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# A RANK 2 VECTOR BUNDLE ON $\mathbb{P}^4$ WITH 15,000 SYMMETRIES

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THE MOTIVATION for this paper was to look for rank 2 vector bundes $\mathscr{F}$ on $\mathbb{P}^n$ for $n \geqq 4$ which are not direct sums of lines bundles. Schwarzenberg [14], found many such bundles on $\mathbb{P}^2$ and one of us [4] found quite a few on $\mathbb{P}^3$ although already they seem to be "rarer". In this paper, we construct *one* on $\mathbb{P}^4$. It seems quite plausible that there are none on $\mathbb{P}^n$ if $n$ is large enough. The question is closely related to the existence of non-singular subvarieties $X^{n-2} \subset \mathbb{P}^n$ of dimension $n-2$ which are not complete intersections:

$$X = H_1.H_2.$$

If $\mathscr{F}$ is an indecomposable rank 2 vector bundle and $n \geq 3$ then for $k \gg 0$, a general section $s \in \Gamma(\mathscr{F}(k))$ will vanish on a non-singular $X^{n-2}$ which is not a complete intersection; conversely, if $X^{n-2} \subset \mathbb{P}^n$ is non-singular and $n \geq 6$, a recent result of Barth and Larsen ([1] and [9]) shows that the line bundle $\Omega_X^{n-2}$ is isomorphic to $\mathcal{O}_X(k)$ for some $k$, from which it follows readily that $X$ is the zero-set of a section of a rank 2 bundle $\mathscr{F}$. And if $X$ is not a complete intersection, then $\mathscr{F}$ is indecomposable. Now interestingly enough, it seems as far as we know that classical procedures and classical examples yield non-singular $X^{n-2}$'s in $\mathbb{P}^n$, which are not complete intersections, *only if* $n \leq 5$.

The vector bundle constructed here has a 4-dimensional space of sections almost all of which vanish on a non-singular $X_s \subset \mathbb{P}^4$ which is an abelian surface. We first found the bundle by establishing that such $X_s$'s had to exist and then constructing $\mathscr{F}$ from $X_s$ as an extension. However by then applying the general "Postnikov" construction of [3], we found a much more direct description of $\mathscr{F}$. The theory of the bundle $\mathscr{F}$ and of the surfaces $X_s$ is united by the fact that both are acted on by the Heisenberg group $H$ (an irreducible 2-step nilpotent subgroup of $SL_5(\mathbb{C})$ of order 125: cf. §1) which is well known from the theory of theta functions; $\mathscr{F}$ is acted on also by the normalizer $N$ of $H$, of order 15,000. We have developed all our results by keeping track of the action of $N$ at every stage and using the character table of $N$ where necessary. This is a quick efficient method although unfortunately not very illuminating. Our main results are as follows: we construct the bundle $\mathscr{F}$ in §2 and note immediately by examining its Chern classes that it is indecomposable; in §4 we find the cohomology of $\mathscr{F}(n)$ for every $n$; in §5, we prove that the zero-sets $X_s$ of its general sections are abelian and that conversely all abelian surfaces in $\mathbb{P}^4$ arise in this way; in §6, we show that as a corollary we get an explicit birational map between a certain moduli space of abelian surfaces and $\mathbb{P}^3$. We have put the character table of $SL_2(\mathbb{Z}_5)$ in an appendix for easy reference.

## §1. THE HEISENBERG GROUP IN DIMENSION FIVE

The purpose of this section is to review in a special case a configuration of groups studied recently by Weil [16] (cf. also Igusa [7, Chap. 1]; Mumford [12, §1]) and closely related to the theory of theta functions and abelian varieties. Weil's construction starts from an arbitrary locally compact abelian group $A$, but we take $A = \mathbb{Z}_5$, the cyclic group of order 5, and proceed as follows:

Let

$$V = \mathrm{Map}(\mathbb{Z}_5, \mathbb{C}),$$

be the vector-space of complex-valued functions on $\mathbb{Z}_5$. Note that $V$ has a natural $\mathbb{Q}$-rational structure given by the $\mathbb{Q}$-subspace $\mathrm{Map}(\mathbb{Z}_5, \mathbb{Q})$. Let $\varepsilon = e^{2\pi i/5} \in \boldsymbol{\mu}_5$, the group of 5th roots of 1. The *Heisenberg group*

$$H \subset SL_5(\mathbb{C})$$

is the subgroup generated by $\sigma$ and $\tau$, given by

$$\sigma x(i) = x(i+1)$$

$$\tau x(i) = \varepsilon^i x(i)$$

for all $x \in V$. Explicitly, $H$ is the set of matrices

$$A_{ij} = (\varepsilon^{ai+b} \cdot \delta_{i,\, j+c})$$

and has order 125. An an algebraic group, $H$ is defined over $\mathbb{Q}$, but it only splits over $\mathbb{Q}(\varepsilon)$. The Galois group $\Theta$ of $\mathbb{Q}(\varepsilon)$ over $\mathbb{Q}$ acts on $H$. Let $\theta \in \Theta$ be the generator given by $\theta(\varepsilon) = \varepsilon^2$ (so that $\theta^2 =$ complex conjugation). We shall sometimes use the notation $'$ to indicate the action of $\theta$. The group $H$ has center $C$ equal to $\boldsymbol{\mu}_5 \cdot I_V$ and is a central extension:

$$1 \to \boldsymbol{\mu}_5 \to H \to \mathbb{Z}_5 \times \mathbb{Z}_5 \to 1, \tag{1.1}$$

where $\sigma$, $\tau$ in $H$ are mapped to $(1, 0)$, $(0, 1)$. The action of $\Theta$ preserves this sequence and $\theta$ acts on $\mathbb{Z}_5 \times \mathbb{Z}_5$ by $(n, m) \to (n, 2m)$.

$V$ is clearly an irreducible $H$-module, and it gives rise to three more by the action of $\Theta$: let $V_i$ be the representation obtained from $V$ by composing $H \to \mathrm{Aut}\, V$ with $\theta^i$. The trace $\langle h, V_i\rangle$ of an element $h \in H$ on $V_i$ is given by:

$$\langle \varepsilon^r I_V, V_i\rangle = 5 \cdot \varepsilon^{2^i r}, \qquad \langle h, V_i\rangle = 0 \qquad (h \in H - C). \tag{1.2}$$

It follows that the four representations $V_i$ are inequivalent. These plus the 25 characters of $\mathbb{Z}_5 \times \mathbb{Z}_5$ exhaust the irreducible representations of $H$ since the sum of squares of their degrees is 125, the order of $H$.

Let $\phi\colon \mathbb{Z}_5 \times \mathbb{Z}_5 \to H$ be the section of (1.1) given by:

$$\phi(m, n) = \varepsilon^{2mn}\sigma^m\tau^n,$$

and define $\omega\colon \boldsymbol{\mu}_5 \times (\mathbb{Z}_5 \times \mathbb{Z}_5) \to H$ by:

$$\omega(\alpha, z) = \alpha \cdot \phi(z).$$

Then $\omega$ is bijective and the group law on $H$ goes over to the law of composition:

$$(a, z)\cdot(a', z') = (aa'B(z, z'), z + z')$$

where

$$B(m, n; m', n') = \varepsilon^{3(mn'-m'n)}$$

is a $\boldsymbol{\mu}_5$-valued skew-symmetric form on $\mathbb{Z}_5 \times \mathbb{Z}_5$. Note that all automorphisms of $H$ preserve the sequence (1.1) and since $B(z, z')^2 \cdot I_V$ is the commutator of $(a, z)$ and $(a', z')$, they preserve the form $B$.

Let $N$ be the normalizer of $H$ in $SL_5(\mathbb{C})$. Each element of $N$ induces by conjugation an automorphism of $H$, hence an automorphism of $\mathbb{Z}_5 \times \mathbb{Z}_5$ preserving $B$. But the group of such automorphisms is isomorphic to $SL_2(\mathbb{Z}_5)$, hence we get a homomorphism:

$$\alpha: N \to SL_2(\mathbb{Z}_5).$$

The kernel of $\alpha$ is just $H$ itself because (a) any automorphism of $H$ which is the identity on $C$ and on $H/C$ is in fact inner, and (b) since the representation $V$ is irreducible, $C$ is the centralizer of $H$ in $SL_2(\mathbb{Z}_5)$. Moreover $\alpha$ is surjective. If $x \in SL_2(\mathbb{Z}_5)$, define $\gamma_x$: $H \to H$ by

$$\gamma_x \omega(a, z) = \omega(a, x(z)).$$

Since $x$ preserves $B$, the mapping $\gamma_x$ is an automorphism. The new representation of $H$ on $V$ obtained by composing with $\gamma_x$ is equivalent to $V$ since $\gamma_x$ is the identity on $C$ and so leaves the character fixed. So $x$ is induced by an element of $N$, in fact an element of $N \cap SL_5(\mathbb{Q}(\varepsilon))$. Thus $\alpha$ is surjective and $N \subset SL_5(\mathbb{Q}(\varepsilon))$, hence $\Theta$ acts on $N$ and the action of $N$ on $V$ induces actions on each $V_i$.

Since $\gamma_x \cdot \gamma_y = \gamma_{xy}$ and $\gamma_x$ is induced by a member of $N$ determined up to multiplication by elements of $C$, it follows that $N/C$ is a semi-direct product $(H/C)\cdot SL_2(\mathbb{Z}_5)$. Let $X$ be the inverse image in $N$ of the factor $SL_2(\mathbb{Z}_5)$. Then $X$ is a central extension of $SL_2(\mathbb{Z}_5)$ by $C$. But the group of Schur multipliers $H^2(SL_2(\mathbb{Z}_5), \mathbb{C}^*)$ is zero [5, p. 645], hence $X$ is a product $C\cdot SL_2(\mathbb{Z}_5)$ and the full group $N$ is a semi-direct product $H\cdot SL_2(\mathbb{Z}_5)$.

