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Characteristic polynomials of graphs having a semifree action

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Abstract

J.H. Kwak and J. Lee (Linear and Multilinear Algebra 32 (1992) 61–73) computed the characteristic polynomial of a finite graph G having an abelian automorphism group which acts freely on G. For a finite weighted symmetric pseudograph G having an abelian automorphism group which acts semifreely on G, K. Wang (Linear Algebra Appl. 51 (1983) 121–125) showed that the characteristic polynomial of G is factorized into a product of a polynomial associated to the orbit graph and a polynomial associated to the free part of the action. But he did not explicitly compute the characteristic polynomial of such a graph G. In this paper, we introduce a new method to construct a finite pseudograph G having an automorphism group which acts semifreely on G, and obtain an explicit formula to compute the characteristic polynomial of such a graph by using the construction method. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let G be a finite connected undirected pseudograph with vertex set $V(G)$ and edge set $E(G)$, and let D be a finite connected pseudo-digraph with vertex set $V(D)$ and directed edge set $E(D)$, where a connected pseudo-digraph is a digraph whose

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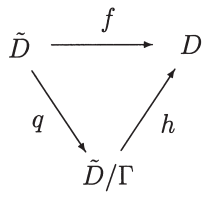
underlying graph is a connected pseudograph. Let $A(G)$ and $A(D)$ denote the adjacency matrices of the undirected graph G and the digraph D, respectively. We also denote by $\Phi(G;\lambda)$ and $\Phi(D;\lambda)$ the characteristic polynomials $\det(\lambda I - A(G))$ and $\det(\lambda I - A(D))$, respectively (see [1]). A digraph D is *symmetric* if $A(D)$ is symmetric. By $|X|$, we denote the cardinality of a finite set X. We say that G admits a Γ-action if there is a group homomorphism from Γ to $\mathrm{Aut}(G)$. For each $v \in V(G)$, let $\Gamma_v = \{\gamma \in \Gamma \mid \gamma(v) = v\}$ be the isotropy subgroup of v, and $\mathrm{Fix}_\Gamma = \{v \in V(G) \mid \Gamma_v = \Gamma\}$. We call Fix_Γ the *fixed part* of $V(G)$. We say that Γ acts semifreely on G if for each $v \in V(G)$, Γ_v is either the trivial group or the full group Γ, and for each $e \in E(\langle \mathrm{Fix}_\Gamma \rangle)$, $\gamma(e) = e$ for all $\gamma \in \Gamma$, where $\langle \mathrm{Fix}_\Gamma \rangle$ is the subgraph induced by the fixed part Fix_Γ. In [6], Wang defined that Γ acts freely on G if Γ acts *semifreely* on G and $\mathrm{Fix}_\Gamma = \emptyset$. Notice that even if Γ acts semifreely on G and $\mathrm{Fix}_\Gamma = \emptyset$, there might exist an edge e in $E(G)$ such that $\gamma(e) = e$ for some non-identity element γ in Γ. In this paper, we say that Γ acts freely on G if Γ acts freely on both $V(G)$ and $E(G)$. We use the same terminology when Γ acts on a digraph D. Notice that if a digraph D has no loops, then Γ acts freely on a digraph D according to Wang's definition if and only if Γ acts freely on a digraph D according to our definition.

A digraph $\tilde{D}$ is called a *covering graph* of D if there exists a direction preserving map $f : \tilde{D} \to D$ with the following properties: $f|_{V(\tilde{D})} : V(\tilde{D}) \to V(D)$ and $f|_{E(\tilde{D})} : E(\tilde{D}) \to E(D)$ are surjective and for each $\tilde{v} \in V(\tilde{D})$, f maps the set of edges originating at $\tilde{v}$ one-to-one onto the set of edges originating at $f(\tilde{v})$, and f maps the set of edges terminating at $\tilde{v}$ one-to-one onto the set of edges terminating at $f(\tilde{v})$. We call such a map $f : \tilde{D} \to D$ a *covering* and D the *base graph*. A covering $f : \tilde{D} \to D$ is *regular* if there exists a group Γ of graph automorphisms of $\tilde{D}$ acting freely on $\tilde{D}$ and a graph isomorphism $h : \tilde{D}/\Gamma \to D$ such that the diagram



commutes, i.e., $h \circ q = f$, where q is the quotient map. Convert the graph G to a digraph $\vec{G}$ by replacing each edge e of G with a pair of oppositely directed edges, say e^+ and e^-. We then say that the digraph $\vec{G}$ is *associated* with G. By e^{-1} we mean the reverse edge to an edge $e \in E(\vec{G})$. We denote the directed edge e of G by uv if the initial and the terminal vertices of e are u and v, respectively. Note that the adjacency matrix of graph G is the same as that of digraph $\vec{G}$, i.e., $A(G) = A(\vec{G})$ (see Fig. 1).

