ choose a sequence $\{B_n\}$ of integral operators with L^∞ kernels such that supp $B_n \subseteq S \times S$ for each n and which converges ultraweakly to B. Let $b_n(x, y)$ be the kernel of the integral operator B_n. Then by [2, Proposition 2.2.7]

$$\mathrm{tr}(B_n T_f) = \int_G \int_G b_n(x, y) \hat{f}_c(x, y) \, dx \, dy$$

$$= \int_G \int_G b_n(x, y) \hat{f}(x, y) \, dx \, dy$$

$$= \int_G \int_G b_n(x, y) \left(\int_\Gamma \int_\Gamma f(\alpha, \beta) \alpha(-x) \beta(-y) \, d\alpha \, d\beta \right) dx \, dy$$

$$= \int_\Gamma \int_\Gamma f(\alpha, \beta) \left(\int_G \int_G b_n(x, y) \alpha(-x) \beta(-y) \, dx \, dy \right) d\alpha \, d\beta$$

$$= \int_\Gamma \int_\Gamma f(\alpha, \beta) k_n(\alpha, -\beta) \, d\alpha \, d\beta$$

$$= \int_\Gamma \int_\Gamma f(\alpha, \beta) \hat{F}_{B_n}(\alpha, \beta) \, d\alpha \, d\beta$$

$$= F_{B_n}(\hat{f}).$$

Here $k_n(\alpha, \beta)$ represents $\mathscr{F} B_n \mathscr{F}^{-1}$ on $L^2(\Gamma)$. A previous argument involving Lebesgue's dominated convergence theorem implies that $\int_\Gamma \int_\Gamma f(\alpha, \beta) k_n(\alpha, \beta) \, d\alpha \, d\beta$ converges to $\int_\Gamma \int_\Gamma f(\alpha, \beta) k(\alpha, \beta) \, d\alpha \, d\beta$ (here $k(\alpha, \beta)$ represents $\mathscr{F} B \mathscr{F}^{-1}$ on $L^2(\Gamma)$). Since $\mathrm{tr}(B_n T_f) \to \mathrm{tr}(BT_f)$ we have that $\mathrm{tr}(BT_f) = F_B(\hat{f})$.

Now consider the operator $M_{\hat{c}} C_D$ on $L^2(G)$, where $M_{\hat{c}}$ is multiplication by $\hat{c}$ and C_D is convolution by the compactly supported pseudomeasure D. It is evident that $M_{\hat{c}} C_D$ has compact support. Say supp $M_{\hat{c}} C_D \subseteq K \times K$. Let $\{B_\alpha\}$ be a net of pseudo-integral operators which converges ultraweakly to $M_{\hat{c}} C_D$. We may assume that supp $B_\alpha \subseteq K \times K$ for every α. Then $\{F_{B_\alpha}\}$ is a net of measures which converges in the weak* topology to $F_{M_{\hat{c}} C_D}$. But then $((F_{B_\alpha}))_\varphi$ is a net of measures which converges in the weak* toplogy to $(F_{M_{\hat{c}} C_D})_\varphi$. But $(F_{M_{\hat{c}} C_D})_\varphi$ is the pseudomeasure which sends $\hat{f}_1(x) \hat{f}_2(y)$ to

$(\int_G \hat{c}(x)\hat{f}_1(x)\,dx)\cdot D(\hat{f}_2(y))$ (tensor product). It is evident that $(F_{M_{\hat{c}}C_D})_\varphi$ is not synthesizable. Hence $M_{\hat{c}}C_D \notin \mathcal{Ol}_{\min}(\varphi(G \times E))$.

Next we wish to emphasize the important role played by the sets $\varphi(G \times E)$ in the theory of CSL algebras. Let G be a separable locally compact abelian group and let Σ be a closed subsemigroup of G containing 0. A pre-order is defined as in [2, p. 497]. Thus $x \leqslant y$ iff $y - x \in \Sigma$. The graph of this pre-order is $\{(x, x+s): x \in G, s \in \Sigma\} = \varphi(G \times -\Sigma)$. The CSL algebra associated with this pre-order is denoted by $\text{Alg}(\Sigma)$. It is evident that $\text{Alg}(\Sigma) = \mathcal{Ol}_{\max}(\varphi(G \times -\Sigma))$. Hence we have

THEROREM 2.1.8. *The operator algebra* $\text{Alg}(\Sigma)$ *is synthetic iff* Σ *is a set of spectral synthesis in the sense of harmonic analysis.*

Unfortunately it seems to be unknown whether there is a closed semigroup Σ which is not a set of spectral synthesis.