Next, look at the dual representation $V_i^*$ of $V_i$: since $N$ acts on each $V_i$ by unitary representations, $V_i^*$ is isomorphic as $N$-module to the complex conjugate $\overline{V}_i$, i.e. to $V_{i+2}$.

Finally, look at the representation of $N$ in $V_i \otimes V_i^*$. $C$ acts trivially here so we have a representation of $N/C$. For all $x \in V_i$, $l \in V_i^*$, put

$$F_{x\otimes l}(h) = l(hx).$$

This gives a map

$$F: V_i \otimes V_i^* \to \mathrm{Map}(H, \mathbb{C})$$

which is easily seen to be injective, with image the space $W_i$ of functions $f$ on $H$ such that

$$f(\alpha h) = \alpha^{2^i}\cdot f(h), \qquad \alpha \in \boldsymbol{\mu}_5 I_V.$$

Moreover, for every $n \in N$, let $n$ induce by conjugation the automorphism $n^*$ of $H$. Then

$$\begin{aligned} F_{n(x\otimes l)}(h) &= F_{nx\otimes l\cdot n^{-1}}(h) \\ &= l\cdot n^{-1}(hnx) \\ &= l(n^*(h)x) \\ &= F_{x\otimes l}(n^*h) \end{aligned}$$

so $F$ transforms the cation of $n$ on $V_i \otimes V_i^*$ to the cation $f \mapsto f \cdot n^*$ on $\mathbb{C}$-valued functions on $H$. Now it is easy to check that if

$$\tilde{f}(\omega(\alpha, z)) = \alpha^{2i} \cdot f(\omega(\alpha, 2z)),$$

then $f \mapsto \tilde{f}$ is an isomorphism of $W_i$ and $W_{i+1}$ commuting with the action of $N$: therefore the four representations $V_i \otimes V_i^*$ are all equivalent, so we may as well work with $V \otimes V^*$. This space has a decomposition $\mathbb{C} \oplus Z$ where $Z$ is the subspace of trace zero. One sees immediately that as an $H/C$-module $Z$ is the sum of the 24 non-trivial (linear) characters of $H/C$. Since $N$ acts transitively on these, $Z$ is irreducible as $N/C$-module. Its character $\zeta$ has values in $\mathbb{Q}$ since $Z$ is equivalent to all its conjugates.

To summarize our conclusions, we have found groups:

$$\begin{array}{ccccccc}
\text{order } 5 & & \text{order } 125 & & \text{order } 15{,}000 & & \\
C & \subset & H & \subset & N & \subset & SL_5(\mathbb{Q}(\varepsilon)). \\
& \underbrace{\hphantom{C \subset H}}_{H/C} & & & & & \\
\wr\Vert & \wr\Vert & & & \wr\Vert & & \\
\boldsymbol{\mu}_5 & \mathbb{Z}_5 \times \mathbb{Z}_5 & & & H \cdot SL_2(\mathbb{Z}_5) & &
\end{array} \tag{1.3.}$$

It is not hard to work out the explicit matrices representing elements of $N$. They turn out to be of two types:

$$A_{ij} = \pm \frac{\varepsilon^{ai^2+bij+cj^2+di+ej+f}}{\sqrt{5}} \qquad (a, \ldots, f \in \mathbb{Z}_5,\, b \neq 0)$$

and

$$A_{ij} = \pm\, \varepsilon^{ai^2+bi+c} \delta_{i,dj+e} \qquad (a, \ldots, e \in \mathbb{Z}_5,\, d \neq 0)$$

(the sign being adjusted to make the determinant $+1$). It will be necessary to identify some special elements of $N$ of the second type for the purpose of computation. Look at the elements $\iota, \mu, \nu \in SL_5(\mathbb{Q}(\varepsilon))$ given by

$$\iota x(i) = x(-i)$$
$$\mu x(i) = -x(2i)$$
$$\nu x(i) = \varepsilon^{i^2} \cdot x(i)$$

$$\begin{aligned}
&\langle \iota, V_i \rangle = 1 \\
&\langle \mu, V_i \rangle = -1,\ \langle \mu^2, V_i \rangle = 1 \\
&\langle \nu, V_i \rangle = \theta^i(\eta - \eta'),\ \langle \nu^2, V_i \rangle = \theta^i(\eta' - \eta)
\end{aligned} \tag{1.4}$$

where $\eta = \varepsilon + \varepsilon^4$, $\eta' = \varepsilon^2 + \varepsilon^3$. Conjugating $\sigma$ and $\tau$, we find

$$\begin{aligned}
\iota^{-1}\sigma\iota &= \sigma^{-1}, & \iota^{-1}\tau\iota &= \tau^{-1} \\
\mu^{-1}\sigma\mu &= \sigma^2, & \mu^{-1}\tau\mu &= \tau^3 \\
\nu^{-1}\sigma\nu &= \sigma\tau^2 \bmod C, & \nu^{-1}\tau\nu &= \tau.
\end{aligned}$$

Thus $\iota$, $\mu$, $\nu \in N$ and their images $\bar{\iota}$, $\bar{\mu}$, $\bar{\nu}$ in $\mathrm{SL}_2(\mathbb{Z}_5)$ are:

$$\begin{aligned} \bar{\iota} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \bar{\mu} &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ \bar{\nu} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1.5}$$

## §2. THE BUNDLE $\mathscr{F}$

Let $\mathbb{P}$ be the projective space representing the one-dimensional subspaces of $V^i$. Since $V$ is given an underlying rational vector space, $\mathbb{P}$ is to be regarded as the complexification of a scheme over $\mathbb{Q}$. In particular it is meaningful to speak of coherent sheaves and their homomorphisms as being defined over specified subfields of $\mathbb{C}$.

Write $\mathcal{O}$ for the sheaf of local rings of $\mathbb{P}$ and $\mathcal{O}(1)$ for the canonical positive invertible sheaf on $\mathbb{P}$. The general linear group acts on $\mathcal{O}(1)$ and the space of sections $\Gamma(\mathcal{O}(1))$ is canonically isomorphic to $V^*$ the dual of $V$. Regard $V$ as a sheaf over Spec $\mathbb{C}$. The external tensor product $\mathcal{O}(1) \otimes_{\mathbb{C}} V$ is a sheaf on $\mathbb{P}$ and $\Gamma(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(V, V)$. Let $\partial$ in $\Gamma(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ correspond to $I_V$. The Koszul complex $\mathscr{K}$ is the exterior algebra $\Lambda^*(\mathcal{O}(1) \otimes_{\mathbb{C}} V)$ with multiplication by $\partial$ as differential:

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to \mathcal{O}(2) \otimes \Lambda^2 V \to \mathcal{O}(3) \otimes \Lambda^3 V \to \mathcal{O}(4) \otimes \Lambda^4 V \to \mathcal{O}(5) \otimes \Lambda^5 V \to 0.$$

The quotient $\mathcal{O}(1) \otimes V/\mathcal{O}$ is isomorphic to the tangent sheaf $\mathscr{T}$ to $\mathbb{P}$ and the sheaf of cycles $\mathrm{Im}(\mathcal{O}(i) \otimes \Lambda^i V) \subset \mathcal{O}(i+1) \otimes \Lambda^{i+1} V$ is isomorphic to the $i$th exterior power $\Lambda^i \mathscr{T}$ of $\mathscr{T}$. For the construction of the bundle $\mathscr{F}$ the relevant part of $\mathscr{K}$ is:

$$\overbrace{\mathcal{O}(2) \otimes_{\mathbb{C}} \Lambda^2 V}^{rk\ 10} \xrightarrow{p_0} \overbrace{\Lambda^2 \mathscr{T}}^{rk\ 6} \xrightarrow{q_0} \overbrace{\mathcal{O}(3) \otimes_{\mathbb{C}} \Lambda^3 V}^{rk\ 10}. \tag{2.1}$$

Note also that $\mathscr{K}$ has a symmetric pairing

$$\mathscr{K}^i \otimes \mathscr{K}^{5-i} \to \mathcal{O}(5) \otimes_{\mathbb{C}} \Lambda^5 V \cong \mathcal{O}(5),$$

given by $x \otimes y \to (x \wedge y)_5$, and that this induces the natural pairing $\Lambda^i \mathscr{T} \otimes \Lambda^{4-i} \mathscr{T} \to \mathcal{O}(5)$ and is compatible with the action of $SL_5(\mathbb{C})$. Note that with respect to these pairings $q_0 = p_0{}^*(5)$.

The $H$-modules $\Lambda^2 V$ and $2V_1$ are isomorphic since $\langle \varepsilon I_V, \Lambda^2 V \rangle = 10\varepsilon^2$ and by (1.2) $2V_1$ is the only representation of degree 10 for which this is possible. In identifying $\Lambda^2 V$ and other such spaces as $N$-modules, the reader should use the general observation:

(2.2) *Let $Y$, $Z$ be representation spaces for a group $G$ and let $K$ be a normal subgroup of $G$. Suppose that $Y$ is irreducible as a $K$-module and that $Z \cong nY$ as $K$-modules, then* $\mathrm{Hom}_K(Y, Z)$ *is a $G/K$-module, the evaluation mapping $Y \otimes \mathrm{Hom}_K(Y, Z) \to Z$ is an isomorphism of $G$-modules, and $Z$ is irreducible as a $G$-module if and only if* $\mathrm{Hom}_K(Y, Z)$ *is irreducible as a $G/K$-module.*

In the present case, put $W = \mathrm{Hom}_H(V_1, \Lambda^2 V)$. It is a representation of $N/H$ of degree 2, and the trace of $\bar{v}$ (the image of $v$ in $N/H$) is

$$\langle \bar{v}, W \rangle = (3 + \varepsilon + \varepsilon^4)/(1 + 2\varepsilon^2 + 2\varepsilon^3) = \eta'.$$

It follows that $W$ has character $\chi_2$ (cf. character table in Appendix).