Notice that Γ acts semifreely on G with $\mathrm{Fix}_\Gamma = \emptyset$ iff Γ acts freely on $\vec{G}$. We say that a graph $\tilde{G}$ is a covering of G if $\vec{\tilde{G}}$ is a covering of $\vec{G}$ as digraphs. Moreover, $\vec{G}$ is always symmetric. It is clear that the complete graph K_2 is not a covering of any smaller graph. But $\vec{K}_2$ can be presented as a covering of a directed loop with one vertex (see Fig. 2).

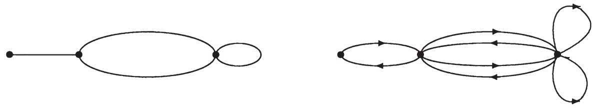


Fig. 1. G and $\vec{G}$.

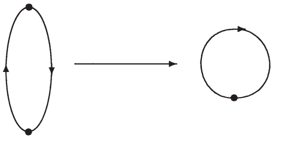


Fig. 2. $\vec{K}_2$ covers a directed loop.

Let Γ be a finite group. A Γ-voltage assignment on G is a function $\Phi : E(\vec{G}) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all e in $E(\vec{G})$. The *derived* graph $G \times_\phi \Gamma$, derived by a Γ-voltage assignment ϕ, has $V(G) \times \Gamma$ as its vertex set and $E(G) \times \Gamma$ as its edge set, where (e, g) joins from (u, g) to $(v, \phi(e)g)$ if $e = uv \in E(\vec{G})$. For convenience, a vertex (u, g) is denoted by u_g and an edge (e, g) by e_g. The voltage group Γ acts naturally on $G \times_\phi \Gamma$ as follows: for every $g \in \Gamma$, let $\Phi_g : G \times_\phi \Gamma \to G \times_\phi \Gamma$ denote the graph automorphism defined by $\Phi_g(v_{g'}) = v_{g'g^{-1}}$ on vertices and $\Phi_g(e_{g'}) = e_{g'g^{-1}}$ on edges. Then the natural map $p : G \times_\phi \Gamma \to G \times_\phi \Gamma/\Gamma \simeq G$ is a $|\Gamma|$-fold regular covering. Gross and Tucker [3] showed that every regular covering of G arises from a voltage assignment on G. Similarly, we can show that every regular covering of a digraph D can be constructed by the same method.

In this paper, we introduce a new method to construct a finite pseudograph G which admits a semifree Γ-action, and obtain an explicit formula to compute the characteristic polynomial of such a graph by using the construction method. The previous works on this direction can be found in [2,4,5].

2. A construction of a Γ-graph

Throughout this paper, by a *Γ-graph G* (resp. D) we mean a graph (resp. digraph D) which admits a semifree Γ-action. In this section, we introduce a method to construct a Γ-graph. Let D be a pseudo-digraph. For a subset S of $V(D)$, we denote by $\langle S\rangle$ the subgraph of D induced by S, and for a pair of subsets S_1 and S_2 of $V(D)$, by $E(S_1, S_2)$ the set of all directed edges $e = uv$ such that $u \in S_1$ and $v \in S_2$. Then, for a subset S of $V(D)$, $E(D) = E(\bar{S}, \bar{S}) \cup E(S, S) \cup E(\bar{S}, S) \cup E(S, \bar{S})$, where $\bar{S} = V(D) - S$.

For a Γ-voltage assignment ϕ on the subgraph $\langle S\rangle$ of D, we define a new digraph $D \times_{(\phi,S)} \Gamma$ as follows. We adjoin an extra element, say ∞, to the group Γ with the property that $\gamma\infty = \infty = \infty\gamma$ for each $\gamma \in \Gamma \cup \{\infty\}$. Notice that $\Gamma \cup \{\infty\}$ is a semigroup. The vertex set $V(D \times_{(\phi,S)} \Gamma)$ is $(S \times \Gamma) \cup (\bar{S} \times \{\infty\})$ and let there be a directed edge from (u, α) to (v, β) if (i) $uv \in E(\bar{S}, \bar{S})$ and $\alpha = \beta = \infty$; (ii) $uv \in E(S, S)$, $\alpha, \beta \in \Gamma$ and $\phi(uv)\alpha = \beta$; (iii) $uv \in E(\bar{S}, S)$, $\alpha = \infty$ and $\beta \in \Gamma$; or (iv) $uv \in E(S, \bar{S})$, $\alpha \in \Gamma$ and $\beta = \infty$. We call $D \times_{(\phi,S)} \Gamma$ the *derived digraph* by a subset S of $V(D)$ and a Γ-voltage assignment ϕ on the subgraph $\langle S\rangle$ or simply, the *derived digraph*.