For any closed sets E_1, E_2 in G it is possible to show that the tensor product formula $\text{Alg}(P_{E_1}) \bar{\otimes} \text{Alg}(P_{E_2}) = \text{Alg}(P_{E_1} \otimes P_{E_2})$ holds iff $\mathcal{Ol}_{\max}(\varphi(G \times E_1)) \bar{\otimes} \mathcal{Ol}_{\max}(\varphi(G \times E_2)) = \mathcal{Ol}_{\max}(\varphi(G \times G \times E_1 \times E_2))$. Now suppose that G is compact, $S \in \mathcal{Ol}_{\max}(\varphi(G \times E_1))$, and $T \in \mathcal{Ol}_{\max}(\varphi(G \times E_2))$. Then $S \otimes T \in \mathcal{Ol}_{\max}(G \times G \times E_1 \times E_2)$ and each diagonal of $S \otimes T$ (after Fourier transform) defines a pseudomeasure supported in $E_1 \times E_2$. In fact each diagonal of $S \otimes T$ defines a pseudomeasure which is a tensor product of pseudomeasures supported in E_1 and E_2, respectively. If both E_1 and E_2 are sets of spectral synthesis then such a tensor product can be synthesized by measures. Hence each diagonal of $S \otimes T$ is in $\mathcal{Ol}_{\min}(\varphi(G \times G \times E_1 \times E_2))$. We have seen that this implies $S \otimes T \in \mathcal{Ol}_{\min}(\varphi(G \times G \times E_1 \times E_2))$. For a noncompact group G the argument can be modified to yield the same conclusion. We can now prove

THEOREM 2.1.9. *Let* E_1, E_2 *be sets of spectral synthesis in a separable locally compact abelian group* G. *Then* $\text{Alg}(P_{E_1}) \bar{\otimes} \text{Alg}(P_{E_2}) = \text{Alg}(P_{E_1} \otimes P_{E_2})$ *iff* $E_1 \times E_2$ *is a set of spectral synthesis in* $G \times G$.

Proof. If $\text{Alg}(P_{E_1}) \bar{\otimes} \text{Alg}(P_{E_2}) = \text{Alg}(P_{E_1} \otimes P_{E_2})$ then $\mathcal{Ol}_{\max}(\varphi(G \times E_1)) \bar{\otimes} \mathcal{Ol}_{\max}(\varphi(G \times E_2)) = \mathcal{Ol}_{\max}(\varphi(G \times G \times E_1 \times E_2))$. Since $\mathcal{Ol}_{\max}(\varphi(G \times E_1)) \bar{\otimes} \mathcal{Ol}_{\max}(\varphi(G \times E_2)) \subseteq \mathcal{Ol}_{\min}(\varphi(G \times G \times E_1 \times E_2))$ by discussion above, it follows that $\mathcal{Ol}_{\min}(\varphi(G \times G \times E_1 \times E_2)) = \mathcal{Ol}_{\max}(\varphi(G \times G \times E_1 \times E_2))$. Hence $E_1 \times E_2$ must be a set of spectral synthesis.

Conversely suppose that $E_1 \times E_2$ is a set of spectral synthesis. Then $\mathcal{Ol}_{\min}((G \times G \times E_1 \times E_2)) = \mathcal{Ol}_{\max}(\varphi(G \times G \times E_1 \times E_2))$. Suppose T_μ is a pseudointegral operator in $\mathcal{Ol}_{\min}(\varphi(G \times G \times E_1 \times E_2))$. Then after Fourier transformation each diagonal of T_μ is a measure supported in $E_1 \times E_2$. Such a measure is the w*-limit of a net of measures each of which is a linear combination of tensor products of measures supported in E_1