Since $N/H$ is perfect, $W$ is unimodular. So $W$ has an invariant skew symmetric pairing defined over $\mathbb{Q}$ and this form is unique up to a scale factor. Let

$$f\colon V_1 \to \Lambda^2 V \otimes W$$

be the $N$-homomorphism determined by this form, and let

$$g\colon \Lambda^3 V \otimes W \to V_3 (= V_1{}^*)$$

be the dual of $f$ composed with the canonical mapping $\Lambda^3 V \otimes W \cong \Lambda^3 V \otimes W^*$. Combining these with (2.1) gives the sequence of sheaf homomorphisms

$$\begin{aligned} \mathcal{O}(2) \otimes V_1 \xrightarrow{1_{\mathcal{O}} \otimes f} \mathcal{O}(2) \otimes \Lambda^2 V \otimes W \xrightarrow{p_0 \otimes 1_W} \Lambda^2 \mathcal{T} \otimes W \xrightarrow{q_0 + 1_W} \\ \mathcal{O}(3) \otimes \Lambda^3 V \otimes W \xrightarrow{1_{\mathcal{O}} \otimes g} \mathcal{O}(3) \otimes V_3 . \end{aligned} \tag{2.3}$$

This sequence is defined over $\mathbb{Q}$.

Let

$$p\colon \overbrace{\mathcal{O}(2) \otimes V_1}^{rk\ 5} \to \overbrace{\Lambda^2 \mathcal{T} \otimes W}^{rk\ 12}, \qquad q\colon \overbrace{\Lambda^2 \mathcal{T} \otimes W}^{rk\ 12} \to \overbrace{\mathcal{O}(3) \otimes V_3}^{rk\ 5}$$

be the composites of the first two and last two morphisms in 2.3. Note that $q \cong p^*(5)$. We shall prove that $qp = 0$ and $p, q$ are locally split. From this it follows that $\mathcal{F} = \mathrm{Ker}\, q/\mathrm{Im}\, p$ is locally free of rank 2 and defined over $\mathbb{Q}$. The bundle $\mathcal{F}$ is our goal.

To prove that $qp = 0$ it is sufficient, since $\mathcal{O} \otimes V_1$ is generated by its sections, to show that $\Gamma(qp(-2)) = 0$ and to prove that $p, q$ are locally split it is sufficient that $p$ is — for then $q$ is locally split by duality. The first assertion follows immediately from

LEMMA 2.4. *Let $U$ be the symmetric square representation $S^2 W$ of degree 3, and let $W'$ be the representation obtained by acting on $W$ with the Galois automorphism $\theta$.*

(i) *$\Gamma(\Lambda^2\mathcal{T}(-2)) \cong \Lambda^2 V \cong V_1 \otimes W$, $\Gamma(\Lambda^2\mathcal{T}(-2) \otimes W) \cong V_1 \oplus (V_1 \otimes U)$, and $\Gamma(p(-2))$ is equivalent to the inclusion $V_1 \to V_1 \oplus (V_1 \otimes U)$.*

(ii) *$\Gamma(\mathcal{O}(1) \otimes V_3) \cong (V_1 \otimes U) \oplus (V_1 \otimes W')$ and $\Gamma(q(-2))$ is equivalent to the homomorphism $V_1 \oplus (V_1 \otimes U) \to (V_1 \otimes U) \oplus (V_1 \otimes W')$ induced by the identity on $V_1 \otimes U$.*

*Proof.* (i) The first isomorphism follows from (2.1), and the second is just the evaluation mapping. The third isomorphism now follows from the decomposition $W \otimes W = \mathbb{C} \oplus U$. Finally, since $\Gamma(\Lambda^2\mathcal{T}(-2)) \cong \Lambda^2 V$, the mapping $\Gamma(p(-2))$ is equivalent to $f$ which is just the mapping induced by $\mathbb{C} \to \mathbb{C} \oplus U$.

(ii) First note that $\Gamma(\mathcal{O}(1) \otimes V_3) \cong V^* \otimes V_3$. The character of $V^* \cong V_3$ as an $H$-module is given by

$$\langle h, V^* \otimes V_3 \rangle = 0 (h \in H - C), \quad \langle \varepsilon^i 1, V^* \otimes V_3 \rangle = 25\varepsilon^{2i}.$$

So $V^* \otimes V_3 \cong 5V_1$ as an $H$-module. Put $X = \operatorname{Hom}_H(V_1, V^* \otimes V_3)$. The formulas (1.4) show that

$$\langle \bar{\mu}, X \rangle = -1, \langle \bar{\mu}^2, X \rangle = 1, \langle \bar{\nu}, X \rangle = \eta - \eta'.$$

The first of these shows that $X$ must have an irreducible component $X^1$ of degree 3, the second that the remaining component $X^2$ is irreducible of degree 2, and the third that $X^1$, $X^2$ have characters $\chi_3$, $\chi_2'$. Since $\chi_2^2 = \chi_3 + \chi_1$ and $\chi_2' = \theta\chi_2$ it follows that $X^1 \cong U$ and $X^2 \cong W'$.

To prove the statement about $\Gamma q(-2)$ it is sufficient to show that $\Gamma q(-2) \neq 0$, for $V_1$, $V_1 \otimes U$, $V_1 \otimes W$ are irreducible. But $q \cong p^*(5)$ and $p \neq 0$ by (i). So $q \neq 0$, and $\Gamma q(-2) \neq 0$ since $\Gamma(\mathcal{O}(1) \otimes V_3)$ generates $\mathcal{O}(1) \otimes V_3$. *Q.E.D.*

It remains to be proven that $p(-2)$ and hence $p$ splits locally. Let $v$ be a non-zero element of $V$ and write $\hat{v}$ for the corresponding point of $\mathbb{P}$. We must show that the induced map on the vector bundle fibres:

$$p(-2)_{\hat{v}}: V_1 \to \Lambda^2 T_{\hat{v}} \otimes \mathcal{O}(-2)_{\hat{v}} \otimes W$$

is injective ($T_{\hat{v}}$ = tangent space to $\mathbb{P}$ at $\hat{v}$). But via $p_o$

$$\Lambda^2 T_{\hat{v}} \otimes \mathcal{O}(-2)_{\hat{v}} \cong \Lambda^{2+}/v \wedge V,$$

so in view of (2.3), the injectivity of $p(-2)_{\hat{v}}$ is equivalent to:

LEMMA 2.5. *For all nonzero $v \in V$ and $t \in V_1$, the element $f(t) \notin (v \wedge V) \otimes W$.*

*Proof.* Let $v_i$ be the element of $V$ defined by $v_i(j) = \delta_{ij}$ and put

$$z_0^+ = v_2 \wedge v_3, z_1^+ = v_3 \wedge v_4, z_2^+ = v_4 \wedge v_0, z_3^+ = v_0 \wedge v_1, z_4^+ = v_1 \wedge v_2$$
$$z_0^- = v_1 \wedge v_4, z_1^- = v_2 \wedge v_0, z_2^- = v_3 \wedge v_1, z_3^- = v_4 \wedge v_2, z_4^- = v_0 \wedge v_3.$$

The linear mappings $w^+: V_1 \to \Lambda^2 V$, $w^-: V_1 \to \Lambda^2 V$ defined by $w^+(v_i) = z_i^+$, $w^-(v_i) = z_i^-$ are $H$-homomorphisms, and form a base for $W$. We wish to show that for non-zero $v \in V$, $t \in V_1$, the equations

$$w^+(t) = v \wedge y^+, w^-(t) = v \wedge y^-$$

are contradictory. But these imply that

$$w^+(t) \wedge w^-(t) = 0,$$

and if $t = \sum a_i v_i$, then one computes that

$$w^+(t) \wedge w^-(t) = \sum_{i=0}^{4} (-1)^i a_i^2 v_0 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_4$$

hence $t = 0$. *Q.E.D.*

This completes the proof that $\mathcal{F}$ is a locally free sheaf of rank 2. To show that $\mathcal{F}$ is indecomposable it is sufficient to verify that its total Chern class $c(\mathcal{F})$ is irreducible. By definition

$$c(\mathcal{F}) = c(\Lambda^2 \mathcal{T})^2 \cdot c(\mathcal{O}(2))^{-5} \cdot c(\mathcal{O}(3))^{-5}.$$

Let $h$ be the positive generator for the Chow ring of $\mathbb{P}$. It follows that

$$c(\mathcal{F}) = ((1+2h)^{10}(1+h)^{-5})^2(1+2h)^{-5}(1+3h)^{-5}$$

(where the first factor comes from the resolution of $\Lambda^2\mathcal{T}$ given by the Koszul complex). Hence

$$c(\mathcal{F}) = 1 + 5h + 10h^2.$$

Applying the Riemann–Roch theorem or directly from the definition of $\mathcal{F}$, one also computes the Hilbert polynomial:

$$\chi(\mathcal{F}(n-5)) = \tfrac{1}{12}(n^2-1)(n^2-24).$$

## §3. THE INVARIANT QUINTICS

This section is preliminary to the computation of $\Gamma(\mathcal{F})$ and the proof of the non-singularity of the zero set of a general section. The main results are the determination of the $N/H$-module $\Gamma_H(\mathcal{O}(5))$ of $H$-invariants of $\Gamma(\mathcal{O}(5))$ and the sheaf of ideals $\mathcal{L}$ in $\mathcal{O}$ generated by the subspace $\Gamma_H(\mathcal{O}(5))$. In the next section we show that this subspace is isomorphic to the second exterior power of $\Gamma(\mathcal{F})$, however the present section does not depend on this fact.