Now, we define a Γ-action on the derived digraph $D \times_{(\phi,S)} \Gamma$ by $\gamma(v, \alpha) = (v, \alpha\gamma^{-1})$ for all $\gamma \in \Gamma$ and $(v, \alpha) \in V(D \times_{(\phi,S)} \Gamma)$. Then $D \times_{(\phi,S)} \Gamma$ is a Γ-graph such that the fixed part Fix_Γ is $\bar{S} \times \{\infty\}$. Moreover, for each $(v, \gamma) \in S \times \Gamma$ the isotropy subgroup $\Gamma_{(v,\gamma)}$ is the trivial subgroup of Γ, i.e., each element of $S \times \Gamma$ is not fixed by any non-identity element of Γ. We call the quotient map $p : D \times_{(\phi,S)} \Gamma \to (D \times_{(\phi,S)} \Gamma)/\Gamma \cong D$ defined by $p(v, \alpha) = v$ for each $(v, \alpha) \in V(D \times_{(\phi,S)} \Gamma)$ the *natural projection*. Fig. 3 illustrates this construction.

Notice that if S is the full set $V(D)$, then the derived digraph $D \times_{(\phi,S)} \Gamma$ is a regular covering of D, and if S is the empty set, then the derived digraph $D \times_{(\phi,S)} \Gamma$ is just the digraph D. For a given Γ-graph D, let $S = (V(D) - \mathrm{Fix}_\Gamma)/\Gamma \in V(D/\Gamma)$. Then the quotient map $p : \langle V(D) - \mathrm{Fix}_\Gamma\rangle \to \langle S\rangle$ is a Γ-covering and there exists a voltage assignment ϕ on $\langle S\rangle$ such that $\langle S\rangle \times_\phi \Gamma = \langle V(D) - \mathrm{Fix}_\Gamma\rangle$. Now, it is clear that D is isomorphic to the derived digraph $(D/\Gamma) \times_{(\phi,S)} \Gamma$. We summarize our discussions as follows.

Theorem 1. *Let Γ be a finite group and D a finite pseudo-digraph. Then D is a Γ-graph if and only if there exist a subset S of $V(D/\Gamma)$ and a Γ-voltage assignment ϕ on the subgraph $\langle S\rangle$ of D/Γ such that D is isomorphic to the derived digraph $(D/\Gamma) \times_{(\phi,S)} \Gamma$.*

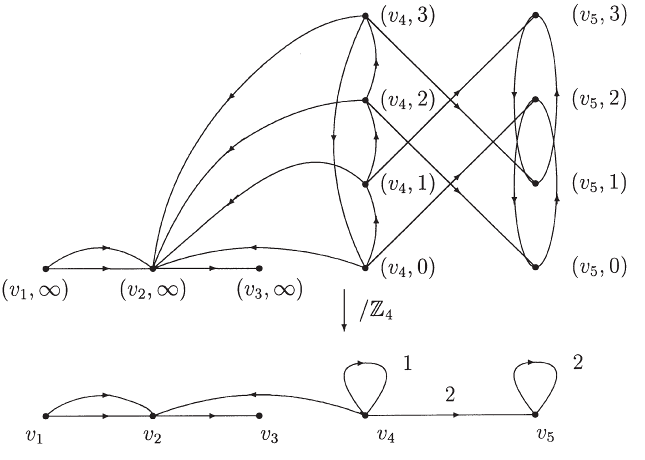


Fig. 3. $D \times_{(\phi,S)} \mathbb{Z}_4$ with $S = \{v_4, v_5\}$.

Notice that for some undirected pseudograph G, $\vec{G}/\Gamma$ need not be the digraph associated with a undirected graph even though $\vec{G}$ is a Γ-graph. Now, we consider a construction method of a Γ-graph G which is undirected. For a Γ-graph G, the digraph $\vec{G}$ is symmetric and also a Γ-graph. This implies that the quotient graph $\vec{G}/\Gamma$ is symmetric. Hence, to construct a Γ-graph, it is suffice to consider the base graph D of our construction as a symmetric digraph. A subset S of $V(D)$ is called a *$\mathscr{P}$-subset* if the number of directed loops based at each vertex in $\bar{S} = V(D) - S$ is even. Notice that if the derived digraph $D \times_{(\phi,S)} \Gamma$ is symmetric, then the subset S of $V(D)$ must be a $\mathscr{P}$-subset. For our purpose, we define a symmetric Γ-voltage assignment ϕ on the subgraph $\langle S\rangle$ induced by a $\mathscr{P}$-subset S of $V(D)$ as follows.

Definition 1. Let D be a finite symmetric connected digraph and S a $\mathscr{P}$-subset of $V(D)$. A Γ-voltage assignment ϕ on $\langle S\rangle$ is said to be *symmetric* if

(i) for each directed loop e based at v_i in S, there exists another directed loop e' based at v_i in S such that $\phi(e') = \phi(e)^{-1}$ if $\phi(e)$ is not of order 2;

(ii) for each directed edge $e = v_iv_j$ $(i \neq j)$ in $E(S, S)$ there exists $e' = v_jv_i$ in $E(S, S)$ such that $\phi(e') = \phi(e)^{-1}$.