and E_2, respectively. Therefore $T_\mu \in \mathcal{C}_{\max}(\varphi(G \times E_1)) \,\bar{\otimes}\, \mathcal{C}_{\max}(\varphi(G \times E_2))$. Hence $\mathcal{C}_{\min}(\varphi(G \times G \times E_1 \times E_2)) \subseteq \mathcal{C}_{\max}(\varphi(G \times E_1)) \,\bar{\otimes}\, \mathcal{C}_{\max}(\varphi(G \times E_2))$ so $\mathcal{C}_{\max}(\varphi(G \times E_1)) \,\bar{\otimes}\, \mathcal{C}_{\max}(\varphi(G \times E2)) = \mathcal{C}_{\max}(\varphi(G \times G \times E_1 \times E_2))$. Therefore $\mathrm{Alg}(P_{E_1}) \,\bar{\otimes}\, \mathrm{Alg}(P_{E_2}) = \mathrm{Alg}(P_{E_1} \otimes P_{E_2})$.

COROLLARY 2.1.10. *If semigroups* Σ_1, Σ_2 *are sets of spectral synthesis in* G *then* $\mathrm{Alg}(\Sigma_1) \,\bar{\otimes}\, \mathrm{Alg}(\Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$ *iff* $\Sigma_1 \times \Sigma_2$ *is a set of spectral synthesis in* $G \times G$.

The importance of Corollary 2.1.10 and Theorem 2.1.8 lies in that they enable us to deal with synthesis in $\mathrm{Alg}(\Sigma)$ and the tensor product formula $\mathrm{Alg}(\Sigma_1) \,\bar{\otimes}\, \mathrm{Alg}(\Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$ when the semigroup(s) have zero Haar measure by using methods from harmonic analysis. For example, it is known that any closed subgroup H_1 of G is a set of spectral synthesis [30]. Hence we can conclude that the von Neumann algebra $\mathrm{Alg}(H_1)$ is synthetic and that for any closed subgroup H_2 of G, $\mathrm{Alg}(H_1) \,\bar{\otimes}\, \mathrm{Alg}(H_2) = \mathrm{Alg}(H_1 \times H_2)$.

2.2 *Spectral Synthesis in Infinite Tensor Products*

For each $n = 1, 2, \ldots$ let (X_n, P_n, dx_n) be a pre-ordered probability measure space. That is, X_n is a compact metric space, P_n is a closed pre-order on X_n, and dx_n is a probability measure on X_n. We can define a closed pre-order $P_1 \otimes P_2 \otimes \cdots$ on the compact metric space $X = X_1 \times X_2 \times \cdots$

$$(x_1, x_2, \ldots) \leqslant (y_1, y_2, \ldots)$$

iff $x_n \leqslant y_n$ for $n = 1, 2, \ldots$. If we equip X with the infinite product measure $dx_1 \, dx_2 \cdots$ then $(X, P_1 \otimes P_2 \otimes \cdots, dx_1 \, dx_2 \cdots)$ is a pre-ordered probability measure space. The associated CSL algebra is denoted by $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$. We now give necessary and sufficient conditions for spectral synthesis in $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$.

First we recall a few facts concerning functions in $L^2(dx_1 \, dx_2 \cdots)$ which depend only on a finite number of coordinates

$$f(x_1, x_2, \ldots) = g(x_1, x_2, \ldots, x_n).$$

There are natural inclusions $L^2(dx_1) \subset L^2(dx_1 \, dx_2) \subset \cdots$ and $\bigcup_{n=1}^{\infty} L^2(dx_1 \cdots dx_n)$ is dense in $L^2(dx_1 \, dx_2 \cdots)$ since the continuous functions in the former are dense in $C(X)$.

Define $P_n : L^2(dx_1 \, dx_2 \cdots) \to L^2(dx_1 \cdots dx_n)$ by

$$(P_n f)(x_1, \ldots, x_n) = \int f(x_1, \ldots, x_n, x_{n+1}, \ldots) \, dx_{n+1} \, dx_{n+2} \cdots.$$

Then

$$\|P_n\| = 1$$

$$P_n^2 = P_n$$

and P_n is the identity on $L^2(dx_1 \cdots dx_n)$. Therefore P_n is the orthogonal projection onto $L^2(dx_1 \cdots dx_n)$.