Write $a_i$ for the character of $\Lambda^i V$ as an $N$-module, $h_i$ for the character of $S^i V$, and $a_i^*$, $h_i^*$ for the characters of the duals. Then

$$\begin{aligned} a_i^* &= \bar{a}_i = \theta^2 a_i \\ h_i^* &= \bar{h}_i = \theta^2 h_i \end{aligned} \tag{3.1}$$

since the representations are unitary. Also

$$a_i^* = a_{5-i} \tag{3.2}$$

since the representation is unimodular. As in §1, decompose $V \otimes V^*$ into $\mathbb{C} \oplus Z$ and let $\zeta$ be the character of $Z$. It follows that $h_1 \cdot \theta^2 h_1 = \theta h_1 \cdot \theta^3 h_1 = 1 + \zeta$.

LEMMA 3.3. (i) $a_1 = h_1$

(ii) $a_2 = \chi_2 \cdot \theta h_1$

(iii) $a_3 = \chi_2 \cdot \theta^3 h_1$

(iv) $a_4 = \theta^2 h_1$

(v) $h_2 = \chi_3' \cdot \theta h_1$

(vi) $h_3 = (\chi_5 + \chi_2') \cdot \theta^3 h_1$

(vii) $h_4 = (\chi_4 + \chi_4^{\#} + \chi_3 + \chi_3') \cdot \theta^2 h_1$

(viii) $h_5 = (\chi_3 + \chi_3') + \zeta \cdot (\chi_3 + \chi_3' - 1)$

(ix) $h_1 \theta h_1 = (\chi_3 + \chi_2') \cdot \theta^3 h_1$.

*Proof.* (ii) follows from Lemma (2.4) and then (i), (iii) and (iv) follows from (3.1) and (3.2). To prove ($v$), note that (2.2) implies $h_2 = \chi \cdot \theta h_1$ for some character $\chi$ of $N/H$. A simple computation shows that $\chi(\iota) = 3$ and $\chi(\nu) = -\eta$. So since $\chi$ nas degree 3, $\chi = \chi_3'$. Now use the well-known formula:

$$h_i = a_1 h_{i-1} - a_2 h_{i-2} + \cdots - (-1)^i a_i h_0,$$

plus the identity (ix) proven in (2.4) and (vi), (vii) and (viii) follow by computing characters via the character table in the Appendix. *Q.E.D.*

The first of the main results of this section follows at once from part (viii) of this lemma:

THEOREM 3.5. *The character of* $\Gamma_H(\mathcal{O}(5))$ *is* $\chi_3 + \chi_3'$ *and its dimension is* 6.

Let $y_i$ be the $i$th coordinate function on $V$, $(y_i(x) = x(i))$. The monomials

$$y_0^5,\ y_0^3 y_1 y_4,\ y_0^3 y_2 y_3,\ y_0^2 y_2^2 y_1,\ y_0^2 y_1^2 y_3,\ \prod_{i=0}^{4} y_i,$$

are invariants of $\tau$, and the six forms

$$S = \sum y_i^5,\ Q,\ Q',\ R,\ R',\ Y = 5\prod y_i$$

obtained by summing these monomials over the powers of $\sigma$ are invariants of $H$. Since they are linearly independent they form a base for $\Gamma_H(\mathcal{O}(5))$.

Another natural basis of $\Gamma_H(\mathcal{O}(5))$ is obtained as follows: the group $H/C$ has six proper subgroups and these subgroups are permuted triply transitively by $N$. The fixed point set in $\mathbb{P}$ of the subgroup $\{C, \tau C, \tau^2 C, \tau^3 C, \tau^4 C\}$ is just the simplex of reference. The six simplexes determined in this way by the six subgroups we call the *fundamental simplexes*. Each of them determines, up to a scalar multiple, the quintic whose zero set consists of five three-dimensional faces of the simplex. The subspace of $\Gamma_H(\mathcal{O}(5))$ that these quintics span is invariant under both $N$ and $\Theta$. So Theorem 3.5 shows that these six quintics also form a base for $\Gamma_H(\mathcal{O}(5))$.

Now let $L$ be the set of common zeros of the polynomials in $\Gamma_H(\mathcal{O}(5))$, and let $L_i$ be its intersection with $y_i = 0$. But

$$\begin{aligned}
S(0, y_1, \ldots, y_4) &= y_1^5 + y_2^5 + y_3^5 + y_4^5 \\
Q(0, y_1, \ldots, y_4) &= y_2 y_3 (y_2^2 y_1 + y_3^2 y_4) \\
Q'(0, y_1, \ldots, y_4) &= y_1 y_4 (y_1^2 y_3 + y_4^2 y_2) \\
R(0, y_1, \ldots, y_4) &= y_2 y_3 (y_1^2 y_3 + y_4^2 y_2) \\
R'(0, y_1, \ldots, y_4) &= y_1 y_4 (y_2^2 y_1 + y_3^2 y_4).
\end{aligned}$$

These equations define the set $L_0$ and it is straightforward to check that it consists of precisely the five lines

$$y_2 + \varepsilon^r y_3 = \varepsilon^{2r} y_1 + y_4 = 0 \qquad (*)$$

plus the 20 points

$$\begin{aligned}
y_1 = y_2 = 0,\ y_3 &= \varepsilon^r y_4 \\
y_1 = y_3 = 0,\ y_2 &= \varepsilon^r y_4 \\
y_4 = y_2 = 0,\ y_3 &= \varepsilon^r y_1 \\
y_4 = y_3 = 0,\ y_2 &= \varepsilon^r y_1.
\end{aligned}$$

The set of points of $L_i$ is just $\sigma^i L_0$, the same set with $y_i = 0$ after a cyclic permutation of the coordinates. Taking the union over all $i$, it follows that the set $L$ consists of the 25 skew lines:

$$y_i = y_{i+2} = \varepsilon^r y_{i+3} = \varepsilon^{2r} y_{i+1} + y_{i+4} = 0, \qquad (0 \leqq i, r \leqq 4). \tag{3.6}$$

We claim that the *scheme* $\mathcal{O}/\mathcal{L}$ is this set of 25 skew lines with reduced structure sheaf, hence is a regular scheme. If $x \in L$ lies on only one face of each of the fundamental simplices, then the ideal $\mathcal{L}_x$ is defined by six linear forms in the $y_i$'s, so $\sqrt{\mathcal{L}_x} = \mathcal{L}_x$. On the other hand, say $x \in L$ lies on a 2-dimensional face of at least one fundamental simplex. Then $x$ is necessarily on a 1-dimensional face or edge of this simplex (as you see by intersecting the line (3.6) with a typical 2-dimensional face $y_0 = y_1 = 0$ on the simplex of reference). But the edges of two distinct fundamental simplices do not intersect: in fact $N$ permutes the fundamental simplices triply transitively and one may readily calculate that the edges of $\prod_i y_i = 0$ and $\prod_i (y_0 + \varepsilon^i y_1 + \varepsilon^{2i} y_2 + \varepsilon^{3i} y_3 + \varepsilon^{4i} y_4) = 0$ do not intersect. Therefore $x$ is singular on at most one fundamental simplex. This shows that $\mathcal{L}_x$ is generated by five linear forms and one of higher degree. If these five linear forms met in a plane, $L$ would contain more than one line in this plane, i.e. $L$ would have two components that met. Since this is false, $\mathcal{L}_x$ is generated by the five linear forms and $\sqrt{\mathcal{L}_x} = \mathcal{L}_x$ again.

## §4. THE SPACES $H^i(\mathcal{F}(n))$

Let $\psi^i(n)$ be the character of $H^i(\mathcal{F}(n))$ as a representation of $N$. To determine these characters we write $\mathcal{G}$ for the cokernel of $p$ (cf. §2) and consider the exact sequences

$$\begin{gathered} 0 \to \mathcal{O}(2) \otimes V_1 \to \Lambda^2 \mathcal{T} \otimes W \to \mathcal{G} \to 0 \\ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{O}(3) \otimes V_3 \to 0. \end{gathered} \tag{4.1}$$

Using the well-known values for the cohomology of $\mathcal{O}(n)$ and $\Lambda^2(\mathcal{T}(n))$ (which is just $\Omega^2(5+n)$, $\Omega^i$ being the shear of $i$-forms on $\mathbb{P}$) we get

$$\begin{aligned} &0 \to \Gamma(\mathcal{O}(n+2)) \otimes V_1 \to \Gamma(\Lambda^2 \mathcal{T}(n)) \otimes W^n \to \Gamma(\mathcal{G}(n)) \to 0; \\ &H^1(\mathcal{G}(n)) = (0); \\ &H^2(\mathcal{G}(n)) = (0) \text{ if } n \neq -5,\ H^2(\mathcal{G}(-5)) \cong W; \\ &0 \to \Gamma(\mathcal{F}(n)) \to \Gamma(\mathcal{G}(n)) \to \Gamma(\mathcal{O}(n+3)) \otimes V_3 \to H^1(\mathcal{F}(n)) \to 0; \\ &\psi^2(n) = 0 \text{ if } n \neq -5,\ \psi^2(-5) = \chi_2. \end{aligned} \tag{4.2}$$

Now since $\Lambda^2 \mathcal{F} \cong \mathcal{O}(5) \cong \Omega^4(10)$, Serre duality asserts that $H^i(\mathcal{F}(n))$ and $H^{4-i}(\mathcal{F}(-10-n))$ are dual. We deduce using (4.2)

$$\begin{aligned} \psi^1(n) = 0 \quad \text{if} \quad n \leqq -4, \\ \psi^3(n) = 0 \quad \text{if} \quad n \geqq -6. \end{aligned} \tag{4.3}$$