Now, we aim to discuss a method to construct an undirected Γ-graph G. Let D be a finite connected symmetric digraph and S a $\mathscr{P}$-subset of $V(D)$. Let ϕ be symmetric Γ-voltage assignment on the subgraph $\langle S\rangle$. Then it is clear that the derived digraph $D \times_{(\phi,S)} \Gamma$ is symmetric. Since for each vertex (v, γ) of the derived digraph $D \times_{(\phi,S)} \Gamma$, the number of directed loops at (v, γ) is even. This implies that $D \times_{(\phi,S)} \Gamma = \vec{G}$ for some Γ-graph G.

Conversely, if $D = \vec{G}/\Gamma$ for some undirected Γ-graph G, then $S = (V(D) - \mathrm{Fix}_\Gamma)/\Gamma$ is a $\mathscr{P}$-subset of $V(D)$ and there exists a symmetric Γ-voltage assignment ϕ on $\langle S\rangle$ such that the derived digraph $D \times_{(\phi,S)} \Gamma$ is isomorphic to $\vec{G}$. We summarize our discussions as follows.

Theorem 2. *Let Γ be a finite group and D a finite connected symmetric digraph. Let S be a subset of $V(D)$ and ϕ a Γ-voltage assignment on the subgraph $\langle S\rangle$ of D. Then the derived digraph $D \times_{(\phi,S)} \Gamma = \vec{G}$ for some Γ-graph G if and only if S is a $\mathscr{P}$-subset of $V(D)$ and ϕ is symmetric.*

3. Adjacency matrices of Γ-graphs

Let D be a finite connected pseudo-digraph and S a subset of $V(D)$. For convenience, let $H_{11}(S) = (V(D),\ E(\bar{S},\ \bar{S}))$, $H_{12}(S) = (V(D),\ E(\bar{S}, S))$, $H_{21}(S) = (V(D),\ E(S, \bar{S}))$ and $H_{22}(S) = (V(D),\ E(S, S))$, where $\bar{S} = V(D) - S$. Then

$$A(D) = A(H_{11}(S)) + A(H_{12}(S)) + A(H_{21}(S)) + A(H_{22}(S)).$$

Let $V(D) = \{v_1, v_2, \dots, v_{|V(D)|}\}$ and $S = \{v_{|V(D)|-|S|+1}, \dots, v_{|V(D)|}\}$. Then the adjacency matrices of $H_{11}(S)$, $H_{12}(S)$, $H_{21}(S)$ and $H_{22}(S)$ are presented as follows.

$$A(H_{11}(S)) = \begin{bmatrix} A_{11}(S) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A(\langle \bar{S} \rangle) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A(H_{12}(S)) = \begin{bmatrix} 0 & A_{12}(S) \\ 0 & 0 \end{bmatrix},$$

$$A(H_{21}(S)) = \begin{bmatrix} 0 & 0 \\ A_{21}(S) & 0 \end{bmatrix},$$

$$A(H_{22}(S)) = \begin{bmatrix} 0 & 0 \\ 0 & A_{22}(S) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A(\langle S \rangle) \end{bmatrix}.$$

For each $1 \leqslant i,\ j \leqslant 2$, we call $A_{ij}(S)$ the *supporting matrix* of the subgraph $H_{ij}(S)$ of D.

Let Γ be a group and ϕ a Γ-voltage assignment on the induced subgraph $\langle S \rangle$ of a finite connected pseudo-digraph D. For each $\gamma \in \Gamma$, let $\langle S \rangle_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\langle S \rangle$ whose directed edge set is $\phi^{-1}(\gamma)$ so that the digraph $\langle S \rangle$ is the edge-disjoint union of spanning subgraphs $\langle S \rangle_{(\phi,\gamma)}$, $\gamma \in \Gamma$.

We define an order relation $\leqslant$ on the vertex set $V(D \times_{(\phi,S)} \Gamma)$ of the derived digraph $D \times_{(\phi,S)} \Gamma$ as follows: for any two vertices (v_i, α) and (v_j, β) of $D \times_{(\phi,S)} \Gamma$, $(v_i, \alpha) \leqslant (v_j, \beta)$ if and only if (i) $\alpha = \infty$ and $\beta \in \Gamma$, (ii) $\alpha, \beta \in \Gamma$ and $\alpha \leqslant \beta$ or (iii) $\alpha = \beta$ and $i \leqslant j$.