We can write

$$L^2(dx_1 \, dx_2 \cdots) = L^2(dx_1 \cdots dx_n) \otimes L^2(dx_{n+1} \, dx_{n+2} \cdots)$$

and given

$$A \in \mathscr{L}(L^2(dx_1 \cdots dx_n))$$

$$B \in \mathscr{L}(L^2(x_{n+1} \, dx_{n+2} \cdots))$$

we obtain $A \otimes B \in \mathscr{L}(L^2(dx_1 \, dx_2 \cdots))$ as follows,

$$A \otimes B(f(x_1, ..., x_n) \, g(x_{n+1}, ...)) = A \otimes B(f \otimes g) = (Af) \otimes (Bg).$$

LEMMA 2.2.1.   *If* $A \in \text{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$ *and* $1 = identity$ *on* $L^2(dx_{n+1} \, dx_{n+2} \cdots)$ *then*

$$A \otimes 1 \in \text{Alg}(P_1 \otimes P_2 \otimes \cdots)$$

Proof.   Recall the discussion of tensor products in Section 1.3. We have $A \otimes 1 \in \text{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n) \bar{\otimes} \text{Alg}(P_{n+1} \otimes \cdots) \subseteq \text{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n \otimes \cdots).$

LEMMA 2.2.2.   *Let* A, B *be pseudo-integral operators on* $L^2(X, \mu)$ *and* $L^2(Y, \gamma)$, *respectively. Then* $A \otimes B$ *is a pseudo-integral operator on* $L^2(X \times Y, \mu \times \gamma) = L^2(X, \mu) \otimes L^2(Y, \gamma)$.

Proof.   We prove the result in the case that both A, B take nonnegative functions into nonnegative functions. The general case follows from this.

According to [31, Theorem 3.1], $A \otimes B$ is a pseudo-integral operator if it takes nonnegative functions into nonnegative functions. Let $f(x, y) \geqslant 0$ be in $L^2(X \times Y, \mu \times \gamma)$. Then we can find a sequence of functions of the form

$$\sum_{k=1}^{n} f_k(x) \, g_k(y)$$

such that

$$f_k \in L^2(X, \mu), \qquad k = 1, 2, ..., n$$

$$g_k \in L^2(Y, \gamma), \qquad k = 1, 2, ..., n$$

and

$$f_k, g_k \geqslant 0 \qquad \text{for} \quad k = 1, 2, ..., n,$$

which converges in L^2-norm to f. It follows that

$$A \otimes B \left(\sum_{k=1}^{n} f_k(x) \, g_k(y) \right) \to A \otimes B(f).$$

But

$$A \otimes B \left(\sum_{k=1}^{n} f_k(x) \, g_k(y) \right) = \sum_{k=1}^{n} (Af_k) \otimes (Bg_k) \geqslant 0.$$

Therefore $A \otimes B(f) \geqslant 0$ and $A \otimes B$ is a pseudo-integral operator.

LEMMA 2.2.3. *Let* $T \in \mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$. *Then* $P_n T P_n$ *restricted to* $L^2(dx_1 \cdots dx_n)$ *is in* $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$.

Proof. Let U, V be open sets in $X_1 \times X_2 \times \cdots \times X_n$ such that $U \times V$ is disjoint from the graph of the pre-order $P_1 \otimes P_2 \otimes \cdots \otimes P_n$ and let $f, g \in L^2(dx_1 \cdots dx_n)$ live in U, V, respectively. We must show that

$$(P_n T P_n f, g) = 0.$$

If we now view f, g as elements of $L^2(dx_1 \, dx_2 \cdots)$, the inner product becomes

$$(TP_n f, P_n g) = (Tf, g).$$

Furthermore f, g live in the open sets

$$U^1 = U \times X_{n+1} \times X_{n+2} \times \cdots$$
$$V^1 = V \times X_{n+1} \times X_{n+2} \times \cdots,$$

respectively. One sees that $U^1 \times V^1$ is disjoint from the graph of $P_1 \otimes P_2 \otimes \cdots$ so $(Tf, g) = 0$. Hence $P_n T P_n \in \mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$.