Since $\Gamma(\Lambda^2 \mathcal{T}(-3)) = (0)$ we further deduce from (4.2)

$$\psi^1(-3) = \theta^3 h_1,\ \psi^3(-7) = \theta h_1. \tag{4.4}$$

From Lemma 2.4 (ii) the image of $\Gamma(\mathcal{G}(-2))$ in $\Gamma(\mathcal{O}(1)) \otimes V_3$ has dimension 15. But the first exact sequence of (4.2) shows that the dimension of $\Gamma\mathcal{G}(-2)$ is 15. So from the second exact sequence and Serre duality we deduce that

$$\begin{aligned} \psi^0(n) &= 0 \quad \text{if} \quad n \leqq -2 \\ \psi^4(n) &= 0 \quad \text{if} \quad n \geqq -8. \end{aligned} \tag{4.5}$$

We can now calculate $\psi^1(-2)$ from the exact sequences of (4.2) together with (4.5) and we find

$$\psi^1(-2) = \chi_2' \cdot \theta h_1, \psi^3(-8) = \chi_2' \cdot \theta^3 h_1. \tag{4.6}$$

Consider now the characters $\psi^i(j)$ for $i = 0, 1$ and $j = 0, -1$. The exact sequences of (4.2) together with Lemma 3.3 give

$$\begin{aligned} \psi^0(-1) - \psi^1(-1) &= -\chi_2' \cdot h_1, \\ \psi^0(0) - \psi^1(0) &= \chi_4 - \chi_2', \end{aligned} \tag{4.7}$$

and in particular it follows that

$$\psi^0(0) \geqq \chi_4, \psi^1(0) \geqq \chi_2', \tag{4.8}$$

where the inequality means that the difference between the two sides is the character of a representation. Also, since $\Gamma(\mathscr{F}(n))$ is a subspace of $\Gamma(\mathscr{G}(n))$, the exact sequences show that

$$\begin{aligned} \psi^0(-1) &\leqq (\chi_4^{\#} + \chi_4 + \chi_5)h_1, \\ \psi^0(0) &\leqq \chi_4 + \chi_5 + (\chi_4 + \chi_5 - \chi_2)\zeta. \end{aligned} \tag{4.9}$$

Since $\Lambda^2 \mathscr{F} \cong \mathscr{O}(5)$ there are homomorphisms

$$\lambda(-1): \Lambda^2\Gamma(\mathscr{F}(-1)) \to \Gamma(\mathscr{O}(3)), \lambda: \Lambda^2\Gamma(\mathscr{F}) \to \Gamma\mathscr{O}(5).$$

LEMMA 4.10 *Let $\mathscr{S}$ be a locally free sheaf of rank 2 on $\mathbb{P}$ such that $\Gamma(\mathscr{S}(-1)) = 0$, let $\gamma: \Lambda^2\Gamma(\mathscr{S}) \to \Gamma\Lambda^2(\mathscr{S})$ be the canonical homomorphism, and let $A$ be any subspace of $\Gamma(\mathscr{S})$. Then*

$$\dim \gamma(\Lambda^2 A) \geqq 2 \dim A - 3.$$

*Proof.* Since the Grassman cone in $\Lambda^2 A$ has dimension $2 \dim A - 3$ it is sufficient to show that the only element of this cone in Ker $\gamma$ is the zero.

Suppose that $\gamma(s \wedge t) = 0$ for some $s$, $t$ in $A$. Assume $s$, $t$ are not both zero, then they generate a subsheaf $\mathscr{L}$ of $\mathscr{S}$ with rank 1. Since $\mathscr{L}$ is torsion-free its bidual is an invertible sheaf. So $s$, $t$ are contained in a subsheaf isomorphic to $\mathscr{O}(r)$ for some $r$. As $\Gamma(\mathscr{S}(-1)) = 0$, it follows that $r \leqq 0$. Hence $s$, $t$ are proportional and $s \wedge t = 0$. *Q.E.D*

First consider $\lambda(-1)$. Suppose that $\psi^0(-1) \neq 0$. Since the terms of the first inequality of (4.9) are irreducible characters with degrees at least 20, it follows that $\dim \Gamma(\mathscr{F}(-1)) \geqq 20$ and so by the lemma $\dim \Gamma(\mathscr{O}(3)) \geqq 37$. Since $\dim \Gamma\mathscr{O}(3) = 35$ it follows that

$$\psi^0(-1) = 0, \psi^1(-1) = \chi_2' \cdot h_1. \tag{4.11}$$

Now consider $\lambda$. Take $A$ to be the subspace $\Gamma_H(\mathscr{F})$ of $H$-invariant sections. From Theorem 3.5, $\dim \Gamma_H(\mathscr{O}(5)) = 6$. So the lemma shows that $\dim A < 5$. Together with (4.8) and (4.9) this shows that the character of $\Gamma_H(\mathscr{F})$ is $\chi_4$. Applying the lemma again shows that $\dim \lambda(A) \geqq 5$. So since $\Gamma_H(\mathscr{O}(5))$ has character $\chi_3 + \chi_3'$ it follows that $\lambda$ is an isomorphism from $\Lambda^2\Gamma_H(\mathscr{F})$ to $\Gamma_H(\mathscr{O}(5))$.

We claim that in fact $\Gamma_H(\mathscr{F}) = \Gamma(\mathscr{F})$. If this is not true, then as a representation of

$H$, the space $\Gamma(\mathscr{F})$ must contain all the non-trivial characters of $H\ C$ at least once (by (4.9)), hence $\dim \Gamma(\mathscr{F}) \geqq 28$. Let $A$, $B \subseteq \mathbf{P}^4$ be two hyperplanes with homogeneous equations $a$, $b$ and consider the exact sequence

$$0 \to \mathscr{F}(-2) \xrightarrow{(a,b)} \mathscr{F}(-1) \oplus \mathscr{F}(-1) \xrightarrow{(b,-a)} \mathscr{F} \to \mathscr{F}_{A \cdot B} \to 0.$$

We find, since $\Gamma(\mathscr{F}(-1)) = 0$,

$$\dim \Gamma(\mathscr{F}_{A \cdot B}) \geqq \dim \Gamma(\mathscr{F}) - \dim \operatorname{Ker}[(a, b) \text{ on } H^1(\mathscr{F}(-2))].$$

But from (4.2), $H^1(\mathscr{F}(-1))$ is generated by $H^0(\mathscr{O}(1)) \otimes H^1(\mathscr{F}(-2))$. Note that $h^1(\mathscr{F}(-2)) = 10$, $h^1(\mathscr{F}(-1)) = 10$ (by (4.6), (4.11)). So if $a$, $b$ are sufficiently generic the image $a \cdot H^1(\mathscr{F}(-2)) + b \cdot H^1(\mathscr{F}(-2))$ in $H^1(\mathscr{F}(-1))$ has dimension at least 4. Therefore $\dim \Gamma \mathscr{F}_{A \cdot B} \geqq 28 - 10 + 4 = 22$. But let $s$ be a non-zero section of $\mathscr{F}$. Its zero set $X_s$ is a surface (since $\Gamma \mathscr{F}(-1) = 0$) and non-empty (since $c_2(\mathscr{F}) \neq 0$), so we can choose $A$, $B$ so that $A \cdot B \cdot X_s$ is a non-empty finite set of points. Then $\mathscr{F}_{A \cdot B}/s\mathscr{O}_{A \cdot B}$ is a torsion-free rank 1 sheaf on $A \cdot B$, and hence isomorphic to $\mathscr{J} \cdot \mathscr{O}_{A \cdot B}(n)$ for some sheaf of ideals $\mathscr{J}$ defining $A \cdot B \cdot X_s$. Computing Chern classes we find $n = 5$. So, since $A \cdot B \cdot X_s$ is non-empty, $\dim \Gamma(\mathscr{F}_{A \cdot B}/s\mathscr{O}_{A \cdot B}) < 21$, and finally

$$21 \leqq \dim \Gamma(\mathscr{F}_{A \cdot B}) - 1 \leqq \dim \Gamma(\mathscr{F}_{A \cdot B}/s\mathscr{O}_{A \cdot B}) < 21,$$

which is a contradiction. So, taking account of (4.7), we have

$$\psi^0(0) = \chi_4, \ \psi^1(0) = \chi_2'. \tag{4.12}$$

Finally we claim $\psi^1(n) = 0$ if $n \geqq 1$. By Castelnuovo's lemma [11], it suffices to prove that $\psi^1(1) = 0$. By (4.2) the cup product $\alpha: \Gamma(\mathscr{O}(1)) \otimes H^1(\mathscr{F}) \to H^1(\mathscr{F}(1))$ is surjective. On the other hand $N$ acts irreducibly on $\Gamma(\mathscr{O}(1)) \otimes H^1(\mathscr{F})$ by (2.2). Therefore either $\psi^1(1) = 0$ or $\alpha$ is an isomorphism. But $\Gamma(\mathscr{O}(1)) \otimes H^1(\mathscr{F}(-1)) \to H^1(\mathscr{F})$ is also surjective so for some $a \in \Gamma(\mathscr{O}(1))$, $\sigma \in H^1(\mathscr{F}(-1))$, it follows that $a \cup \sigma \neq 0$. Since $\dim \Gamma(\mathscr{O}(1)) > \dim H^1(\mathscr{F})$, it follows that $b \cup \sigma = 0$ for some other non-zero $b$. Therefore $\alpha(b, a \cup \sigma) = 0$ and $\alpha$ is not injective.