Now, under this order relation, we can show that

$$A(p^{-1}(H_{11}(S))) = \begin{bmatrix} A(\langle \bar{S} \rangle) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A(p^{-1}(H_{12}(S))) = \begin{bmatrix} 0 & A_{12}(S) & \cdots & A_{12}(S) \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$A(p^{-1}(H_{21}(S))) = \begin{bmatrix} 0 & 0 \\ A_{21}(S) & 0 \\ \vdots & \vdots \\ A_{21}(S) & 0 \end{bmatrix}, \quad A(p^{-1}(H_{22}(S))) = A(\langle S \rangle \times_\phi \Gamma),$$

where $p : D \times_{(\phi,S)} \Gamma \to D$ is the natural projection. Since

$$\begin{aligned} A(D \times_{(\phi,S)} \Gamma) = {} & A(p^{-1}(H_{11}(S))) + A(p^{-1}(H_{12}(S))) \\ & + A(p^{-1}(H_{21}(S))) + A(p^{-1}(H_{22}(S))), \end{aligned}$$

$$A(D \times_{(\phi,S)} \Gamma) = \left[\begin{array}{c|c} A(\bar{S}) & A_{12}(S) \cdots A_{12}(S) \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ A_{21}(S) & \\ \vdots & A(\langle S\rangle \times_\phi \Gamma) \\ A_{21}(S) & \end{array}\right].$$

To simplify the adjacency matrix $A(D \times_{(\phi,S)} \Gamma)$ of the derived digraph $D \times_{(\phi,S)} \Gamma$, we define the tensor product $A \otimes B$ of two matrices A and B by the matrix obtained from B when every element b_{ij} is replaced by the matrix Ab_{ij}. Now, by the virtue of the properties of the tensor product of matrices, we have following theorem.

Theorem 3. *Let Γ be a finite group and D a finite connected pseudo-digraph. Let S be a subset of $V(D)$ and ϕ a Γ-voltage assignment on the subgraph $\langle S\rangle$ of D. Then the adjacency matrix $A(D \times_{(\phi,S)} \Gamma)$ of the derived digraph $D \times_{(\phi,S)} \Gamma$ is*

$$A(D \times_{(\phi,S)} \Gamma) = \begin{bmatrix} A(\langle\bar{S}\rangle) & A_{12}(S) \otimes J \\ A_{21}(S) \otimes J^{\mathrm{t}} & \sum_{\gamma\in\Gamma} A(\langle S\rangle_{(\phi,\gamma)}) \otimes P(\gamma) \end{bmatrix},$$

where J is $[1\ 1 \cdots 1]$, $P(\gamma)$ is the $|\Gamma| \times |\Gamma|$ permutation matrix associated with $\gamma \in \Gamma$ and A^{t} is the transpose of a matrix A.

Let Γ be an abelian group. By the classification of finite abelian groups, Γ is isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$ for some $n_i \geqslant 2$ with $n_{i-1}|n_i$. For each $k = 1, \ldots, s$, let ρ_k denote a generator of the cyclic group $\mathbb{Z}_{n_k}$ so that $\mathbb{Z}_{n_k} = \{\rho_k^0, \rho_k^1, \ldots, \rho_k^{n_k-1}\}$. We define an order relation $\leqslant$ on the cyclic group $\mathbb{Z}_n = \{\rho^0, \ldots, \rho^{n-1}\}$ by $\rho^i \leqslant \rho^j$ if and only if $i \leqslant j$. Under this order relation, we can see that for a generator ρ of a cyclic group $\mathbb{Z}_n$, the permutation matrix $P(\rho)$ associated with ρ is expressed as follows:

$$P(\rho) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Notice that the order relation defined on a cyclic group $\mathbb{Z}_n$ gives an order relation on the product of cyclic groups Γ. For example, if $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, then $({\rho_1}^i, {\rho_2}^h) \leqslant ({\rho_1}^j, {\rho_2}^k)$ if and only if either $h < k$ or $h = k$ and $i \leqslant j$. Under this order relation, we can see that

$$P(\rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_s^{m_s}) = P(\rho_1)^{m_1} \otimes P(\rho_2)^{m_2} \otimes \cdots \otimes P(\rho_s)^{m_s}$$

for each $(\rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_s^{m_s}) \in \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$.

Since $P(\gamma)$ is diagonalizable for each $\gamma \in \Gamma$ and Γ is abelian, the matrices $P(\gamma), \gamma \in \Gamma$, are simultaneously diagonalizable. More precisely, there exists a unitary matrix M_Γ of order $|\Gamma|$ such that $M_\Gamma P(\gamma) M_\Gamma^{-1}$ is a diagonal matrix for each $\gamma \in \Gamma$ and $J M_\Gamma = \left[\sqrt{|\Gamma|}\ 0 \cdots 0\right]$. For convenience, let $\eta_k = \exp(2\pi \mathrm{i}/n_k)$ for each $k = 1, \ldots, s$. Then for each $\gamma = (\rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_s^{m_s})$ in $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$, we have

$$
\begin{aligned}
&M_\Gamma P(\gamma) M_\Gamma^{-1} \\
&\quad = \mathrm{Diag}[1, \lambda_{(\gamma,1)}, \ldots, \lambda_{(\gamma,|\Gamma|-1)}] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_{(\gamma,1)} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{(\gamma,|\Gamma|-1)} \end{bmatrix},
\end{aligned}
$$