THEOREM 2.2.4. *The operator algebra* $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$ *is synthetic iff for each* n, $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$ *is synthetic.*

Proof. Assume that for every n, $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$ is synthetic. Let $T \in \mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots)$. By Lemmas 2.2.1 and 2.2.3 the operators

$$T_n = (P_n T P_n) \otimes 1 \in \mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots).$$

Since $\mathrm{Alg}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$ is synthetic, there is a net $\{A_\alpha\}$ of pseudo-integral operators such that

$$A_\alpha \xrightarrow{\text{ultra weakly}} P_n T P_n.$$

Hence

$$A_\alpha \otimes 1 \xrightarrow{\text{uw}} P_n T P_n \otimes 1.$$

By Lemma 2.2.2, $A_\alpha \otimes 1$ is a pseudo-integral operator, so $T_n \in A_{\min}(P_1 \otimes P_2 \otimes \cdots)$. If we show that

$$T_n \xrightarrow{\text{strongly}} T,$$

then $T \in A_{\min}(P_1 \otimes P_2 \otimes \cdots)$, and the algebra will be synthetic.

Since $P_n \to^s I$ we know that $P_n T \to^s T$. Furthermore $\|P_n T P_n\| \leqslant \|T\|$ and so $\|(P_n T P_n) \otimes 1\| \leqslant \|T\|$ for all n. Pick $f_m \in L^2(dx_1 \cdots dx_m)$. Then for sufficiently large n,

$$T_n f = ((P_n T P_n) \otimes 1) f = (P_n T P_n) f = P_n T f \to T f.$$

Therefore $\lim_{n \to \infty} T_n f = T f$ on a dense subset of $L^2(dx_1 \, dx_2 \cdots)$ and so $T_n \to^s T$.

COROLLARY 2.2.5 [2, p. 498]. *The operator algebras* $\mathrm{Alg}(2^\infty, \leqslant, m_p)$ *are synthetic.*

Proof. The partial order on 2^∞ is the partial order $P \otimes P \otimes \cdots$, where P is the obvious order on the space $\{-1, 1\}$. For each n the algebra $\mathrm{Alg}(P \otimes P \otimes \cdots P)$ is a CSL algebra on a finite-dimensional Hilbert space and is synthetic. Hence $\mathrm{Alg}(2^\infty, \leqslant, m_p)$ is synthetic.

COROLLARY 2.2.6. *The algebras* $\mathrm{Alg}(2^\infty, \leqslant, m_p)$ *are doubly generated.*

We would like to conclude by listing some open problems which we feel are important for future progress.

(i) Let G be a locally compact abelian group, Σ a closed sub-semigroup of G, and μ a Borel measure on G. Define a closed pre-order on G by

$$x \leqslant y \quad \text{iff} \quad y - x \in \Sigma.$$

Then $(G, \mu, \leqslant)$ is a standard pre-ordered measure space. The algebra $\mathrm{Alg}(G, \mu, \leqslant)$ is denoted by $\mathrm{Alg}(G, \Sigma, \mu)$ [3, part 6].

When does $\mathrm{Alg}(G, \Sigma, \mu) \, n.\mathscr{K} \neq 0$?

When is $\mathrm{Alg}(G, \Sigma, \mu)$ synthetic?

(ii) If the compact operators are dense in $\text{Alg}(P)$, are the Hilbert–Schmidt operators dense in $\text{Alg}(P)$? This is the extension of Theorem 1.4.3 to the ideal of compact operators.

(iii) Are the operator algebras $\text{Alg}(2^\infty, \leqslant, m_p) + \mathcal{K}$ norm-closed?

(iv) Let $K_1 \subseteq X \times X$, $K_2 \subseteq Y \times Y$ be closed sets. Define $K_1 \otimes K_2 \subset X \times Y \times X \times Y$ to be the set

$$\{(x_1, y_1, x_2, y_2): (x_1, x_2) \in K_1, (y_1, y_2) \in K_2\}.$$

Note that if K_1 is the graph of a pre-order P_1 and K_2 is the graph of a pre-order P_2 then $K_1 \otimes K_2$ is the graph of the pre-order $P_1 \otimes P_2$.

$$\text{Is } A_{\min}(K_1) \bar{\otimes} A_{\min}(K_2) = A_{\min}(K_1 \otimes K_2)?$$

This would imply that the tensor product formula holds in the class of synthetic CSL algebras iff the tensor product of two synthetic algebras is synthetic.