We summarize our calculations as follows:

TABLE OF $\dim H^i(\mathscr{F}(n-5))$

| $n$ | $H^0$ | $H^1$ | $H^2$ | $H^3$ | $H^4$ |
|---|---|---|---|---|---|
| $n \geqq 6$ | $\frac{(n^2-1)(n^4-24)}{12}$ | 0 | 0 | 0 | 0 |
| 5 | 4 | 2 | 0 | 0 | 0 |
| 4 | 0 | 10 | 0 | 0 | 0 |
| 3 | 0 | 10 | 0 | 0 | 0 |
| 2 | 0 | 5 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 2 | 0 | 0 |
| $-1$ | 0 | 0 | 0 | 0 | 0 |
| $-2$ | 0 | 0 | 0 | 5 | 0 |
| $-3$ | 0 | 0 | 0 | 10 | 0 |
| $-4$ | 0 | 0 | 0 | 10 | 0 |
| $-5$ | 0 | 0 | 0 | 2 | 4 |
| $n \leqq -6$ | 0 | 0 | 0 | 0 | $\frac{(n^2-1)(n^2-24)}{12}$ |

## §5. THE ZERO SETS $X_s$, $s \in \Gamma(\mathscr{F})$

THEOREM 5.1. *For almost all $s \in \Gamma(\mathscr{F})$, the zero set of $s$ is a non-singular surface $X_s \subset \mathbb{P}$ of degree 10; when $X_s$ is nonsingular, it is an abelian surface.*

*Proof.* Let $Q$ be the projective space associated to $\Gamma(\mathscr{F})$, and let $Z$ be the subvariety of $Q \times \mathbb{P}$ represented by pairs $(s, x)$ $(s \in \Gamma(\mathscr{F}),\ x \in \mathbb{P})$ such that $s(x) = 0$. Since $\Lambda^2\Gamma(\mathscr{F}) \cong \Gamma_H(\mathcal{O}(5))$ the sheaf $\mathscr{F}$ is generated by $\Gamma(\mathscr{F})$ except at points of the set of 25 skew lines $L$ whose ideal is generated by $\Gamma_H(\mathcal{O}(5))$ (see §3). It follows that $Z$ is a fibre-bundle over $\mathbb{P} - L$, in particular it is non-singular over $\mathbb{P} - L$. Applying Sard's theorem [15] to the projection $Z \to Q$ shows that the zero variety $X_s = \{x \mid s(x) = 0\}$ of a general section $s$ is a surface that is non-singular except possibly at points of $L$.

Let $x \in L$ and let $e_1$, $e_2$ be a basis of the free rank two $\mathcal{O}_x$-module $\mathscr{F}_x$. Each $s \in \psi(\mathscr{F})$ can be written

$$s = s_1 e_1 + s_2 e_2, \qquad s_i \in \mathcal{O}_x,$$

so that if $s, t \in \Gamma(\mathscr{F})$,

$$s \wedge t = (s_1 t_2 - s_2 t_1) e_1 \wedge e_2 .$$

If for every $s \in \Gamma(\mathscr{F})$, $s_1(x) = s_2(x) = 0$, then for every $s$ and $t$, $s \wedge t$ would vanish to second order at $x$. Using again the fact that $\Lambda^2\Gamma(\mathscr{F}) \cong \Gamma_H(\mathcal{O}(5))$ and that $\Gamma_H(\mathcal{O}(5))$ generates the ideal of $L$, this is impossible. We may therefore choose $e_1$, $e_2$ so that $e_1$ is an element of $\Gamma(\mathscr{F})$. Write out a basis of $\Gamma(\mathscr{F})$ locally:

$$\begin{aligned} s &= e_1 \\ t &= f e_1 + u e_2 \\ t' &= f' e_1 + u' e_2 \qquad \text{where } f, f', f'', u, u', u'' \in \mathcal{O}_x . \\ t'' &= f'' e_1 + u'' e_2 \end{aligned}$$

Then

$$\begin{aligned} s \wedge t &= u \cdot e_1 \wedge e_2 \\ s \wedge t' &= u' \cdot e_1 \wedge e_2 \\ s \wedge t'' &= u'' \cdot e_1 \wedge e_2 \end{aligned}$$

and $t^{(i)} \wedge t^{(j)}$ vanishes at $x$ to 2nd order.

Therefore $u$, $u'$, $u''$ must generate the ideal of $L$ at $x$, i.e. their differentials are independent at $x$. But if $\lambda s + \mu t + \mu' t' + \mu'' t''$ is a general section in $\Gamma(\mathscr{F})$, so that $(\lambda, \mu, \mu', \mu'')$ are homogeneous coordinates in $Q$, then $Z$ is described above points near $x$ by the equations:

$$\begin{aligned} \lambda &= -(\mu f + \mu' f' + \mu'' f'') \\ 0 &= \mu u + \mu' u' + \mu'' u'' \end{aligned}$$

which are easily seen to define a non-singular subvariety of $Q \times \mathbb{P}$. Thus $Z$ is everywhere non-singular, hence by Sard's theorem so is the set $X_s$ of zeros of a generic section $s$ of $\mathscr{F}$.

To prove that $X(= X_s)$ is abelian of degree 10 note that its normal bundle $N$ in $\mathbb{P}$ is isomorphic to $\mathscr{F} \otimes \mathcal{O}_X$. So the Chern class $c(N)$ of $N$ (in the Chow ring of $X$) is just the restriction to $X$ of $1 + 5h + 10h^2$. Since $\mathbb{P}$ has Chern class $1 + 5h + 10h^2 + \cdots$, the Chern class of $X$ is 1. So the canonical class $K_X$ is zero and the Euler characteristic $c_2(X)$ is zero. This characterizes abelian surfaces [8, §6]. Since $c_2(N)$ is just the self intersection of $X$, the degree of $X$ is 10. *Q.E.D*

THEOREM 5.2. *Every abelian surface $Z \subset \mathbb{P}$ is projectively equivalent to the zero set of some section $s$ of $\mathscr{F}$.*

*Proof.* Let $\mathscr{L} = \mathscr{O}(1) \otimes \mathscr{O}_Z$. Since the Chern class of $Z$ is 1 that of its normal bundle is the restriction of $1 + 5h + 10h^2$. As above it follows that $Z$ has degree 10. Choose an origin on $Z$ and let $H(\mathscr{L})$ be the subgroup

$$\{z \in Z \mid T_z{}^* \mathscr{L} \cong \mathscr{L}\}$$

where $T_z$ is just the translation by $z$ (cf. [13, §13]). Since $H(\mathscr{L})$ has order $(\deg Z/2)^2$ and carries a non-degenerate alternating form,

$$H(\mathscr{L}) \cong \mathbb{Z}_5 \times \mathbb{Z}_5.$$

Further the Riemann–Roch theorem for abelian varieties (*ibid.*) shows that $\dim \Gamma(\mathscr{L}) = 5$, and Lefschetz's theorem implies that $Z$ cannot lie in a subspace of $\mathbb{P}$ (otherwise $Z$ would be simply-connected). So the mapping

$$\phi : \Gamma \mathscr{O}(1) \to \Gamma(\mathscr{L})$$

is necessarily an isomorphism. Applying the results of [12, §1] it follows that when $Z$ is embedded in $\mathbb{P}^4$ by the complete linear system $\Gamma(\mathscr{L})$ and a suitable isomorphism is chosen between $\mathbb{P}^4$ of §2 then $Z$ is invariant under the action of the Heisenberg group introduced in §1. But since $\phi$ is an isomorphism this is just the composition of our given embedding and a projective transformation, i.e. after a projective transformation we may assume that $Z$ is invariant under $H$. Actually we can go a bit further: if we choose an origin $O$ in $Z$ with respect to which $\mathscr{L}$ is symmetric then the map $x \mapsto -x$ for this origin extends to a projective transformation $\iota_0$ of $\mathbb{P}$, leaving $Z$ fixed, normalizing the action of $H/C$ and so that $\iota_0 \cdot \eta \cdot \iota_0{}^{-1} = \eta^{-1}$ for $\eta \in H/C$. Therefore $\iota_0$ must be induced by the element $\iota$ of $N$ introduced in §1, and $Z$ is invariant under $H$ and $\iota$.

Next, look at the natural map:

$$\psi : \Gamma_H(\mathbb{P}, \mathscr{O}(5)) \to \Gamma_H(Z, \mathscr{L}^5).$$

The group $H(\mathscr{L})$ acts on the line bundle $\mathscr{L}^5$, hence there is a line bundle $\mathscr{M}$ on $Y = Z/H(\mathscr{L})$ such that $\pi^*\mathscr{M} \cong \mathscr{L}^5$ ($\pi : Z \to Y$ the natural homomorphism). Then $\Gamma(Y, \mathscr{M}) \cong \Gamma_H(Z, \mathscr{L}^5)$ and $\deg \mathscr{M} = \deg \mathscr{L}^5/\deg \pi = 5$, so $\dim \Gamma(Y, \mathscr{M}) = 5$. In fact, under the symmetry $x \mapsto -x$, the space $\Gamma(Y, \mathscr{M})$ breaks up into the sum of an eigenspace of dimension 3 and one of dimension 2 (cf. [12, §2]; note that the action of $x \mapsto -x$ on $\Gamma(Y, \mathscr{M})$ is only well determined up to sign, so we have no obvious way of labelling one eigenspace "even" and the other "odd"). Since $\iota$ is the identity on $\Gamma_H(\mathbb{P}, \mathscr{O}(5))$, the image of $\psi$ is contained in one of these eigenspaces. Therefore $\dim \ker(\psi) \geqq 3$, *i.e.* at least three independent quintics of §3 contain $Z$.