where $\lambda_{(\gamma,j)} = \prod_{k=1}^{s} ({\eta_k}^{m_k})^{j_k}$ if $j = (n_1 n_2 \cdots n_{s-1}) j_s + (n_1 n_2 \cdots n_{s-2}) j_{s-1} + \cdots + (n_1) j_2 + j_1$ for some $j_k = 0, 1, \ldots, n_k - 1$. Then, by a simple computation, we can show that

$$
\begin{aligned}
&(I_m \oplus (I_n \otimes M_\Gamma))\, A(D \times_{(\phi,S)} \Gamma)\, (I_m \oplus (I_n \otimes M_\Gamma))^{-1} \\
&\quad = \begin{bmatrix} A(\langle \bar{S} \rangle) & \sqrt{|\Gamma|} A_{12}(S) \\ \sqrt{|\Gamma|} A_{21}(S) & A(\langle S \rangle) \end{bmatrix} \oplus \left[\bigoplus_{j=1}^{|\Gamma|-1} \sum_{\gamma \in \Gamma} \lambda_{(\gamma,j)} A\left(\langle S \rangle_{(\phi,\gamma)}\right) \right],
\end{aligned}
$$

where $m = |\bar{S}|$ and $n = |S|$. Hence, we have the following theorem.

Theorem 4. *Let Γ be a finite group and D a finite connected pseudo-digraph. Let S be a subset of $V(D)$ and ϕ a Γ-voltage assignment on the subgraph $\langle S \rangle$ of D. If Γ is abelian, then the adjacency matrix $A(D \times_{(\phi,S)} \Gamma)$ of $D \times_{(\phi,S)} \Gamma$ is similar to*

$$
\begin{bmatrix} A(\langle \bar{S} \rangle) & \sqrt{|\,\Gamma\,|} A_{12}(S) \\ \sqrt{|\,\Gamma\,|} A_{21}(S) & A(\langle S \rangle) \end{bmatrix} \oplus \left[\bigoplus_{j=1}^{|\Gamma|-1} \sum_{\gamma \in \Gamma} \lambda_{(\gamma,j)} A(\langle S \rangle_{(\phi,\gamma)}) \right],
$$

where $A_{12}(S)$ and $A_{21}(S)$ are the supporting matrices of $H_{12}(S)$ and $H_{21}(S)$, respectively, and $\bar{S} = V(D) - S$.

It is clear that if D is a symmetric connected digraph, then the transpose $A(H_{12}(S))^t$ of the matrix $A(H_{12}(S))$ is $A(H_{21}(S))$ for each subset S of $V(D)$, i.e., $A_{12}(S)^t = A_{21}(S)$. If S is a $\mathscr{P}$-subset of $V(D)$ and ϕ a symmetric Γ-voltage assignment on $\langle S \rangle$, then

$$
\left[A(\langle S \rangle_{(\phi,\gamma)})\right]^{\mathrm{t}} = A(\langle S \rangle_{(\phi,\gamma^{-1})}) \quad \text{for each } \gamma \in \Gamma.
$$

It follows from Theorem 2 that every Γ-graph G can be described as a derived digraph $D \times_{(\phi,S)} \Gamma$, where D is a symmetric digraph and ϕ is a symmetric Γ-voltage assignment on the subgraph $\langle S \rangle$ of D induced by a $\mathscr{P}$-subset S. Now, by

Theorems 2 and 4, we can find a similar form of the adjacency matrix of a Γ-graph G as follows.

Corollary 1. *Let Γ be a finite abelian group and D a finite symmetric connected pseudo-digraph. Let S be a $\mathscr{P}$-subset of $V(D)$ and ϕ a symmetric Γ-voltage assignment on the subgraph $\langle S\rangle$ of D. Then the adjacency matrix $A(D\times_{(\phi,S)}\Gamma)$ of the derived digraph $D\times_{(\phi,S)}\Gamma$ is similar to*

$$\begin{bmatrix} A(\langle\bar{S}\rangle) & \sqrt{|\,\Gamma\,|}A_{12}(S) \\ \sqrt{|\,\Gamma\,|}A_{12}(S)^{\mathrm{t}} & A(\langle S\rangle) \end{bmatrix} \oplus \left[\bigoplus_{j=1}^{|\Gamma|-1}\sum_{\gamma\in\Gamma}\lambda_{(\gamma,j)}A(\langle S\rangle_{(\phi,\gamma)})\right],$$

where $\bar{S}=V(D)-S$ and $A_{12}(S)$ is the supporting matrix of the graph $H_{12}(S)=(V(D),E(\bar{S},S))$.

4. Computation formulas

In this section, we aim to find a similar form of the adjacency matrix $A(D\times_{(\phi,S)}\Gamma)$ of the derived digraph $D\times_{(\phi,S)}\Gamma$ to make it easy to compute the characteristic polynomial of $D\times_{(\phi,S)}\Gamma$, and then obtain a computational formula for the characteristic polynomial of a Γ-graph G, by using our construction.