ACKNOWLEDGMENTS

REFERENCES

1. W. ARVESON, The harmonic analysis of automorphism groups, *in* "Operator Algebras and Applications," Proceedings of Symposia in Pure Mathematics, Vol. 38, Amer. Math. Society, Providence, RI, 1982.
2. W. ARVESON, Operator algebras and invariant subspaces, *Ann. of Math.* **100** (1974), 433–532.
3. W. ARVESON, Perturbation theory for groups and lattices, *J. Funct. Anal.* **53** (1983), 22–73.
4. W. ARVESON, "Ten Lectures on Operator Algebras," Regional Conference Series in Mathematics, No. 55, Amer. Math. Soc., Providence, RI, 1984.
5. J. J. BENEDETTO, "Spectral Synthesis," Academic Press, New York/London, 1975.
6. G. BROWN AND E. HEWITT, Continuous singular measures with small Fourier–Stieltjes transform, *Adv. in Math.* **37** (1980), 27–60.
7. K. R. DAVIDSON, Similarity and compact perturbations of nest algebras.
8. K. DAVIDSON AND C. K. FONG, An operator algebra which is not closed in the Calkin algebra, *Pacific J. Math.* **72** (1977), 57–58.
9. R. E. EDWARDS, "Fourier Series," Vols. I and II, 2nd Ed., Springer-Verlag, New York, 1982.
10. T. FALL, W. ARVESON, AND P. MUHLY, Perturbations of nest algebras, *J. Oper. Theory* **1** (1979), 137–150.

11. J. FROELICH, Compact operators in the algebra of a partially ordered measure space, *J. Oper. Theory*, **10** (1983), 353–355.
12. J. FROELICH, "Compact Operators, Invariant Subspaces and Spectral Synthesis," Dissertation, University of Iowa, 1984.
13. C. C. GRAHAM, Harmonic analysis and LCA groups, *in* "Studies in Harmonic Analysis," Math. Assoc. Amer., Washington, D.C., 1976.
14. C. C. GRAHAM AND O. C. MCGEHEE, "Essays in Commutative Harmonic Analysis," Springer-Verlag, New York, 1979.
15. P. R. HALMOS, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **5** (1970), 887–933.
16. A. HOPENWASSER, Compact operators in the radical of a reflexive operator algebra, *J. Oper. Theory* **2** (1979), 127–129.
17. A. HOPENWASSER, The radical of a reflexive operator algebra, *Pacific J. Math.* **65** (1976), 375–392.
18. A. HOPENWASSER AND D. LARSON, The carrier space of a reflexive operator algebra, *Pacific J Math.* **81** (1979), 417–434.
19. A. HOPENWASSER, C. LAURIE, AND R. MOORE, Reflexive algebras with completely distributive subspace lattices, *J. Oper. Theory* **11** (1984), 91–108.
20. A. HOPENWASSER AND J. KRAUS, Tensor products of reflexive algebras, II, preprint.
21. J. P. KAHANE AND R. SALEM, "Ensembles Parfaits et Series Trigonometriques," Hermann, Paris, 1963.
22. J. KRAUS, Tensor products of reflexive algebras, preprint.
23. J. KRAUS, The slice map problem for σ-weakly closed subspaces of Von Neumann algebras, *Trans. Amer. Math. Soc.*, in press.
24. J. KRAUS, W*-Dynamical systems and reflexive operator algebras, *J. Oper. Theory* **8** (1982), 181–194.
25. J. KRAUS, Abelian operator algebras and tensor products, preprint.
26. C. LAURIE, On density of compact operators in reflexive algebras, *Indiana Univ. Math. J.* **30** (1981), 1–16.
27. C. LAURIE AND W. LONGSTAFF, A note on rank-one operators in reflexive algebras, *Proc. Amer. Math. Soc.* **89** (1983), 293–297.
28. H. RADJAVI AND P. ROSENTHAL, Invariant subspaces, *Ergeb. Math. Grenzgeb.* **77** (1973).
29. J. R. RINGROSE, "Compact Non-self-adjoint Operators," Mathematical Studies, No. 35, Van Nostrand, London, 1971.
30. W. RUDIN, Fourier Analysis on Groups," Interscience, New York, 1962.
31. A. R. SOUROUR, Pseudo-integral operators, *Trans. Amer. Math. Soc.* **253** (1979), 339–363.