Consider the map

$$\Lambda^2\Gamma(\mathscr{F}) \xrightarrow{\approx} \Gamma_H(\mathcal{O}(5)).$$

We have proven that there is a subspace $K \subset \Lambda^2\Gamma(\mathscr{F})$ of dim $\geq 3$ consisting of elements that are mapped to zero in $\Gamma(\Lambda^2(\mathscr{F} \otimes \mathcal{O}_Z))$. It follows with a little linear algebra that there are two possibilities:

for some basis $s_1, s_2, s_3, s_4$ of $\Gamma(\mathscr{F})$, either

(a) $s_1 \wedge s_2, s_3 \wedge s_4, s_1 \wedge s_3 - s_2 \wedge s_4 \in K$, or

(b) $s_1 \wedge s_2, s_1 \wedge s_3$ (and a 3rd independent elt.) $\in K$.

Now if $s \wedge t \in K$, and $\bar{s}$, $\bar{t}$ are the restrictions of $s$, $t$ to $Z$, then $\bar{s} = f \cdot \bar{t}$ for some $f \in \mathbb{C}(Z)$. Therefore in case (a),

$$\bar{s}_1 = f \cdot \bar{s}_2 \qquad \bar{s}_4 = f \cdot \bar{s}_3.$$

Let $D$ be the divisors of poles of $f$ and let $\mathscr{M} = \mathcal{O}_Z(D)$. Define

$$\alpha\colon \mathscr{M} + \mathscr{M} \to \mathscr{F} \otimes \mathcal{O}_Z \quad \text{by} \quad (g_1, g_2) \mapsto g_1\bar{s}_2 + g_2\bar{s}_3.$$

Then all four sections $\bar{s}_i$ of $\mathscr{F} \otimes \mathcal{O}_Z$ are images by $\alpha$ of sections of $\mathscr{M} + \mathscr{M}$ (i.e. $\alpha(1, 0)$, $\alpha(0, 1)$, $\alpha(f, 0)$ and $\alpha(0, f)$). But the $s_i$ generate $\mathscr{F}$ everywhere except at the 25 lines $L$. Since $Z$ is abelian, none of these lines is contained in $Z$, hence the $\bar{s}_i$ generate $\mathscr{F} \otimes \mathcal{O}_Z$ at all but a finite set of points. But $\alpha$ is a homomorphism of rank 2 bundles, so the support of its cokernel is defined by the principal ideal (det $\alpha$), and has codimension 1. Therefore $\alpha$ must be an isomorphism. But then $\mathscr{M}^2 \cong \Lambda^2\mathscr{F} \otimes \mathcal{O}_Z \cong \mathscr{D}^5$, hence

$$4(D^2) = 25 \cdot c_1(\mathscr{D})^2 = 25 \cdot \deg Z = 250,$$

contradiction.

In case (b), either $\bar{s}_2 = f \cdot \bar{s}_1$, $\bar{s}_3 = g \cdot \bar{s}_1, f, g \in \mathbb{C}(Z)$ or $\bar{s}_1 = 0$. In the 1st case, as above, we get a homomorphism:

$$\alpha\colon \mathscr{M} \to \mathscr{F} \otimes \mathcal{O}_Z,$$

with three out of the four $\bar{s}_i$'s in the image $\alpha\Gamma(\mathscr{M})$. Then $\bar{s}_4$ generates the cokernel except at a finite set of points:

$$0 \to \mathcal{O}_Z \xrightarrow{\bar{s}_4} \mathscr{F} \otimes \mathcal{O}_Z/\alpha\mathscr{M} \to \underset{\text{finite support}}{\mathscr{G}} \to 0.$$

By elementary homological algebra, extensions of $\mathscr{G}$ by a line bundle split. Using this twice we find $\mathscr{G}$ must be zero, and we have

$$0 \to \mathscr{M} \xrightarrow{\alpha} \mathscr{F} \otimes \mathcal{O}_Z \to \mathcal{O}_Z \to 0,$$

hence $c_2(\mathscr{F} \otimes \mathcal{O}_Z) = Z \cdot c_2(\mathscr{F}) = 0$, which is absurd. Thus $\bar{s}_1 = 0$, i.e. $Z \subset$ zeroes of $s_1$. Since $\deg Z = 10 = \deg X_{s_1}$, it follows that $Z = X_{s_1}$. *Q.E.D.*

## §6. CONNECTIONS WITH MODULI

The bundle $\mathscr{F}$ can be used to give an explicit representation of a certain moduli space for 2-dimensional abelian varieties. We first recall some standard results in the theory of moduli of abelian varieties:

(a) Let $g \geq 1$ be the dimension;

(b) Let $\mathfrak{H}_g$ = Siegel upper $\frac{1}{2}$-space of $g \times g$ symmetric matrices $\Omega$, $\operatorname{Im}\Omega > 0$,

$$\cong \mathrm{Sp}(2g, \mathbb{R})/\text{maximum compact } K;$$

(c) Fix a sequence $\delta$ of positive integers $\delta_1, \ldots, \delta_g$ such that $\delta_i$ divides $\delta_{i+1}$;

(d) Sp $(2g, \mathbb{Q})$ acts on $\mathbb{Q}^{2g}$, fixing the form

$$A(e_i, e_{g+j}) = \delta_{ij},\ A(e_i, e_j) = A(e_{g+i}, e_{g+j}) = 0, \qquad 1 \leq i, j \leq g;$$

(e) Let $L_\delta$ = the sublattice $\mathbb{Z}^g \times \prod_{i=1}^{g} \delta_i \mathbb{Z}$ of $\mathbb{Z}^{2g}$,

$L_\delta^\perp$ = the lattice $\prod_{i=1}^{g}(1/\delta_i)\mathbb{Z} \times \mathbb{Z}^g$, characterized as the set of $x \in \mathbb{Q}^{2g}$ such that $A(x, y) \in \mathbb{Z}$, all $y \in L_\delta$;

(f) On $L_\delta^\perp/L_\delta$, put the multiplicative symplectic form

$$e_\delta(x, y) = e^{2\pi i A(x, y)}.$$

(g) Let $\Gamma(\delta_1, \ldots, \delta_g)_0 = \{X \in \mathrm{Sp}(2g, \mathbb{Q}) \mid X(L_\delta) = L_\delta\}$;

$\Gamma(\delta_1, \ldots, \delta_g) = \{X \in \mathrm{Sp}(2g, \mathbb{Q}) \mid X(L_\delta) = L_\delta \text{ and } X = \text{id. on } L_\delta^\perp/L_\delta\}$;

(h) Then the analytic quotient spaces have the significance

$$\mathfrak{U}_\delta^{(0)} \underset{\text{def}}{=} \mathfrak{H}_g/\Gamma(\delta_1, \ldots, \delta_g)_0 = \left\{\begin{array}{l}\text{moduli space of pairs } (X, \lambda),\ X \text{ a} \\ g\text{-dimensional abelian variety, } \lambda\text{:} \\ X \to \hat{X} \text{ a polarization such that} \\ \qquad \ker(\lambda) \cong \prod_{i=1}^{g} (\mathbb{Z}/\delta_i\mathbb{Z})^2.\end{array}\right\}$$

$$\mathfrak{U}_\delta \underset{\text{def}}{=} \mathfrak{H}_g/\Gamma(\delta_1, \ldots, \delta_g) = \left\{\begin{array}{l}\text{moduli space of triples } (X, \lambda, \alpha), \\ (X, \lambda) \text{ as above, and} \\ \qquad \alpha\colon \ker(\lambda) \xrightarrow{\cong} L_\delta^\perp/L_\delta \\ \text{a symplectic isomorphism with} \\ \text{respect to } e_\lambda \text{ and } e_\delta.\end{array}\right\}$$

(i) $\mathfrak{U}_\delta$ and $\mathfrak{U}_\delta^{(0)}$ have natural structures of quasi-projective varieties;

(j) Note that the finite "symplectic" group $\Gamma(\delta)_0/\Gamma(\delta)$ acts on $\mathfrak{U}_\delta$ and $\mathfrak{U}_\delta^{(0)}$ is the quotient $\mathfrak{U}_\delta/[\Gamma(\delta)_0/\Gamma(\delta)]$.

Now if $\lambda : X \to \hat{X}$ is a polarization, let $L_\lambda$ denote one of the corresponding invertible sheaves—all such are isomorphic after a translation. The result can now be stated:

THEOREM 6.1. *Let*

$$\mathfrak{U}^*_{(5,1)} = \left\{\begin{array}{l}\textit{the Zariski-open set of points of } \mathfrak{U}_{(5,1)} \\ \textit{corresponding to triples } (X, \lambda, \alpha) \textit{ such} \\ \textit{that } L_\lambda \textit{ is very ample}\end{array}\right\}.$$

*Let*

$$\mathbb{P}(\Gamma(\mathscr{F}))^* = \left\{\begin{array}{l}\textit{the Zariski-open subset of } \mathbb{P}(\Gamma(\mathscr{F})) \textit{ of spaces} \\ \textit{of sections } \mathbb{C}\cdot s, \textit{ whose zero sets } X_s \textit{ are non-} \\ \textit{singular}\end{array}\right\}.$$

*Then* $\mathfrak{U}^*_{(5,1)} \cong \mathbb{P}(\Gamma(\mathscr{F}))^*$, *the action of* $\Gamma(5,1)_0/\Gamma(5,1) \cong SL_2(\mathbb{Z}_5)$ *on* $\mathfrak{U}_{(5,1)}$ *corresponding to the action of* $N/H \cong SL_2(\mathbb{Z}_5)$ *on* $\mathbb{P}(\Gamma(\mathscr{F}))$.