Let $\mathbb{C}$ denote the field of complex numbers. A weighted digraph is a pair $D_\omega=(D,\omega)$, where D is a digraph and $\omega : E(D)\to\mathbb{C}$ is a function. Given any weighted pseudo-digraph, the adjacency matrix $A(D_\omega)=(a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij}=\sum_{e\in E(\{v_i\},\{v_j\})}\omega(e).$$

The characteristic polynomial $\det(\lambda I-A(D_\omega))$ is denoted by $\Phi(D_\omega;\lambda)$ and is called the characteristic polynomial of D_ω.

Let Γ be a finite abelian group. For a $\mathscr{P}$-subset S of $V(D)$ and a symmetric Γ-voltage assignment ϕ on the subgraph $\langle S\rangle$, we define a function $\omega_0(\phi) : E(D)\to\mathbb{C}$ by

$$\omega_0(\phi)(e)=\begin{cases}1 & \text{if } e\in E(\bar{S},\bar{S})\cup E(S,S),\\ \sqrt{|\,\Gamma\,|} & \text{if } e\in E(\bar{S},S)\cup E(S,\bar{S}),\end{cases}$$

and define a function $\omega_k(\phi) : E(\langle S\rangle)\to\mathbb{C}$ by $\omega_k(\phi)(e)=\lambda_{(\phi(e),k)}$ for each $k=1,2,\ldots,|\Gamma|-1$.

Then, by Corollary 1, the adjacency matrix $A(D\times_{(\phi,S)}\Gamma)$ of the derived digraph $D\times_{(\phi,S)}\Gamma$ is similar to

$$A(D_{\omega_0(\phi)}) \oplus \left(\bigoplus_{j=1}^{|\Gamma|-1} A(\langle S \rangle_{\omega_j(\phi)}) \right).$$

Hence we have the following.

Theorem 5. *Let Γ be a finite abelian group and D a finite symmetric connected pseudo-digraph. Let S be a $\mathscr{P}$-subset of $V(D)$ and ϕ a symmetric Γ-voltage assignment on the subgraph $\langle S \rangle$ of D. Then the characteristic polynomial $\Phi(D \times_{(\phi,\, S)} \Gamma; \lambda)$ of the derived digraph $D \times_{(\phi,\, S)} \Gamma$ is*

$$\Phi(D \times_{(\phi, S)} \Gamma; \lambda) = \Phi(D_{\omega_0(\phi)}; \lambda) \times \prod_{j=1}^{|\Gamma|-1} \Phi\left(\langle S \rangle_{\omega_j(\phi)}; \lambda\right).$$

Now, we aim to find an explicit computational formula for the characteristic polynomial $\Phi(D \times_{(\phi,\, S)} \Gamma; \lambda)$ of the derived digraph $D \times_{(\phi,\, S)} \Gamma$. Notice that the matrix $A(D_{\omega_0(\phi)})$ is Hermitian and $A(\langle S \rangle_{\omega_j(\phi)})$ is also Hermitian for each $j = 1, \ldots, n-1$. For a Hermitian matrix A, let G be the simple graph associated with A, i.e., the number of vertices of G is equal to the number of columns (or rows) of A and there is an edge between two vertices v_s and v_t of G if and only if the (s, t)-entry of A is not zero. Let $G_0, G_1, \ldots, G_{n-1}$ be the simple graphs associated with the Hermitian matrices $A(D_{\omega_0(\phi)}), A(\langle S \rangle_{\omega_1(\phi)}), \ldots, A(\langle S \rangle_{\omega_{n-1}(\phi)})$, respectively. We define a function $\mu_0(\phi) : E(\vec{G}_0) \cup V(\vec{G}_0) \to \mathbb{C}$ as follows: $\mu_0(\phi)(v_s)$ is the (s, s)-entry of $A(D_{\mu_0(\phi)})$ for each $v_s \in V(\vec{G}_0)$ and $\mu_0(\phi)(v_s v_t)$ is the (s, t)-entry of $A(D_{\mu_0(\phi)})$ for each $v_s v_t \in E(\vec{G}_0)$. Then $A(G_0, \mu_0(\phi)) = A(D_{\omega_0(\phi)})$, where the adjacency matrix $A(G_0, \mu_0(\phi)) = (a_{ij})$ is defined by $a_{ij} = \mu_0(\phi)(v_i v_j)$ if $i \neq j$ and $a_{ij} = \mu_0(\phi)(v_i)$ if $i = j$. For each $j = 1, \ldots, n-1$, we can also define $\mu_j(\phi) : E(\vec{G}_j) \cup V(\vec{G}_j) \to \mathbb{C}$ so that $A(\vec{G}_j, \mu_j(\phi)) = A(\langle S \rangle_{\omega_j(\phi)})$. Notice that for each $i = 0, 1, \ldots, n-1$, $\overline{\mu_i(e)} = \mu_i(e^{-1})$ for each $e \in E(\vec{G}_i)$ and $\mu_i(v_s)$ is a real number for each $v_s \in V(\vec{G}_i)$. Such a weight function is said to be *symmetric* (see [5]). Now, we aim to compute the characteristic polynomials of the weighted graphs $(\vec{G}_i, \mu_i(\phi))$, $j = 0, \ldots, |\Gamma| - 1$. To do this, let G be a simple graph with a symmetric weight function $\omega : E(\vec{G}) \cup V(\vec{G}) \to \mathbb{C}$ on the digraph $\vec{G}$ of G. A subgraph P of G is called an *elementary configuration* if each of its components is either a cycle C_m $(m \geqslant 3)$, K_2 or K_1. We denote E_k as the set of all elementary configurations of G having k vertices, for each k. For an elementary configuration P, let $\kappa(P)$ denote the number of components of P, $C(P)$ the set of all cycles C_m $(m \geqslant 3)$ in P, and $I_v(P)$ and $I_E(P)$ the set of all isolated vertices and edges in P, respectively. Notice that for each cycle C in $C(P)$, there exist two directed cycles, say C^+ and C^-, in $\vec{G}$. By a slight modification of the method used in [5], we can show the following theorem.