*Proof.* The idea is to set up a set-theoretic map from $\mathbb{P}(\Gamma(\mathscr{F}))^*$ to $\mathfrak{U}^*_{(5,1)}$; verify that it is a morphism and is bijective; and apply Zariski's Main Theorem. To define the map, start with a one-dimensional subspace $\mathbb{C}\cdot s \subset \Gamma(\mathscr{F})$. This determines uniquely its zero-set $X_s$. This variety carries a line bundle, $\mathcal{O}_{X_s}(1)$, and is invariant under the group $H/C$. Strictly speaking, $X_s$ is not yet an abelian variety, since it has no distinguished origin. We can either choose any point $x \in X_s$ as origin, or if we wish to be canonical, replace $X_s$ by its "double dual":

$$X_s' = \mathrm{Pic}^0(\mathrm{Pic}^0 X_s),$$

$$(\mathrm{Pic}^0 = \text{connected component of Grothendieck's Picard scheme}).$$

In this case, $X_s$ is canonically a principal homogeneous space over $X_s'$. In both cases, $\mathcal{O}_{X_s}(1)$ induces a polarization $\lambda$ on $X_s$ (or $X_s'$). And the automorphisms induced by $H/C$ are the translations by the points of $\ker(\lambda)$, so we get an isomorphism

$$\alpha\colon \ker(\lambda) \xrightarrow{\approx} H/C \underset{\text{def}}{=} \mathbb{Z}_5 \times \mathbb{Z}_5 = L^{\perp}_{(5,1)}/L_{(5,1)}.$$

This is a point of $\mathfrak{U}^*_{(5,1)}$. The fact that this is a morphism comes from checking that the above construction can be carried out universally leading to an abelian scheme $\mathfrak{X}$ over $\mathbb{P}(\Gamma(\mathscr{F}))^*$, plus a polarization $\Lambda\colon \mathfrak{X} \to \hat{\mathfrak{X}}$ plus an isomorphism of $\ker(\Lambda)$ with the constant group scheme $\mathbb{Z}_5 \times \mathbb{Z}_5$. This induces a morphism from $\mathbb{P}(\Gamma(\mathscr{F}))^*$ to $\mathfrak{U}^*_{(5,1)}$ by the universal property characterizing coarse moduli spaces (cf. [10, p. 96]). To check that this map is injective, say $\mathbb{C}\cdot s_1$ and $\mathbb{C}\cdot s_2$ lead to isomorphic triples $(X, \lambda, \alpha)$. It follows that there is an isomorphism

$$\phi\colon X_{s_1} \to X_{s_2}$$

such that $\phi^*\mathcal{O}_{X,s_2}(1)$ is algebraically equivalent to $\mathcal{O}_{X,s_1}(1)$ and such that for all $\sigma \in H/C$, if $\sigma$ induces on $X_{s_i}$ translation by $x_i \in X_{s_i}$, then $\phi T_{x_1} = T_{x_2}\phi$. But then changing $\phi$ by a translation, we can assume that $\phi^*(\mathcal{O}_{X,s_2}(1)) \cong \mathcal{O}_{X,s_1}(1)$, hence $\phi$ is the restriction to $X_{s_1}$ of a projective transformation $\tau$. Moreover $\tau$ satisfies $\sigma\tau \equiv \tau\sigma$ on $X_{s_1}$, all $\sigma \in H/C$, hence $\tau\sigma = \sigma\tau$ in $PGL_5(\mathbb{C})$. But $H/C$ is its own centralizer so $\tau \in H/C$. Therefore $X_{s_2} = \tau(X_{s_1}) = X_{s_1}$, hence $\mathbb{C}\cdot s_1 = \mathbb{C}\cdot s_2$. Finally surjectivity follows from (5.2). *Q.E.D.*

A natural question is to analyze how the isomorphism above goes wrong outside the open sets *. We have not worked this out completely, but we state without proof two pretty facts about this:

(a) If the abelian variety $X$ tends to $E_1 \times E_2$, $E_i$ an elliptic curve, so that the polarization tends to $\lambda = 5\lambda_1 + \lambda_2$ ($\lambda_i\colon E_i \xrightarrow{\approx} \hat{E}_i$ the canonical isomorphism), then $\lambda$ is *not* very ample. In fact $L_\lambda$ has a fixed component $F$ and defines the morphism

$$X - F \xrightarrow{p_1} E_1 \xrightarrow{\phi_1} \mathbb{P}^4$$

where $\phi_1$ is the morphism defined by $L_{\lambda_1}$. Let $C_1 = \phi_1(E_1)$, an elliptic quintic curve. Then while $X$ approaches $E_1 \times E_2$, the corresponding section $s$ of $\mathscr{F}$ has a well-defined limit $s_0$

and $X_{s_0}$ is the singular ruled surface with $C_1$ as cuspidal double curve equal to the union of the tangent lines to $C_1$. It follows that at the points $(E_1 \times E_2, 5\lambda_1 + \lambda_2, \alpha) \in \mathfrak{U}_{(5,1)}$ the correspondence with $\mathbb{P}(\Gamma(\mathscr{F}))$ given in (6.1) is still regular, but not biregular, since the image does not depend on $E_2$.

(b) Suppose we compactify $\mathfrak{U}_{(5,1)}$ following Igusa [6] [i.e. take his compactification of $\mathfrak{U}_{(1,1)}$ and normalize it in $\mathbb{C}(\mathfrak{U}_{(5,1)})$ via any of the canonicalmorphisms $\mathfrak{U}_{(5,1)} \to \mathfrak{U}_{(1,1)}$]. Then at some of the points at $\infty$ lying even on the 0-dimensional piece of Satake's compactification, the correspondence remains biregular. The corresponding $X_s$'s depend on one parameter $\alpha \in \mathbb{C} - (0)$ and are unions of five non-singular quadric surfaces as follows:

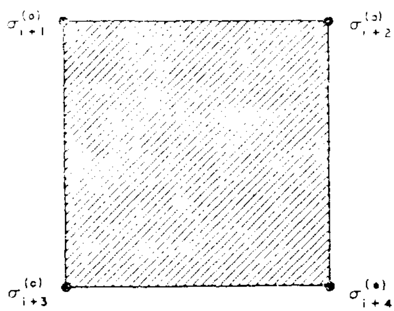
$$X_s = Q_0 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$$
$$Q_i = (\text{locus } Y_i = \alpha Y_{2+i} Y_{3+i} + Y_{1+i} Y_{4+i} = 0).$$

The 10 lines $Y_i = Y_j = Y_k = 0$ $(0 \leqq i < j < k \leqq 4)$ are double lines on $X_s$, and the five points $P_i$ given by $Y_j(P_i) = \delta_{ij}$ are 4-fold points of $X_s$. The whole configuration is readily visualized if you form a $CW$-complex $\Sigma$ as follows:

(a) take a point $\sigma_i^{(0)}$ for each point $P_i$;

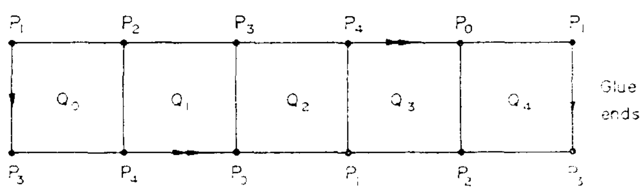
(b) joint $\sigma_i^{(0)}$ and $\sigma_j^{(0)}$ by a 1-simplex $\sigma_{ij}^{(1)}$ corresponding to the double line $\overline{P_i P_j}$, for each $i < j$;

(c) glue in a square $\sigma_i^{(2)}$ corresponding to $Q_i$ filling in the loop




for each $0 \leqq i \leqq 4$ (read the subscripts mod 5).

Then a point or line is on a line or a quadric in $\mathbb{P}$ if and only if the corresponding 0-simplex or 1-simplex is on the corresponding 1-simplex or square in $\Sigma$. The nice thing is that the $\Sigma$ you get is homeomorphic to a 2-dimensional real torus:




Glue top and bottom with horizontal shift

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## APPENDIX

*The character table of* $SL_2(\mathbb{Z}_5)$ [2, p. 160]

Put $\varepsilon = \exp 2\pi\sqrt{-1}/5$, $\eta = \varepsilon + \varepsilon^4$, $\eta' = \varepsilon^2 + \varepsilon^3$, and let $\omega$ be a primitive root of $x^6 = 1$ over $\mathbb{Z}_5$.

| | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ | $\bar{\mu}$ $\begin{pmatrix}2&0\\0&3\end{pmatrix}$ | $\begin{pmatrix}\omega^2&0\\0&\omega^{-2}\end{pmatrix}$ | $\begin{pmatrix}\omega&0\\0&\omega^{-1}\end{pmatrix}$ | $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ | $\bar{\nu}$ $\begin{pmatrix}1&2\\0&1\end{pmatrix}$ | $\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$ | $\begin{pmatrix}-1&2\\0&-1\end{pmatrix}$ | Symbols used in text for these representations |
|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $I$ |
| $\chi_5$ | 5 | 5 | 1 | $-1$ | $-1$ | 0 | 0 | 0 | 0 | |
| $\chi_6$ | 6 | $-6$ | 0 | 0 | 0 | 1 | 1 | $-1$ | $-1$ | |
| $\chi_4$ | 4 | 4 | 0 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | |
| $\chi_4{}^*$ | 4 | $-4$ | 0 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | |
| $\chi_3$ | 3 | 3 | $-1$ | 0 | 0 | $-\eta$ | $-\eta'$ | $-\eta$ | $-\eta'$ | $U$ |
| $\chi_3'$ | 3 | 3 | $-1$ | 0 | 0 | $-\eta'$ | $-\eta$ | $-\eta'$ | $-\eta$ | $U$ |
| $\chi_2$ | 2 | $-2$ | 0 | $-1$ | $+1$ | $\eta$ | $\eta'$ | $-\eta$ | $-\eta'$ | $W$ |
| $\chi_2'$ | 2 | $-2$ | 0 | $-1$ | $+1$ | $\eta'$ | $\eta$ | $-\eta'$ | $-\eta$ | $W'$ |