Theorem 6. *Let G be a finite simple connected graph and $\omega : E(\vec{G}) \cup C(\vec{G}) \to \mathbb{C}$ a symmetric weight function on $\vec{G}$. Then*

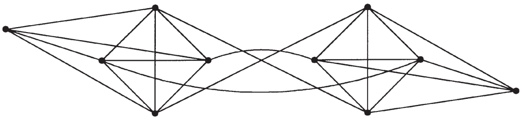


Fig. 4. A $\mathbb{Z}_4$-graph.

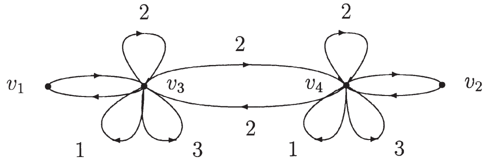


Fig. 5. A symmetric digraph D with a symmetric $\mathbb{Z}_4$-voltage assignment ϕ.

$$\Phi(\vec{G}_\omega;\lambda)=\lambda^{|V(D)|}+\sum_{k=1}^{n}\left[\sum_{P\in E_k}(-1)^{\kappa(P)}2^{|C(P)|}\prod_{u\in I_v(P)}\omega(u)\right.$$
$$\left.\times\prod_{e\in I_E(P)}|\omega(e)|^2\prod_{e\in C(P)}\mathrm{Re}(\omega(C^+))\right]\lambda^{|V(D)|-k},$$

where $\mathrm{Re}(\omega(C^+))$ *is the real part of* $\prod_{e\in C^+}\omega(e)$ *and the product over the empty index set is defined to be* 1.

Example. Let $\Gamma=\mathbb{Z}_4$ and G be the graph depicted in Fig. 4 .

Clearly, G is a $\mathbb{Z}_4$-graph. Then there exist a symmetric digraph D and a symmetric $\mathbb{Z}_4$-voltage assignment ϕ on a $\mathscr{P}$-subset S such that $\vec{G}$ is isomorphic to $D\times_{(\phi,S)}\mathbb{Z}_4$. Such a symmetric digraph D with a symmetric $\mathbb{Z}_4$-voltage assignment ϕ is depicted in Fig. 5. In Fig. 5, the $\mathscr{P}$-subset S of $V(D)$ is $\{v_3, v_4\}$.

By Corollary 1, we can see that the adjacency matrix $A(D\times_{(\phi,S)}\mathbb{Z}_4)$ is similar to

$$A(D_{\omega_0(\phi)})\oplus A(\langle S\rangle_{\omega_1(\phi)})\oplus A(\langle S\rangle_{\omega_2(\phi)})\oplus A(\langle S\rangle_{\omega_3(\phi)})$$
$$=\begin{bmatrix}0&0&2&0\\0&0&0&2\\2&0&3&1\\0&2&1&3\end{bmatrix}\oplus\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}\oplus\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\oplus\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}.$$

Notice that the simple graph G_0 associated with the Hermitian matrix $A(D_{\omega_0(\phi)})$ is the path P_4 on $V(D)$ and for each $i=1,2,3$, the simple graph G_i associated with the Hermitian matrix $A(\langle S\rangle_{\omega_i(\phi)})$ is the complete graph K_2 on two vertices $\{v_3, v_4\}$.

Moreover, for each $i = 0, 1, 2, 3$, it is not hard to find the weight function $\mu_i(\phi)$ on $\vec{G}_i$. Now, by Theorem 6, we have

$$\Phi(G;\lambda) = \left(\lambda^4 - 6\lambda^3 + 24\lambda + 16\right)\left(\lambda^2 + 2\lambda\right)\left(\lambda^2 + 2\lambda\right)\left(\lambda^2 + 2\lambda\right).$$

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